Note on "Wavelets, Gaussian mixtures and Wiener filtering"

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Abstract

In this work we discuss an improvement of the image-denoising wavelet-based method presented by [1]. This method is based on the estimation of the signal power at each wavelet scale and the proportion between signal and background at each scale. These parameters were estimated from the 2^nd^ and 4^th^ moments of the wavelet coefficients at the corresponding scale. In this work, we explored the use of any combination of the 2^nd^, 4^th^ and 6^th^ moments. We show that the 2^nd^ and 6^th^ moments yield better results for the experiments carried out.

1. Introduction

Many are the papers addressing the problem of image denoising using wavelets. In brief, all these algorithms first perform the wavelet transform of the image to denoise, then apply some filter to the wavelet coefficients, and finally take the inverse wavelet transform to restore the denoised image. Most popular wavelet-filtering algorithms are based on thresholding [2] or Wiener filtering [3]. Bijaoui introduced [1] a Bayesian approach to image denoising in wavelet space shown to be superior to a number of previous thresholding or Wiener-filtering algorithms. The technique needs to estimate the signal power at each scale as well as the signal proportion with respect to the background. Bijaoui used the 2^{nd} and 4^{th} moments of the wavelet coefficients at each scale. The rationale for choosing these moments is that low-order moments are more reliably estimated than high-order ones. A second reason is that the equation system to solve is also simpler. We explored the use of the 6^{th} moment. This yields more complicated equations and a higher number of combinations (the algorithm parameters can be obtained either from the 2^{nd} and 4^{th} moments, 2^{nd} and 6^{th}, 4^{th} and 6^{th}, or 2^{nd}, 4^{th} and 6^{th}). We explored all of them and show that the 2^{nd} and 6^{th} is generally a better choice for the experiments carried out. We provide a strategy to choose between the 2^{nd} and 6^{th} moment solution and the 2^{nd} and 4^{th}.

We tested our algorithms on a set of simulated Electron Microscopy images as used in single-particle structural studies of macromolecular complexes [4]. One of the main drawbacks of EM images are their extremely low Signal-to-Noise Ratio (around -3dB).

2. The filtering procedure

The filtering technique used in this article is based on the Bayesian approach proposed by [1]. Let us assume that the image formation model is of the form $y = x + n$, where x is the ideal image, n is random independent noise, and y is the measured image. We assume that the noise is white and normally distributed. Thus, its probability density function (PDF) is given by

$$p(n) = G(n, N) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{n^2}{2N}}$$

where N is the noise variance. Solving the image formation model for n, it can be seen that the conditional PDF of y given x is

$$p(y \mid x) = p(n) = G(y - x, N).$$

It is assumed that the PDF of the ideal image can be expressed as a mixture of zero-mean Gaussians

$$p(x) = \sum_i \alpha_i G(x, S_i).$$

Under these assumptions it can be proved that the PDF of the measured image is given by the convolution of both PDFs

$$p(y) = p(x) * p(n) = \sum_i \alpha_i G(x, S_i + N).$$

Then, the denoising problem is stated as the Bayesian problem of estimating x from y. The *a posteriori* PDF is given by

$$p(x \mid y) = \frac{\sum_i \alpha_i G(x, S_i) G(y - x, N)}{\sum_i \alpha_i G(y, S_i + N)}$$

The filtering is done by taking the *a posteriori* expectation of x

$$E\{x \mid y\} = y\frac{\sum_i \alpha_i \frac{S_i}{S_i + N} G(y, S_i + N)}{\sum_i \alpha_i G(y, S_i + N)} = yw(y)$$

The relationship of this filter with classical Wiener filtering is given in the article by [1].

The previous filter is applied independently to each of the scales of the wavelet transform of the measured image, i.e., the variable x is formed by all those wavelet coefficients belonging to the same scale. Therefore, each scale has its own α_i and S_i parameters while the noise power, N, is common to all the scales since the noise is assumed to be white.

The problem now is to estimate the α_i, S_i and N parameters from the measured images. Bijaoui assumes that there is no signal at the finest scale. Therefore, all the energy at that scale is coming from the noise term. Under this assumption the noise power N can be estimated from that scale and later used for the rest of the scales. Bijaoui proposes a robust estimate of N at the finest scale based on a k-σ clipping strategy.

For the rest of the scales, the original paper affirms that, in practical terms, only two Gaussians are necessary to model the PDF of the measured wavelet coefficients at a given scale

$$p(y) = (1-a)G(y, N) + aG(y, S+N) \qquad \text{(Eq. 1)},$$

where a and S are parameters defining the distribution of y. Notice that a is a measure of the proportion of the area occupied by the signal and the area occupied by the background and S is the power of the signal coefficients. a and S are estimated at each scale through the 2nd (M_2) and 4th-order (M_4) moments of the measured wavelet coefficients at that scale

$$M_2 = (1-a)N + a(S+N)$$
$$M_4 = 3(1-a)N^2 + 3a(S+N)^2$$

that yields

$$S = \frac{\frac{M_4}{3} - N^2}{M_2 - N}$$

$$a = \frac{(M_2 - N)^2}{\frac{M_4}{3} - N^2}$$

In order to guarantee the convex nature of the combination given by Eq. 1 a and S are set to 0 if $M_2 - N < 0$ or if $\frac{M_4}{3} - N^2 < 0$. Accordingly, a is set to 1 if $a > 1$ and, in this case, S is estimated only from the variance M_2. It should be noted that the conditions imposed in the original article do not cover all the space of possibilities since if $3N^2 < M_4 < 3((M_2 - N)^2 + N^2)$, then $S < 0$ and $a > 1$. An appealing choice in this case would be to set S to 0 and a to 1. However, this choice yields the same PDF as $S = a = 0$, i.e., the measured image is compound only of noise.

3. Alternative parameter estimation

We explored the possibility of using other moments to obtain the parameter estimates. In particular we used different combinations of the three equations:

$$M_2 = (1-a)N + a(S+N)$$
$$M_4 = 3(1-a)N^2 + 3a(S+N)^2$$
$$M_6 = 15(1-a)N^3 + 15a(S+N)^3$$

3.1. Solution for M_2 and M_6

The solution for S and a using the equations of M_2 and M_6 is

$$S = \frac{-45N(M_2 - N) + \sqrt{15(N - M_2)(-4M_6 + 15N^2(3M_2 + N))}}{30M_2 + N}$$

$$a = \frac{(M_2 - N)(45N(M_2 - N) + \sqrt{15(N - M_2)(-4M_6 + 15N^2(3M_2 + N))}}{2(M_6 + 15N^2(2N - 3M_2))}$$

If $M_6 < \max\{15(3M_2N^2 - 2N^3), \frac{15}{4}N^2(3M_2 + N)\}$ or $M_2 < N$, then we set a and S to 0. If $S > 0$ and $M_6 \le 15M_2^3$, then we set a to 1, and S is estimated only from the variance M_2.

3.2. Solution for M_4 and M_6

The solution for S and a using the equations of M_4 and M_6 is

$$S = \frac{M_6 - 15M_4N + 30N^3 - \sqrt{M_6^2 + 10M_6N(M_4 - 6N^2) - 75M_4N^2(M_4 - 4N^2)}}{10(M_4 - 3N^2)}$$

$$a = \frac{M_6^2 + M_6(-30N^3 - \sqrt{M_6^2 + 10M_6N(M_4 - 6N^2) - 75M_4N^2(M_4 - 4N^2)}}{30N^3(2M_6 + 15N(N^2 - M_4)}$$
$$- \frac{5N(15M_4^2N - 90M_4N^3 + 90N^5 + M_4\sqrt{M_6^2 + 10M_6N(M_4 - 6N^2) - 75M_4N^2(M_4 - 4N^2)}}{30N^3(2M_6 + 15N(N^2 - M_4)}$$

If $M_4 < 3N^2$ or $M_6 \le \frac{15}{2}(M_4 N - N^3)$ and $M_4 > 3N^2$, then a and S are set to 0. If $S > 0$ and $a > 1$, then a is set to 1 and S is estimated from the 4th order moment.

3.3. Solution for M_2, M_4 and M_6

In this case we are solving three equations for two unknowns. Due to the noisy nature of the equation system this equation system is likely to be inconsistent. In this case we solve the equation system in a Least-Squares sense subject to $S \ge 0$ and $0 \le a \le 1$. The possible solutions for S for this minimization problem must be roots of the following polynomial

$$\begin{aligned} p(S) = \; & ((225N^3 - 15M_6)S^2 + (675N^4 + 9N^2 - 45M_6 N - 3M_4)S \\ & + 675N^5 + 18N^3 - 45M_6 N^2 - 6M_4 N + N - M_2) \\ & ((675N^2 - 225M_4)S^4 + (4050N^3 - 1575M_4 N + 150N + 45M_6 - 150M_2)S^3 \\ & + (8100N^4 - 4050M_4 N^2 + 675N^2 + 270M_6 N - 675M_2 N)S^2 \\ & + (6750N^5 - 4725M_4 N^3 + 975N^3 + 495M_6 N^2 - 1125M_2 N^2 + 3N + 10M_6 - 3M_2)S \\ & + 2025N^6 - 2025M_4 N^4 + 450N^4 + 270M_6 N^3 - 675M_2 N^3 + 3N^2 + 15M_6 N - 6M_2 N + M_4) \end{aligned}$$

whereas the corresponding a is given by

$$a = \frac{M_2 - N + 6M_4 N + 45M_6 N^2 - 18N^3 - 675N^5 + (3M_4 + 45M_6 N - 9N^2 - 675N^4)S + (15M_6 - 225N^3)S^2}{S(1 + 36N^2 + 2025N^4 + (36N + 4050N^3)S + (9 + 3375N^2)S^2 + 1350NS^3 + 225S^4}$$

The referred polynomial always has two roots such that a is equal to 0. We only consider the real non-negative roots such that the corresponding a is between 0 and 1.

4. Results

In order to test the efficacy of the newly proposed parameter estimation equations we denoised 600 projection images (of size 64x64) of the Protein Data Bank [5] model of the bacteriorhodopsin [6] (PDB entry: 1BRD, see Fig. 1). These projection images were computed as line integrals of a voxelized volume created from the atom description provided by the PDB model. The Xmipp package [7] was used to create these projections. We added white Gaussian noise up to a Signal-to-Noise Ratio (SNR) of -3dB. These parameters yield similar images to those obtained in electron cryomicroscopy [4]. An orthonormal wavelet decomposition (with Daubechies 12 as wavelet function [8]) was employed. We applied the denoising procedures in the first four scales ($s = 1,2,3,4$) since further scales had too few wavelet coefficients to make reliable estimates of the statistical moments.

In our experiments we observed that real roots of the polynomial for the solution of M_2, M_4 and M_6 such that $0 < a \le 1$ should be preferred to those where $a = 0$ even if they do not provide the global minimum of the minimization problem. Our approach took the S, a solution ($0 < a \le 1$) such that the objective function was minimum. If such a solution did not exist, then $S = a = 0$ (notice that there are always two solutions for which $a = 0$).

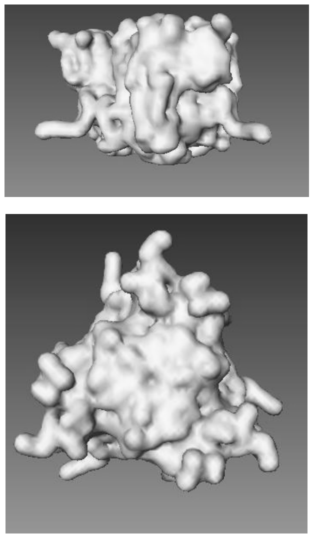

Figura 1. *Side and top view of the isosurface of the bacteriorhodopsin*

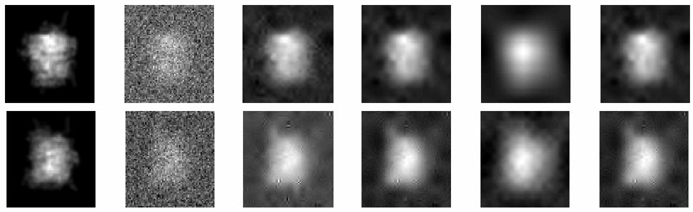

Figura 2. *Each row shows a different example of the results for cryomicroscopy conditions of the different parameter estimation alternatives. From left to right: original image, noisy image, image denoised using M_2 and M_4, image denoised using M_2 and M_6, image denoised using M_4 and M_6, image denoised using M_2, M_4 and M_6.*

Figure 2 shows some examples of the kind of images considered as well as of the application of the different parameter estimation alternatives. Table 1 shows the average SNR for each of the methods and their corresponding standard deviations.

	Noisy image	M_2, M_4	M_2, M_6	M_4, M_6	M_2, M_4, M_6
Average	-2.99	11.22	11.21	8.09	11.31
Std. Dev.	0.11	1.05	1.48	1.06	1.54

Figura 3. *Average and standard deviation of the SNR in dB for the different proposed parameter estimation equations*

Clearly, the denoising performed with M_4, M_6 is inferior to the other three. This is visually confirmed in Figure 2 where it can be seen that these denoised images are too smooth. The average of the rest of performances are statistically undistinguishable with a confidence of 99.5%. Although, the variance of M_2, M_4 is significantly smaller than the ones of M_2, M_6 and M_2, M_4, M_6 with a confidence of 99.5%.

In order to improve the SNR of the denoised image we propose the following strategy: denoise the input image with several parameter estimation procedures; evaluate the energy of each output image; then, select as output the image with smaller energy. We applied this strategy to our experiments with methods M_2, M_4; M_2, M_6 and M_2, M_4, M_6. The strategy selected M_2, M_4, M_6 in 97% of the cases (in 27% of them, the selected image was the one with the best SNR). The SNR achieved by following the strategy was 11.31±1.54 while the best SNR achievable with these three methods was 11.69±0.81. The best SNR achievable is defined as the average of the maximum of the SNR obtained by each of the parameter estimation methods. Since the average performance of M_2, M_4, M_6 is statistically undistinguishable from the other two techniques, not much is gained with respect to the case in which M_2, M_4, M_6 is always used for the parameter estimation.

However, if M_2, M_4, M_6 is excluded from the comparison and the strategy is applied only to M_2, M_4 and M_2, M_6 then this strategy selected in 90% of the cases the best denoising technique which turned out to be M_2, M_6 in 74% of them. The SNR achieved by following the strategy is 11.64±0.81 which is statistically undistinguishable from the best achievable with these two methods (11.65±0.80). The improvement with respect to M_2, M_4 and M_2, M_6 is significantly different with a confidence of 99.5%.

Summarizing, our experiments showed that M_2, M_4 was the choice with the smallest variance among the available possibilities (using the 2nd, 4th and 6th moments of the wavelet coefficients) to estimate the signal parameters in a Bayesian denoising framework. However, it is not always the one with highest performance. Particularly, M_2, M_6 usually shows better performance although in a small number of cases it may produce solutions that are too smooth. We proposed a simple strategy to choose between the results of the current M_2, M_4 approach and the new one based on M_2, M_6. M_4, M_6 always produced too smooth output images. In our experiments, we checked the effect of selecting the output image as the one with smallest variance among the output of a set of bayesian denoised images using different parameter estimation procedures. We showed that this strategy effectively improved the SNR achieved if this comparison was performed between the outputs of M_2, M_4 and M_2, M_6. M_2, M_4, M_6 was excluded from the strategy although it produced output images of similar quality to the one of the other two procedures.

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