Universidad San Pablo - CEU

Dynamic Systems in Biomedical Engineering

Problems

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CEU
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1 Chapter 1

Kreyszig, 1.1.2

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$y' + xe^{-\frac{x^2}{2}} = 0$$

Solution:

$$y' = \frac{dy}{dx} = -xe^{-\frac{x^2}{2}}$$

By separating variables

$$dy = -xe^{-\frac{x^2}{2}} dx$$

and integrating

$$\int dy = \int -xe^{-\frac{x^2}{2}} dx$$

$$\boxed{y = e^{-\frac{x^2}{2}} + C}$$

Kreyszig, 1.1.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$y' = 4e^{-x}\cos(x)$$

Solution:

$$y' = \frac{dy}{dx} = 4e^{-x}\cos(x)$$

By separating variables

$$dy = 4e^{-x}\cos(x)dx$$

and integrating

$$\int dy = \int 4e^{-x}\cos(x)dx$$

Let's integrate by parts:

$$\begin{array}{rcl}
I_1 & = & \int e^{-x}\cos(x)dx \quad [u = e^{-x}, dv = \cos(x)dx] \\
 & = & e^{-x}\sin(x) - \int \sin(x)(-e^{-x}dx) \\
 & = & e^{-x}\sin(x) + \int \sin(x)e^{-x}dx \quad [u = e^{-x}, dv = \sin(x)dx] \\
 & = & e^{-x}\sin(x) + e^{-x}(-\cos(x)) - \int (-\cos(x))(-e^{-x}dx) \\
 & = & e^{-x}\sin(x) - e^{-x}\cos(x) - \int \cos(x)e^{-x}dx \\
 & = & e^{-x}\sin(x) - e^{-x}\cos(x) - I_1 \Rightarrow \\
2I_1 & = & e^{-x}\sin(x) - e^{-x}\cos(x) \Rightarrow \\
I_1 & = & e^{-x}\frac{\sin(x)-\cos(x)}{2}
\end{array}$$

Finally

$$\boxed{y} = 4I_1 + C = \boxed{2(\sin(x) - \cos(x))e^{-x} + C}$$

Kreyszig, 1.1.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$y'' = -y$$

Solution: Let us try a particular solution of the form

$$\begin{aligned} y &= e^{\lambda x} \\ y' &= \lambda e^{\lambda x} \\ y'' &= \lambda^2 e^{\lambda x} \end{aligned}$$

Then, substituting these functions in the ODE

$$\lambda^2 e^{\lambda x} = -e^{\lambda x}$$

$$\lambda^2 = -1 \Rightarrow \lambda = \pm i$$

So the two functions

$$y_1 = e^{ix}$$

and

$$y_2 = e^{-ix}$$

are solutions of the ODE. Actually, any function of the form

$$\boxed{y} = c_1 y_1 + c_2 y_2 = \boxed{c_1 e^{ix} + c_2 e^{-ix}}$$

is also a solution. In fact, it is the general solution of the ODE. Let us check this statement

$$\begin{aligned} y' &= ic_1 e^{ix} - ic_2 e^{-ix} \\ y'' &= -c_1 e^{ix} - c_2 e^{-ix} \end{aligned}$$

Substituting in the ODE

$$y'' = -y$$

$$-c_1 e^{ix} - c_2 e^{-ix} = -(c_1 e^{ix} + c_2 e^{-ix})$$

As can be easily seen the function

$$y = c_1 e^{ix} + c_2 e^{-ix} + C$$

with $C \neq 0$ is not a solution of the ODE

$$-c_1 e^{ix} - c_2 e^{-ix} \neq -(c_1 e^{ix} + c_2 e^{-ix} + C)$$

Kreyszig, 1.1.7

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$y' = \cosh(5.13x)$$

Solution: To solve the proposed ODE we rewrite it as

$$\frac{dy}{dx} = \cosh(5.13x)$$

Consequently

$$dy = \cosh(5.13x)dx$$

Integrating

$$\int dy = \int \cosh(5.13x)dx$$

$$\boxed{y = \frac{1}{5.13}\sinh(5.13x) + C}$$

Kreyszig, 1.1.8

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$y''' = e^{-0.2x}$$

Solution: Let us define

$$y_1 = y'$$
$$y_2 = y_1' = y''$$

Then the ODE can be rewritten as

$$y_2' = e^{-0.2x}$$

whose solution is

$$dy_2 = e^{-0.2x}dx$$

$$y_2 = \frac{1}{-0.2}e^{-0.2x} + c_1 = -5e^{-0.2x} + c_1$$

Now we solve the equation

$$y_1' = y_2 = -5e^{-0.2x} + c_1$$

$$dy_1 = (-5e^{-0.2x} + c_1)dx$$

$$y_1 = 25e^{-0.2x} + c_1 x + c_2$$

And, finally, the equation

$$y' = y_1 = 25e^{-0.2x} + c_1 x + c_2$$

$$dy = (25e^{-0.2x} + c_1 x + c_2)dx$$

$$y = -125e^{-0.2x} + \frac{c_1}{2}x^2 + c_2 x + c_3$$

Since c_1 is an arbitrary constant, we can absorb the $\frac{1}{2}$ factor into c_1, so that the general solution is

$$\boxed{y = -125e^{-0.2x} + c_1 x^2 + c_2 x + c_3}$$

Kreyszig, 1.1.10

Carlos Oscar Sorzano, Aug. 31st, 2014

1. Verify that $y = ce^{-2.5x^2}$ is a solution of the ODE

$$y' + 5xy = 0$$

2. Determine from y the particular solution of the ODE that satisfies the initial condition $y(0) = \pi$.

3. Graph the solution of the IVP.

Solution:

1. Let us calculate y' and substitute it into the ODE

$$\begin{gathered} y' = -5cxe^{-2.5x^2} \\ \left(-5cxe^{-2.5x^2}\right) + 5x\left(ce^{-2.5x^2}\right) = 0 \\ -5cxe^{-2.5x^2} + 5cxe^{-2.5x^2} = 0 \\ 0 = 0 \end{gathered}$$

So y is actually a solution of the ODE.

2. To satisfy the initial condition we need

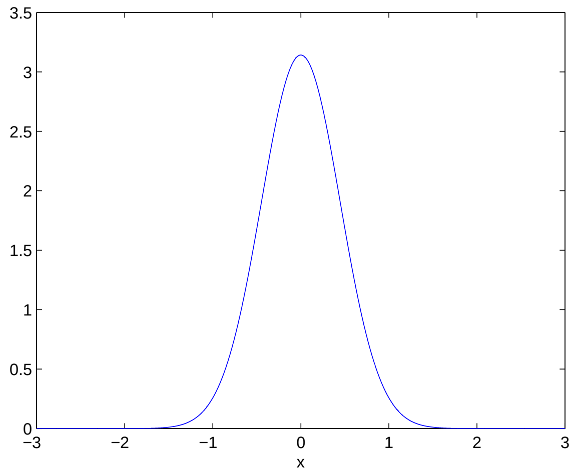
$$y(0) = \pi = ce^{-2.5(0)^2} = ce^0 = c$$

that is, we need $c = \pi$. The particular solution fulfilling the initial condition is

$$\boxed{y_p = \pi e^{-2.5x^2}}$$

3. In MATLAB:

```
x=[-3:0.001:3]; plot(x,pi*exp(-2.5*x.^2)); xlabel('x');
```



Kreyszig, 1.1.12

Carlos Oscar Sorzano, Aug. 31st, 2014

1. Verify that $y^2 - 4x^2 = C$ is a solution of the ODE

$$yy' = 4x$$

2. Determine from y the particular solution of the ODE that satisfies the initial condition $y(1) = 4$.

3. Graph the solution of the IVP.

Solution:

1. Let us differentiate the equation defining the implicit function

$$D_x(y^2 - 4x^2 = C)$$
$$2yy' - 8x = 0$$
$$yy' = 4x$$

that is exactly the ODE, so the implicit function defined by $y^2 - 4x^2 = C$ is actually a solution of the proposed ODE.

2. To satisfy the initial condition $y(1) = 4$ we need

$$y^2 - 4x^2 = C$$

$$(4)^2 - 4(1)^2 = C$$

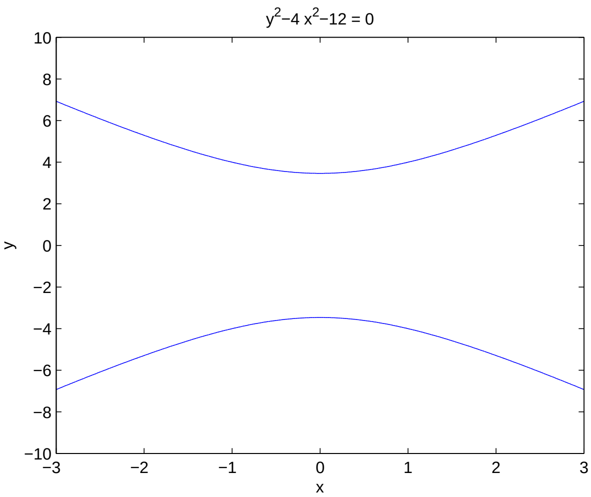
$$C = 16 - 4 = 12$$

So the particular solution satisfying the given initial condition is

$$\boxed{y_p^2 - 4x^2 = 12}$$

3. In MATLAB:

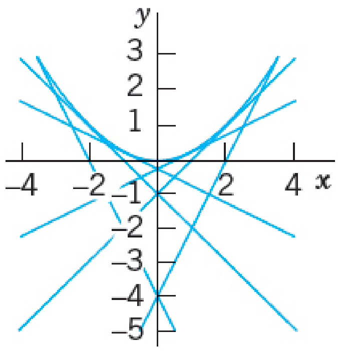
```
h=ezplot('y.^2-4*x.^2-12',[-3 3 -10 10]);
set(h,'Color','b')
```



Kreyszig, 1.1.16

Carlos Oscar Sorzano, Aug. 31st, 2014

An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a **singular solution**. The ODE $(y')^2 - xy' + y = 0$ is of this kind. Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = \frac{1}{4}x^2$. Explain the following figure.



Solution: Let us calculate the derivative of the proposed solution

$$y = cx - c^2 \Rightarrow y' = c$$

Substituting in the ODE

$$(y')^2 - xy' + y = 0$$
$$(c)^2 - x(c) + (cx - c^2) = 0$$
$$0 = 0$$

So the proposed solution is a solution of the ODE. However, the function $y = \frac{1}{4}x^2$ is also a solution as can be easily verified

$$y = \frac{1}{4}x^2 \Rightarrow y' = \frac{1}{2}x$$

$$(y')^2 - xy' + y = 0$$
$$\left(\frac{1}{2}x\right)^2 - x\left(\frac{1}{2}x\right) + \left(\frac{1}{4}x^2\right) = 0$$
$$\frac{1}{4}x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^2 = 0$$
$$0 = 0$$

The explanation of the proposed figure is the following. The different lines correspond to different values of c in the general solution

$$y = cx - c^2$$

The function $y = \frac{1}{4}x^2$ is the upper envelope of all these functions.

Kreyszig, 1.1.18

Carlos Oscar Sorzano, Aug. 31st, 2014

Radium $^{228}_{88}$Ra has a half-life of about 3.6 days.

1. Given 1 gram, how much will still be present after 1 day?
2. After 1 year?

Solution: Radioactive desintegration responds to the linear ODE

$$\frac{dA}{dt} = -Kt$$

whose general solution is

$$A(t) = A(0)e^{-Kt} \quad t > 0$$

Note that the units of K are $[time^{-1}]$. We can also write the general solution as

$$A(t) = A(0)e^{-\frac{t}{\tau}} \quad t > 0$$

where the units of τ are now $[time]$.

A half-life of 3.6 days implies that

$$A(3.6) = \frac{A(0)}{2} = A(0)e^{-\frac{3.6}{\tau}}$$

$$-\log(2) = -\frac{3.6}{\tau}$$

$$\tau = \frac{3.6}{\log(2)} = 5.1937[days]$$

At this point we can answer the two questions:

1. After 1 day there is: $A(1) = A(0)e^{-\frac{1}{\tau}} = 1e^{-\frac{1}{5.1937}} = 0.8249[g]$.
2. After 1 year there is: $A(365) = A(0)e^{-\frac{365}{\tau}} = 1e^{-\frac{365}{5.1937}} = 3 \cdot 10^{-31}[g]$.

Kreyszig, 1.1.19

Carlos Oscar Sorzano, Aug. 31st, 2014

In dropping a stone or an iron ball, air resistance is practically negligible. Experiments show that the acceleration of the motion is constant (equal to $g = 9.80[m/s^2]$, called the acceleration of gravity). Model this as an ODE for $y(t)$, the distance fallen as a function of time t. If the motion starts at time $t = 0$ from rest (i.e., with velocity $v = y' = 0$), show that you obtain the familiar law of free fall

$$y = \frac{1}{2}gt^2$$

Solution: Let us understand the physical meaning of each of the variables involved:

- $y(t)$ is the distance fallen at time t
- $y'(t)$ is the speed of the object at time t

- $y''(t)$ is its acceleration at time t

The fact that acceleration is constant along the fall implies

$$y'' = g$$

Let us define the variable

$$v = y'$$

Then, the free fall ODE can be written as

$$v' = g$$

$$dv = gdt$$

$$v = gt + c$$

But the object is at rest at $t = 0$, that is

$$v(0) = 0 = g(0) + c \Rightarrow c = 0$$

Now we solve the equation

$$v = y'$$

for y

$$dy = vdt = gtdt$$

$$y = \frac{1}{2}gt^2 + c$$

At time $t = 0$ the object had not moved, that is

$$y(0) = 0 = \frac{1}{2}g(0)^2 + c \Rightarrow c = 0$$

Finally, the solution of the falling ODE is

$$\boxed{y = \frac{1}{2}gt^2}$$

Kreyszig, 1.2.4

Carlos Oscar Sorzano, Aug. 31st, 2014

Graph a direction field (by a CAS or by hand) for the ODE

$$y' = 2y - y^2$$

In the field graph several solution curves by hand, particularly those passing through the points $(0,0), (0,1), (0,2), (0,3)$.
Solution: In MATLAB

```
[x,y]=meshgrid(-1:0.25:5,-2:0.25:4);
f = @(x,y) 2*y-y.^2;
dy=feval(f,x,y);
dx=ones(size(dy));
quiver(x,y,dx,dy);
axis([-1 5 -2 4])
```

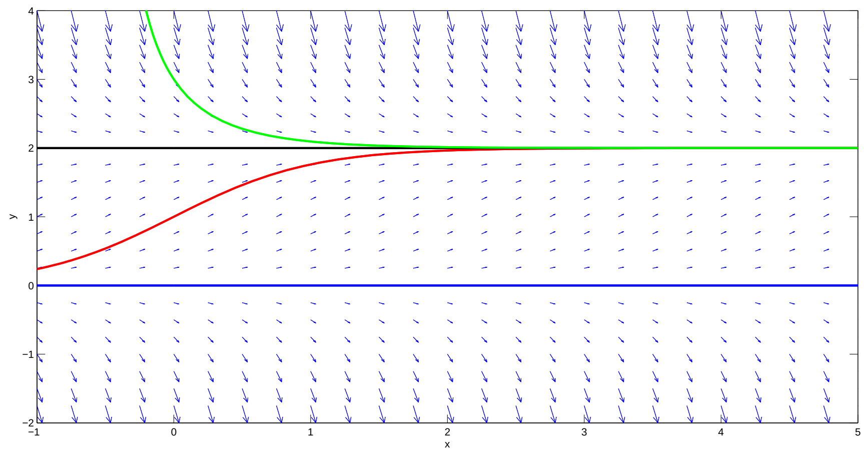
```
xlabel('x')
ylabel('y')
hold on

% (0,0)
[xa,ya] = ode45(f,[0,5],0);
[xb,yb] = ode45(f,[0,-1],0);
plot(xa,ya,'b','LineWidth',2)
plot(xb,yb,'b','LineWidth',2)

% (0,1)
[xa,ya] = ode45(f,[0,5],1);
[xb,yb] = ode45(f,[0,-1],1);
plot(xa,ya,'r','LineWidth',2)
plot(xb,yb,'r','LineWidth',2)

% (0,2)
[xa,ya] = ode45(f,[0,5],2);
[xb,yb] = ode45(f,[0,-1],2);
plot(xa,ya,'k','LineWidth',2)
plot(xb,yb,'k','LineWidth',2)

% (0,3)
[xa,ya] = ode45(f,[0,5],3);
[xb,yb] = ode45(f,[0,-1],3);
plot(xa,ya,'g','LineWidth',2)
plot(xb,yb,'g','LineWidth',2)
```



Kreyszig, 1.2.5

Carlos Oscar Sorzano, Aug. 31st, 2014

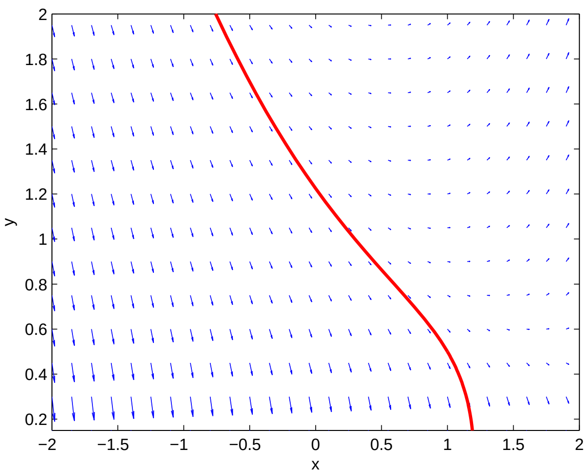
Graph a direction field (by a CAS or by hand) for the ODE

$$y' = x - \frac{1}{y}$$

In the field graph several solution curves by hand, particularly that one passing through the point $(1, \frac{1}{2})$.
Solution: In MATLAB

```
[x,y]=meshgrid(-2:0.15:2,0.15:0.15:2);
f = @(x,y) x-1./y;
dy=feval(f,x,y);
dx=ones(size(dy));
quiver(x,y,dx,dy);
axis([-2 2 0.15 2])
xlabel('x')
ylabel('y')
hold on

% (1,0.5) [xa,ya] = ode45(f,[1,1.2],0.5);
[xb,yb] = ode45(f,[1,-2],0.5);
plot(xa,ya,'r','LineWidth',2)
plot(xb,yb,'r','LineWidth',2)
```



Kreyszig, 1.2.11

Carlos Oscar Sorzano, Aug. 31st, 2014

An ODE is autonomous if it does not show x (the independent variable) explicitly in f

$$y' = f(x, y)$$

For instance,

$$y' = \sin^2(y)$$

$$y' = -5y^{\frac{1}{2}}$$

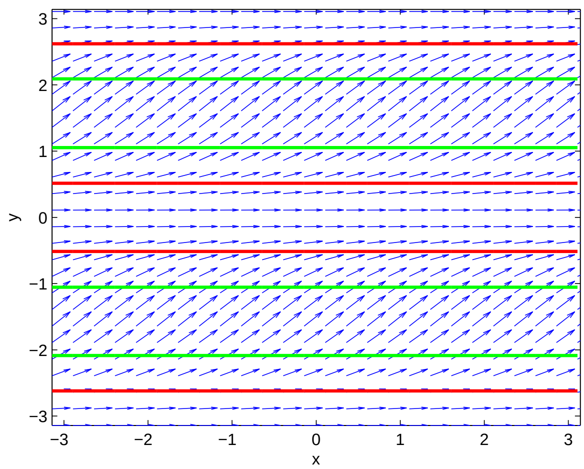
What will the level curves $f(x, y) = const$ (also called **isoclines**, of equal inclination) of an autonomous ODE look like? Give reason.
Solution: They are lines parallel to the x axis, since all points with the same x have the same inclination (slope of the tangent). For example, for the equation

$$y' = \sin^2(y)$$

we would have in MATLAB

```
[x,y]=meshgrid(-pi:0.25:pi,-pi:0.25:pi);
f = @(x,y) (sin(y)).^2;
dy=feval(f,x,y);
dx=ones(size(dy));
quiver(x,y,dx,dy);
axis([-pi pi -pi pi])
xlabel('x')
ylabel('y')
hold on

% Isoclines
contour(x,y,dy./dx,0.25,'r','LineWidth',2)
contour(x,y,dy./dx,0.75,'g','LineWidth',2)
```



Kreyszig, 1.2.15

Carlos Oscar Sorzano, Aug. 31st, 2014

Two forces act on a parachutist, the attraction by the earth mg (m is the mass of person plus equipment, $g = 9.8[m/s^2]$ the acceleration of gravity) and the air resistance, assumed to be proportional to the square of the velocity $v(t)$. Using Newton's second law of motion (mass $\times$ acceleration = resultant of the forces), set up a model (an ODE for $v(t)$). Graph a direction field (choosing m and the constant of proportionality equal to 1). Assume that the parachute opens when $v = 10[m/s]$. Graph the corresponding solution in the field. What is the limiting velocity? Would the parachute still be sufficient if the air resistance were only proportional to $v(t)$?

Solution: The following equation for the velocity v reflects the physical knowledge of the problem

$$mv' = mg - \nu v^2$$

With $m = 1[kg]$ and $\nu = 1[Ns^2/kg]$, we have

$$v' = g - v^2$$

If the parachute opens at $v = 10[m/s]$ it means

$$v(0) = 10$$

we would have in MATLAB (see red curve)

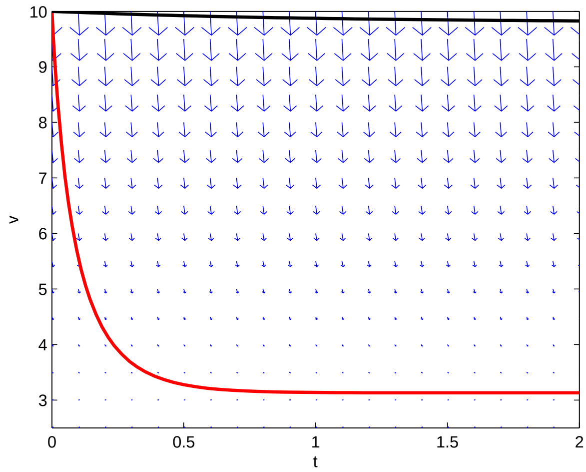
```
[x,v]=meshgrid(0:0.1:2,0:0.5:10);
f = @(x,v) 9.8-v.^2;
dv=feval(f,x,v);
dx=ones(size(dv));
quiver(x,v,dx,dv);
axis([0 2 0 10])
xlabel('t')
ylabel('v')
hold on

% Solution
[t10,v10]=ode45(f,[0 2],10);
plot(t10,v10,'r','LineWidth',2)
```

If the air resistance were proportional to $v(t)$, then (see black curve)

$$v' = g - v$$

It can be seen that the decrease of speed is much slower:



Kreyszig, 1.2.17

Álvaro Martín Ramos, Dec. 25th, 2014

Apply Euler's method to the ODE

$$y' = y$$

with $h = 0.1$ and $y(0) = 1$.
Solution: The method applied to this case would give

$$\begin{aligned}
y_0 &= y(0) = 1 \\
y_1 &= y_0 + hf(x_0, y_0) = 1 + 0.1(y_0) = 1 + 0.1(1) = 1.1 \\
y_2 &= y_1 + hf(x_1, y_1) = 1.1 + 0.1(y_1) = 1.1 + 0.1(1.1) = 1.21 \\
y_3 &= y_2 + hf(x_2, y_2) = 1.11 + 0.1(y_2) = 1.21 + 0.1(1.21) = 1.331 \\
&\ldots
\end{aligned}$$

Kreyszig, 1.2.20

Carlos Oscar Sorzano, Aug. 31st, 2014

Apply Euler's method to the ODE

$$y' = -5x^4y^2 \quad y(0) = 1$$

with $h = 0.2$. The true solution is

$$y = \frac{1}{(1+x)^5}$$

Solution: The method applied to this case would give

$y_0 = y(0) = 1$
$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.2(-5x_0^4y_0^2) = 1 + 0.2(-5(0)^4(1)^2) = 1$
$y_2 = y_1 + hf(x_1, y_1) = 1 + 0.2(-5x_1^4y_1^2) = 1 + 0.2(-5(0.2)^4(1)^2) = 0.9984$
$y_3 = 0.9729$
$y_4 = 0.8502$
$\ldots$

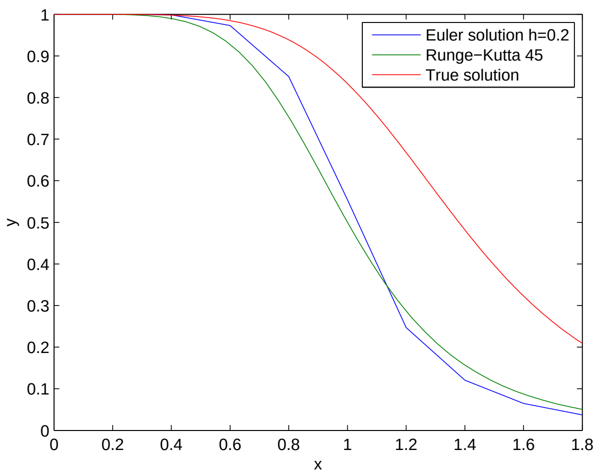
In MATLAB

```
f = @(x,y) -5*x.^4.*y.^2;

% Euler
y=zeros(10,1);
x=zeros(10,1);
x(1)=0; y(1)=1; % y(0)=1
h=0.2;
for k=1:length(y)-1
y(k+1)=y(k)+h*f(x(k),y(k));
x(k+1)=x(k)+h;
end

% ODE45
[xRK,yRK]=ode45(f,[0,1.8],1);

% True solution
xt=0:0.01:1.8;
yt=1./((xt+1).^5);
plot(x,y,xRK,yRK,xt,yt)
legend('Euler solution','Runge-Kutta 45','True solution')
xlabel('x')
ylabel('y')
```



Kreyszig, 1.3.2

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve

$$y^3 + y' + x^3 = 0$$

Solution: We can rearrange the equation as

$$y' = -\frac{x^3}{y^3} = -\left(\frac{x}{y}\right)^3$$

We see that the equation has the form

$$y' = f\left(\frac{y}{x}\right)$$

so that it can be reduced to a separable form by making the change of variables

$$u = \frac{y}{x} \Rightarrow y = xu \Rightarrow y' = u'x + u$$

Substituting in the ODE

$$u'x + u = -\frac{1}{u^3}$$

$$u'x = -\left(u + \frac{1}{u^3}\right) = -\frac{u^4+1}{u^3}$$

Separating variables

$$\frac{u^3}{u^4+1}du = -\frac{1}{x}dx$$

Integrating

$$\int \frac{u^3}{u^4+1}du = -\int \frac{1}{x}dx$$

$$\frac{1}{4}\int \frac{4u^3}{u^4+1}du = -\log|x| + C$$

Solving for u

$$\frac{1}{4}\log|u^4+1| = -\log|x| + C$$

$$\log|u^4+1|^{\frac{1}{4}} = -\log|x| + C$$

$$|u^4+1|^{\frac{1}{4}} = \frac{C}{x}$$

$$u^4 + 1 = \frac{C}{x^4}$$

And undoing the change of variable

$$\left(\frac{y}{x}\right)^4 + 1 = \frac{C}{x^4}$$

$$\boxed{y^4 + x^4 = C}$$

Kreyszig, 1.3.7

Carlos Oscar Sorzano, Nov. 2nd, 2014

Solve

$$xy' = y + 2x^3\sin^2\left(\frac{y}{x}\right)$$

by making the change of variables $\frac{y}{x} = u$
Solution: The change of variables $\frac{y}{x} = u$ implies

$$y = ux$$

$$y' = u'x + u$$

Substituting in the differential equation we get

$$x(u'x + u) = ux + 2x^3 \sin^2(u)$$

$$x^2u' = 2x^3 \sin^2(u)$$

$$\frac{u'}{\sin^2(u)} = 2x$$

$$\frac{du}{\sin^2(u)} = 2xdx$$

Integrating we get

$$-\frac{1}{\tan(u)} = x^2 + C$$

$$\tan(u) = \frac{1}{C - x^2}$$

$$u = \arctan \frac{1}{C - x^2}$$

Undoing the change of variable

$$y = ux = x \arctan \frac{1}{C - x^2}$$

Kreyszig, 1.3.8

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve

$$y' = (y + 4x)^2$$

by making the change of variables $y + 4x = v$
Solution:

$$y + 4x = v \Rightarrow y' + 4 = v' \Rightarrow y' = v' - 4$$

Substituting in the ODE

$$v' - 4 = v^2$$

$$v' = v^2 + 4$$

$$\frac{v'}{v^2 + 4} = 1$$

Separating variables

$$\frac{dv}{v^2 + 4} = dx$$

Integrating

$$\int \frac{dv}{v^2 + 4} = \int dx$$

$$\int \frac{1}{4} \frac{dv}{(\frac{v}{2})^2 + 1} = x + C$$

$$\frac{1}{2} \int \frac{\frac{1}{2} dv}{(\frac{v}{2})^2 + 1} = x + C$$

$$\frac{1}{2} \text{atan} \frac{v}{2} = x + C$$

Solving for v

$$v = 2\tan(2x + C)$$

Undoing the change of variables

$$y + 4x = 2\tan(2x + C)$$

$$\boxed{y = -4x + 2\tan(2x + C)}$$

Kreyszig, 1.3.9

Carlos Oscar Sorzano, June 15th, 2015

Solve

$$xy' = y^2 + y$$

by making the change of variables $u = \frac{y}{x}$.

Solution:

$$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u'x + u$$

Substituting in the ODE

$$x(u'x + u) = (ux)^2 + ux$$

Dividing by x

$$u'x + u = u^2 x + u$$

$$u'x = u^2 x$$

$$u' = u^2$$

$$\frac{du}{u^2} = dx$$

Integrating

$$-u^{-1} = x + C$$

$$u = -\frac{1}{x + C}$$

Undoing the change of variables

$$\frac{y}{x} = -\frac{1}{x + C}$$

Finally,

$$\boxed{y = -\frac{x}{x + C}}$$

Kreyszig, 1.3.19

Carlos Oscar Sorzano, Aug. 31st, 2014

If the growth rate of the number of bacteria at any time t is proportional to the number present at t and doubles in 1 week, how many bacteria can be expected after 2 weeks? After 4 weeks?

Solution: The growth rate of the number of bacteria is $A'(t)$. If it is proportional to the number of bacteria, we have

$$A' = \mu A$$

whose solution can be obtained by separating variables

$$dA = \mu A dt$$

$$\frac{dA}{A} = \mu dt$$

Integrating

$$\log|A| = \mu t + C$$

Solving for A

$$A = Ce^{\mu t}$$

If the number of bacteria doubles every week, we have

$$A(t+7) = 2A(t)$$

$$Ce^{\mu(t+7)} = 2Ce^{\mu t}$$

$$e^{\mu 7} = 2 \Rightarrow \mu = \frac{\log(2)}{7} = 0.0990$$

After 2 weeks we will have

$$\boxed{A(t+14)} = Ce^{\mu(t+14)} = Ce^{\mu t}e^{\mu 14} = A(t)e^{\frac{\log(2)}{7}14} = A(t)e^{2\log(2)} = A(t)(e^{\log(2)})^2 = A(t)2^2 = \boxed{4A(t)}$$

Similarly, after 4 weeks, we will have

$$\boxed{A(t+28)} = A(t)e^{\frac{\log(2)}{7}28} = A(t)e^{4\log(2)} = A(t)(e^{\log(2)})^4 = A(t)2^4 = \boxed{16A(t)}$$

Kreyszig, 1.3.20

Carlos Oscar Sorzano, Aug. 31st, 2014

1. If the birth rate and death rate of the number of bacteria are proportional to the number of bacteria present, what is the population as a function of time.
2. What is the limiting situation for increasing time? Interpret it.

Solution:

1. The following model describes the situation

$$A' = \mu_b A - \mu_d A = (\mu_b - \mu_d)A$$

Similarly to Problem 1.3.19, its solution is

$$A = Ce^{(\mu_b - \mu_d)t} = A(0)e^{(\mu_b - \mu_d)t}$$

2. If $\mu_b = \mu_d$, the number of bacteria stays stable from $t = 0$. If $\mu_b > \mu_d$, the number of bacteria grows exponentially. On the contrary, if $\mu_b < \mu_d$, the number of bacteria exponentially decreases to 0.

Kreyszig, 1.3.23

Carlos Oscar Sorzano, Aug. 31st, 2014

Boyle–Mariotte's law for ideal gases. Experiments show for a gas at low pressure P (and constant temperature) the rate of change of the volume $V(P)$ equals $-\frac{V}{P}$. Solve the model.
Solution: The following ODE models the system

$$V' = -\frac{V}{P}$$

$$\frac{V'}{V} = -\frac{1}{P}$$

Separating variables

$$\frac{dV}{V} = -\frac{dP}{P}$$

Integrating

$$\log|V| = -\log|P| + C = \log\left|\frac{C}{P}\right|$$

$$\boxed{V = \frac{C}{P}}$$

Kreyszig, 1.3.26

Carlos Oscar Sorzano, Aug. 31st, 2014

Gompertz growth in tumors. The Gompertz model is $y' = -Ay\log(y)$ ($A > 0$), where $y(t)$ is the mass of tumor cells at time t. The model agrees well with clinical observations. The declining growth rate with increasing $y > 1$ corresponds to the fact that cells in the interior of a tumor may die because of insufficient oxygen and nutrients. Use the ODE to discuss the growth and decline of solutions (tumors) and to find constant solutions. Then solve the ODE.
Solution: Let us solve the equation

$$y' = -Ay\log(y)$$

$$\frac{dy}{y\log(y)} = -Adt$$

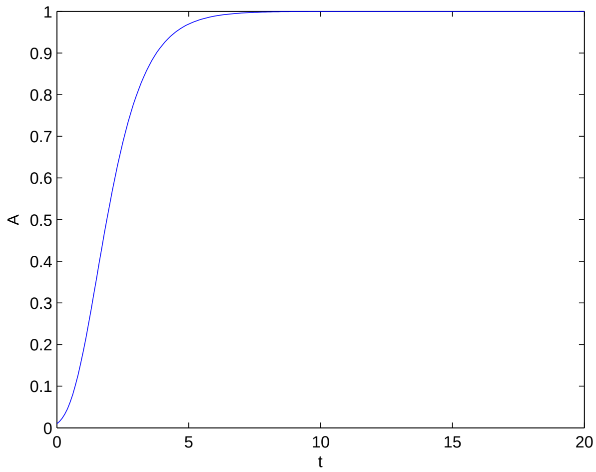
$$\frac{\frac{1}{y}dy}{\log(y)} = -Adt$$

$$\log|\log(y)| = -At + C$$

$$\log(y) = Ce^{-At}$$

$$\boxed{y} = \exp(C\exp(-At)) = \boxed{\exp(\log(y(0))\exp(-At))}$$

The following figure shows the growth for $y(0) = 0.01$ and $A = 1$



Kreyszig, 1.4.4

Carlos Oscar Sorzano, Nov. 2nd, 2014

Solve

$$e^{3\theta}(dr + 3rd\theta) = 0$$

Solution: We rewrite the differential equation as

$$e^{3\theta}dr + 3re^{3\theta}d\theta = 0$$

which is of the form

$$Pdr + Qd\theta = 0$$

To check if it is an exact equation we calculate

$$\frac{\partial P}{\partial \theta} = 3e^{3\theta}$$

$$\frac{\partial Q}{\partial r} = 3e^{3\theta}$$

Since both partial derivatives are equal, the equation is exact and we look for a solution of the form

$$U = \int Pdr + C(\theta) = \int e^{3\theta}dr + C(\theta) = e^{3\theta}r + C(\theta)$$

To determine the constant $C(\theta)$ we differentiate this function with respect to θ

$$\frac{\partial U}{\partial \theta} = 3re^{3\theta} + C'(\theta)$$

and compare it to Q

$$3re^{3\theta} + C'(\theta) = Q$$

$$3re^{3\theta} + C'(\theta) = 3re^{3\theta}$$

$$C'(\theta) = 0$$

Integrating with respect to θ

$$C(\theta) = C$$

Finally, the implicit solution of the differential equation is

$$e^{3\theta} r + C = 0$$

or explicitly

$$r = -Ce^{-3\theta}$$

Kreyszig, 1.4.8

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$e^x(\cos(y)dx - \sin(y)dy) = 0$$

Solution: We may rewrite the ODE as

$$e^x \cos(y)dx - e^x \sin(y)dy = 0$$

That is of the form

$$P(x, y)dx + Q(x, y)dy = 0$$

To see if it is exact we calculate

$$\frac{\partial P}{\partial y} = e^x(-\sin(y))$$

$$\frac{\partial Q}{\partial x} = -e^x \sin(y)$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the ODE is exact. To find the solution, u that satisfies

$$\frac{\partial u}{\partial x} = P \quad \frac{\partial u}{\partial y} = Q$$

we integrate P with respect to x

$$u(x, y) \;\; = \;\; \int e^x \cos(y)dx = \cos(y)e^x + C(y)$$

If we now differentiate u with respect to y we should obtain Q

$$\frac{\partial u}{\partial y} = e^x(-\sin(x)) + C'(y) = -e^x \sin(y) \Rightarrow C'(y) = 0 \Rightarrow C(y) = C$$

So the solution to the problem are all functions of the form

$$u(x, y) = C = \cos(y)e^x \Rightarrow \boxed{y = \operatorname{acos}(Ce^{-x})}$$

Kreyszig, 1.4.9

Álvaro Martín Ramos, Dec. 25th, 2014

Solve the ODE

$$e^{2x}(2\cos(y)dx - \sin(y)dy) = 0$$

Solution: We may rewrite the ODE as

$$e^{2x}2\cos(y)dx - e^{2x}\sin(y)dy) = 0$$

That is of the form

$$P(x, y)dx + Q(x, y)dy = 0$$

To see if it is exact we calculate

$$\frac{\partial P}{\partial y} = -2e^{2x}\sin(y)$$

$$\frac{\partial Q}{\partial x} = -2e^{2x}\sin(y)$$

Since

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

the ODE is exact. To find the solution, u, that satisfies

$$\frac{\partial u}{\partial x} = P$$

$$\frac{\partial u}{\partial y} = Q$$

we integrate P with respect to x

$$u(x, y) = \int e^{2x}2\cos(y)dx = \cos(y)e^{2x} + C(y)$$

If we now differentiate u with respect to y we should obtain Q

$$\frac{\partial u}{\partial y} = -\sin(y)e^{2x} + C'(y) = -e^{2x}\sin(y) \Rightarrow C'(y) = 0 \Rightarrow C(y) = C$$

So the solution to the problem are all functions of the form

$$u(x, y) = C = \cos(y)e^{2x} \Rightarrow \boxed{y = \text{acos}(Ce^{-2x})}$$

Kreyszig, 1.4.10

Carlos Oscar Sorzano, Jan. 13th, 2015

Solve the differential equation

$$ydx + (y + \tan(x + y))dy = 0$$

knowing that $\cos(x + y)$ is an integrating factor.
Solution: Let us multiply the whole equation by $\cos(x + y)$

$$y\cos(x + y)dx + (y\cos(x + y) + \sin(x + y))dy = 0$$

which is of the form

$$P(x, y)dx + Q(x, y)dy = 0$$

Let us check if this is an exact equation:

$$P_y = \frac{\partial P(x,y)}{\partial y} = \cos(x+y) - y\sin(x+y)$$

$$Q_x = \frac{\partial Q(x,y)}{\partial x} = -y\sin(x+y) + \cos(x+y)$$

Since $P_y = Q_x$, the equation is exact. We can solve it by integrating with respect to one of the variables

$$U(x,y) = \int P(x,y)dx = \int y\cos(x+y)dx = y\sin(x+y) + C(y)$$

We now differentiate U with respect to y

$$Q(x,y) = \frac{\partial U(x,y)}{\partial y}$$

$$y\cos(x+y) + \sin(x+y) = \sin(x+y) + y\cos(x+y) + C'(y)$$
$$C'(y) = 0$$
$$C(y) = C$$

Finally, the implicit solution of the differential equation is

$$U(x,y) = 0$$

$$\boxed{y\sin(x+y) + C = 0}$$

Kreyszig, 1.4.11

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$2\cosh(x)\cos(y)dx = \sinh(x)\sin(y)dy$$

Solution: We may rewrite the ODE as

$$2\cosh(x)\cos(y)dx - \sinh(x)\sin(y)dy = 0$$

That is of the form

$$P(x,y)dx + Q(x,y)dy = 0$$

To see if it is exact we calculate

$$\frac{\partial P}{\partial y} = 2\cosh(x)(-\sin(y))$$

$$\frac{\partial Q}{\partial x} = -\cosh(x)\sin(y)$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, the ODE is not exact. For finding an integrating factor, we start by calculating

$$P_y - Q_x = 2\cosh(x)(-\sin(y)) - (-\cosh(x)\sin(y)) = -\cosh(x)\sin(y)$$

We note that
$$\frac{Q_x - P_y}{P} = \frac{\cosh(x)\sin(y)}{2\cosh(x)\cos(y)} = \frac{1}{2}\tan(y)$$
is a function of y, $f(y)$. The integrating factor comes
$$F = \exp\left(\int \frac{1}{2}\tan(y)dy\right) = \exp\left(-\frac{1}{2}\log(\cos(y))\right) = \frac{1}{\sqrt{\cos(y)}}$$
We now multiply the ODE by the integrating factor
$$\frac{1}{\sqrt{\cos(y)}}\left(2\cosh(x)\cos(y)dx - \sinh(x)\sin(y)dy\right) = 0$$
$$2\cosh(x)\sqrt{\cos(y)}dx - \sinh(x)\frac{\sin(y)}{\sqrt{\cos(y)}}dy = 0$$
At this point, the ODE is exact. We find its solution by integrating P with respect to x
$$u(x,y) = \int 2\cosh(x)\sqrt{\cos(y)}dx = 2\sinh(x)\sqrt{\cos(y)} + C(y)$$
Differentiating with respect to y we should obtain FQ
$$\frac{\partial u}{\partial y} = -\sinh(x)\frac{\sin(y)}{\sqrt{\cos(y)}} + C'(y) = -\sinh(x)\frac{\sin(y)}{\sqrt{\cos(y)}} \Rightarrow C'(y) = 0$$
So the solutions of the ODE are of the form
$$u(x,y) = \boxed{C = 2\sinh(x)\sqrt{\cos(y)}}$$

Kreyszig, 1.5.7

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE
$$xy' = 2y + x^3e^x$$
Solution: We may rewrite the ODE as
$$y' - \frac{2}{x}y = x^2e^x$$
That is of the form
$$y' + p(x)y = r(x)$$
This is a linear, non-homogeneous equation, whose solution is given by
$$y_h = e^{-h}(\int e^h r dx + C)$$
where
$$\begin{aligned} h &= \int p dx = -\int \tfrac{2}{x}dx = -2\log|x| \\ e^{-h} &= e^{2\log|x|} = x^2 \\ \int e^h r dx &= \int (x^{-2})(x^2e^x)dx = e^x \end{aligned}$$

Finally

$$\boxed{y = x^2(e^x + C)}$$

Kreyszig, 1.5.13

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$y' = 6(y - 2.5)\tanh(1.5x)$$

Solution: We may rewrite the ODE as

$$y' - 6\tanh(1.5x)y = -15\tanh(1.5x)$$

That is of the form

$$y' + p(x)y = r(x)$$

This is a linear, non-homogeneous equation, whose solution is given by

$$y_h = e^{-h}(\int e^h r dx + C)$$

where

$$\begin{array}{rcl} h &=& \int p dx = -6\int \tanh(1.5x)dx = -6\frac{\log(\cosh(1.5x))}{1.5} = -4\log(\cosh(1.5x)) \\ e^{-h} &=& e^{4\log(\cosh(1.5x))} = \cosh^4(1.5x) \\ \int e^h r dx &=& \int (\cosh^{-4}(1.5x))(-15\tanh(1.5x))dx = \frac{2.5}{\cosh^4(1.5x)} \end{array}$$

Finally

$$\boxed{y} = \cosh^4(1.5x)\left(\frac{2.5}{\cosh^4(1.5x)} + C\right) = \boxed{2.5 + C\cosh^4(1.5x)}$$

Kreyszig, 1.5.15

Carlos Oscar Sorzano, Aug. 31st, 2014

Let H be the homogeneous problem

$$y' + p(x)y = 0$$

and NH be the non-homogeneous problem

$$y' + p(x)y = r(x)$$

Show that the sum of two solutions and of the homogeneous equation (H) is a solution of (H), and so is a scalar multiple for any constant a. These properties are not true for the non-homogeneous problem (NH).

Solution: Let y_1 and y_2 be two solutions of the homogeneous problem so that

$$y_1' + p(x)y_1 = 0$$

$$y_2' + p(x)y_2 = 0$$

Adding both equations we have

$$y_1' + y_2' + p(x)y_1 + p(x)y_2 = 0$$

$$(y_1 + y_2)' + p(x)(y_1 + y_2) = 0$$

This last equation proves that $y_1 + y_2$ is also a solution of the homogeneous problem. Similarly if we multiply the first equation by a we have

$$a(y_1' + p(x)y_1) = 0$$

$$ay_1' + ap(x)y_1 = 0$$

$$(ay_1)' + p(x)(ay_1) = 0$$

which proves that ay_1 is also a solution of the homogeneous problem.

However, this is not true for the non-homogeneous problem. Let us assume that y_1 and y_2 are solutions of the non-homogeneous problem

$$y_1' + p(x)y_1 = r(x)$$

$$y_2' + p(x)y_2 = r(x)$$

Let us check if $y_1 + y_2$ is also a solution. For doing so, we substitute $y_1 + y_2$ into the ODE

$$(y_1 + y_2)' + p(x)(y_1 + y_2) = (y_1' + p(x)y_1) + (y_2' + p(x)y_2) = 2r(x) \neq r(x)$$

The same happens with ay_1

$$(ay_1)' + p(x)(ay_1) = a(y_1' + p(x)y_1) = ar(x) \neq r(x)$$

Kreyszig, 1.5.17

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that the sum of a solution of the non-homogeneous problem and a solution of the homogeneous one is a solution of the non-homogeneous problem.
Solution: Let y_p be a solution of the non-homogeneous problem

$$y_p' + p(x)y_p = r(x)$$

and y_h be a solution of the homogeneous problem

$$y_h' + p(x)y_h = 0$$

Let us check if $y_p + y_h$ is a solution of the non-homogeneous problem

$$(y_p + y_h)' + p(x)(y_p + y_h) = (y_p' + p(x)y_p) + (y_h' + p(x)y_h) = r(x) + 0 = r(x)$$

That is, $y_p + y_h$ is indeed a solution of the non-homogeneous problem.

Kreyszig, 1.5.18

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that the difference of two solutions of the non-homogeneous problem is a solution of the homogeneous problem.
Solution: Let y_{p_1} and y_{p_2} be two solutions of the non-homogeneous problem

$$y_{p_1}' + p(x)y_{p_1} = r(x)$$

$$y'_{p_2} + p(x)y_{p_2} = r(x)$$

Let us check if $y_{p_1} - y_{p_2}$ is a solution of the homogeneous problem

$$(y_{p_1} - y_{p_2})' + p(x)(y_{p_1} - y_{p_2}) = (y'_{p_1} + p(x)y_{p_1}) - (y'_{p_2} + p(x)y_{p_2}) = r(x) - r(x) = 0$$

That is, $y_{p_1} - y_{p_2}$ is indeed a solution of the homogeneous problem.

Kreyszig, 1.5.21

Carlos Oscar Sorzano, Aug. 31st, 2014

Variation of parameter. Another method of obtaining the solution $y = e^{-h}(\int e^h r dx + C)$ of a non-homogeneous problem

$$y' + p(x)y = r(x)$$

results from the following idea. Write the solution of the homogeneous problem as

$$y = Ce^{-\int p dx} = Ce^{-h} = Cy^*$$

where y^* is the exponential function, which is a solution of the homogeneous linear ODE

$$(y^*)' + p(x)y^* = 0$$

Replace the arbitrary constant C in the homogeneous solution with a function u to be determined so that the resulting function $y = uy^*$ is a solution of the nonhomogeneous linear ODE.

Solution: Let us introduce the function uy^* into the non-homogeneous ODE to see the requirements that u must meet

$$\begin{aligned}
(uy^*)' + p(uy^*) &= u'y^* + u(y^*)' + puy^* \\
&= u'y^* + u((y^*)' + py^*) \\
&= u'y^* + u0 \\
&= u'y^* \\
&= r
\end{aligned}$$

That is, we need

$$u'y^* = r \Rightarrow u' = \frac{r}{e^{-h}} = re^h \Rightarrow u = \int re^h dx + C$$

So the solution of the non-homogeneous problem is

$$y = uy^* = \left(\int re^h dx + C\right) e^{-h}$$

Kreyszig, 1.5.24

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve $y' + y = -\frac{x}{y}$

Solution: This is a Bernouilli equation of the form

$$y' + p(x)y = g(x)y^a$$

with $p(x) = 1$, $g(x) = -x$ and $a = -1$. We do the change of variable

$$u = y^{1-a} = y^{1-(-1)} = y^2$$

Differentiating

$$u' = 2yy' = 2y(-y - xy^{-1}) = -2y^2 - 2x = -2u - 2x$$

$$u' + 2u = -2x$$

This is now a linear, non-homogeneous equation system of the form

$$u' + pu = r$$

whose solution is given by

$$h = \int pdx = \int 2dx = 2x$$

$$\begin{aligned} u &= e^{-h}(\textstyle\int e^h r dx + C) = e^{-2x}\left(\int e^{2x}(-2x)dx + C\right) \\ &= e^{-2x}\left(xe^{2x} - \tfrac{1}{2}e^{2x} + C\right) = x - \tfrac{1}{2} + Ce^{2x} \end{aligned}$$

Now we undo the change of variable

$$\boxed{y^2 = \sqrt{x - \frac{1}{2} + Ce^{2x}}}$$

Kreyszig, 1.5.25

Álvaro Martín Ramos, Dec. 25th, 2014

Solve

$$y' = 3.2y - 10y^2$$

Solution: We may rewrite the ODE as

$$y' - 3.2y = -10y^2$$

This is a Bernouilli equation of the form

$$y' + p(x)y = g(x)y^a$$

with $p(x) = -3.2$, $g(x) = -10$ and $a = 2$. We do the change of variable

$$u = y^{1-a} = y^{1-2} = y^{-1}$$

Differentiating

$$u' = (y^{-1})' = -\frac{1}{y^2}y' = -\frac{1}{y^2}(3.2y - 10y^2) = -3.2y^{-1} + 10 = -3.2u + 10$$

$$u' + 3.2u = 10$$

This is now a linear, non-homogeneous equation system of the form

$$u' + pu = r$$

whose solution is given by

$$h = \int pdx = \int (-3.2)dx = -3.2x$$

$$\begin{array}{rcl} u & = & e^{-h}(\int e^h r dx + C) = e^{3.2x}(\int e^{-3.2x} 10 dx + C) \\ & = & e^{3.2x}(\frac{10e^{-3.2x}}{-3.2} + C) = -\frac{10}{3.2} + Ce^{3.2x} \end{array}$$

Now we undo the change of variable

$$\boxed{y} = \frac{1}{u} = \boxed{\frac{1}{-\frac{10}{3.2} + Ce^{3.2x}}}$$

Kreyszig, 1.5.28

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve $2xyy' + (x-1)y^2 = x^2e^x$. Hint: set $z = y^2$

Solution: If we do the change of variable

$$z = y^2 \Rightarrow z' = 2yy'$$

then the ODE is transformed to

$$xz' + (x-1)z = x^2e^x$$

$$z' + \frac{x-1}{x}z = xe^x$$

This is now a linear, non-homogeneous equation system of the form

$$z' + pz = r$$

whose solution is given by

$$h = \int pdx = \int \frac{x-1}{x}dx = x - \log|x|$$

$$\begin{array}{rcl} z & = & e^{-h}(\int e^h r dx + C) = e^{-x+\log|x|}\left(\int e^{x-\log|x|}(xe^x)dx + C\right) \\ & = & e^{-x}x\left(\int e^{2x}dx + C\right) = e^{-x}x\left(\frac{1}{2}e^{2x} + C\right) \\ & = & \frac{x}{2}e^x + Ce^{-x} \end{array}$$

Now we undo the change of variable

$$\boxed{y^2 = \sqrt{\frac{x}{2}e^x + Ce^{-x}}}$$

Kreyszig, 1.5.33

Carlos Oscar Sorzano, Aug. 31st, 2014

Find and solve the model for drug injection into the bloodstream if, beginning at $t = 0$ a constant amount $A[g/min]$ is injected and the drug is simultaneously removed at a rate proportional to the amount of the drug present at time t.

Solution: The ODE

$$A' = K_{in} - K_{out}A \quad A(0) = 0$$

models the system. This can be rewritten as

$$A' + K_{out}A = K_{in}$$

which is a linear, non-homogeneous ODE whose solution is

$$\begin{aligned} h &= \int K_{out} dt = K_{out} t \\ A &= e^{-h}\left(\int e^h r dt + C\right) = e^{-K_{out} x}\left(\int e^{K_o ut} t K_{in} dt + C\right) \\ &= e^{-K_{out} t}\left(K_{in} \tfrac{1}{K_{out}} e^{K_{out}} t + C\right) = \tfrac{K_{in}}{K_{out}} + C e^{-K_{out} t} \end{aligned}$$

Now we impose the initial condition

$$A(0) = 0 = \frac{K_{in}}{K_{out}} + C \Rightarrow C = -\frac{K_{in}}{K_{out}}$$

Finally, the solution is

$$\boxed{A(t) = \frac{K_{in}}{K_{out}}(1 - e^{-K_{out} t})} \quad (t > 0)$$

Kreyszig, 1.5.34

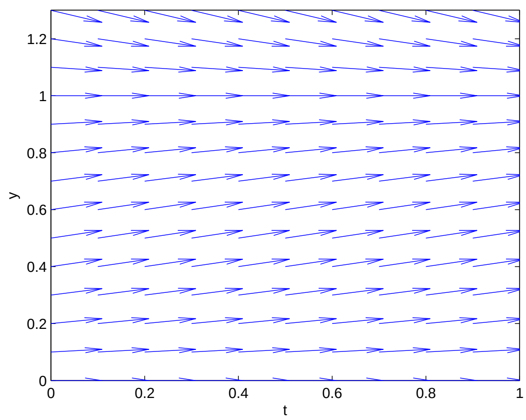
Carlos Oscar Sorzano, Aug. 31st, 2014

A model for the spread of contagious diseases is obtained by assuming that the rate of spread is proportional to the number of contacts between infected and noninfected persons, who are assumed to move freely among each other. Set up the model. Find the equilibrium solutions and indicate their stability or instability. Solve the ODE. Find the limit of the proportion of infected persons as $t \to \infty$ and explain what it means.

Solution: Let us call y the proportion of infected persons. The growth of infected persons is proportional to the number of contacts means that

$$y' = ky(1 - y) \quad y(0) = y_0$$

The two equilibrium solutions are $y = 0$ (unstable) and $y = 1$ (stable) as can be seen in the figure below



We can rewrite the ODE as

$$y' - ky = -ky^2$$

This is a Bernouilli equation of the form

$$y' + py = gy^a$$

with $p = -k$, $g = -k$, $a = 2$. We do the change of variable

$$u = y^{1-a} = y^{1-2} = y^{-1}$$

$$u' = -y^{-2}y' = -y^{-2}(ky - ky^2) = -(ky^{-1} - k) = k - ku$$

$$u' + ku = k$$

This is a linear, non-homogeneous equation whose solution is

$$\begin{array}{rcl} h & = & \int kdt = kt \\ u & = & e^{-h}\left(\int e^h r dt + C\right) = e^{-kt}\left(\int e^{kt} k dt + C\right) = e^{-kt}\left(e^{kt} + C\right) = 1 + Ce^{-kt} \end{array}$$

We undo now the change of variable

$$y = \frac{1}{1 + Ce^{-kt}}$$

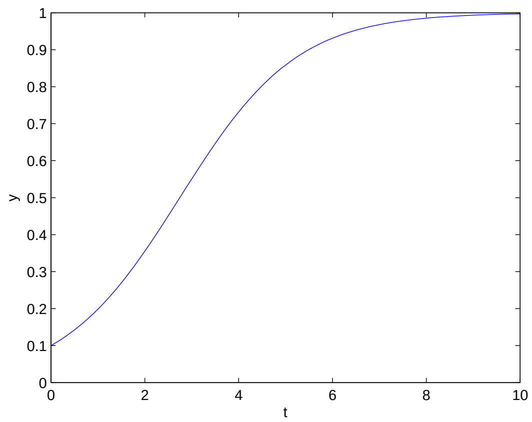
Imposing the initial condition

$$y_0 = \frac{1}{1 + C} \Rightarrow C = \frac{1}{y_0} - 1 = \frac{1 - y_0}{y_0}$$

Finally

$$\boxed{y} = \frac{1}{1 + \frac{1-y_0}{y_0}e^{-kt}} = \boxed{\frac{y_0}{y_0 + (1 - y_0)e^{-kt}}}$$

The following figure shows the curve for $y_0 = 0.1$, $k = 0.8$.



Kreyszig, 1.6.9

Carlos Oscar Sorzano, Jan. 13th, 2015

Which is the set of orthogonal trajectories to the curve family

$$y = ce^{-x^2}$$

Solution: To find the orthogonal trajectories, we differentiate the set of curves

$$y' = -2xce^{-x^2}$$

That we may rewrite as

$$y' = -2xy = f(x, y)$$

The set of orthogonal curves must fulfill

$$\tilde{y}' = -\frac{1}{f(x,\tilde{y})} = \frac{1}{-2x\tilde{y}}$$
$$\tilde{y}\tilde{y}' = -\frac{1}{2x}$$
$$\tilde{y}d\tilde{y} = -\frac{1}{2x}dx$$

Which is a separable differential equation that can be directly integrated

$$\frac{1}{2}\tilde{y}^2 = -\frac{1}{2}\log(x) + C$$
$$\tilde{y}^2 = -\log(x) + C$$
$$e^{\tilde{y}^2} = \frac{C}{x}$$

Finally, the curve family can be rewritten as

$$\boxed{x = Ce^{-\tilde{y}^2}}$$

Kreyszig, 1.6.12

Carlos Oscar Sorzano, Aug. 31st, 2014

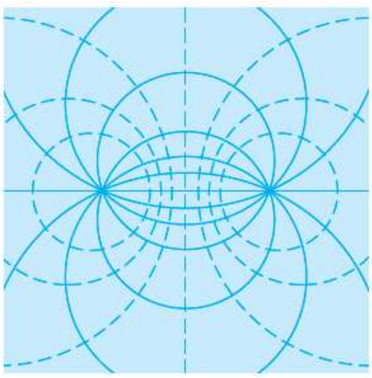
Electric field. The lines of electric force of two opposite charges of the same strength at $(-1, 0)$ and $(1, 0)$ are the circles through $(-1, 0)$ and $(1, 0)$. Show that these circles are given by

$$x^2 + (y - c)^2 = 1 + c^2.$$

Show that the equipotential lines (which are orthogonal trajectories of those circles) are the circles given by

$$(x + c^*)^2 + \tilde{y}^2 = (c^*)^2 - 1$$

(dashed in the following figure).



Solution: The curve

$$x^2 + (y-c)^2 = 1 + c^2$$

is the family of all circles pasing by $(-1,0)$ and $(1,0)$. To show this statement we show that $(-1,0)$ and $(1,0)$ fulfill this equation

$$(-1)^2 + (0-c)^2 = 1 + c^2$$

$$(1)^2 + (0-c)^2 = 1 + c^2$$

Obviously this family is a set of circles.

To find the orthogonal trajectories, we differentiate the curve

$$2x + 2(y-c)y' = 0$$

$$x + (y-c)y' = 0$$

This curve contains the parameter c which should not be there. To eliminate it, we manipulate the original set of curves to get

$$x^2 + y^2 + c^2 - 2yc = 1 + c^2$$

$$x^2 + y^2 - 2yc = 1$$

$$c = \frac{x^2 + y^2 - 1}{2y}$$

So the differential equation becomes

$$x + \left(y - \frac{x^2 + y^2 - 1}{2y}\right) y' = 0$$

$$2yx + \left(2y^2 - (x^2 + y^2 - 1)\right) y' = 0$$

$$2yx + \left(y^2 - x^2 + 1\right) y' = 0$$

$$y' = -\frac{2yx}{y^2 - x^2 + 1} = f(x,y)$$

The set of orthogonal curves must fulfill

$$\tilde{y}' = -\frac{1}{f(x,\tilde{y})} = \frac{\tilde{y}^2 - x^2 + 1}{2\tilde{y}x} = \frac{1}{2x}\tilde{y} + \frac{1}{2}\frac{1-x^2}{x}\tilde{y}^{-1}$$

$$\tilde{y}' - \frac{1}{2x}\tilde{y} = \frac{1}{2}\frac{1-x^2}{x}\tilde{y}^{-1}$$

This ODE is a Bernouilli equation of the form

$$\tilde{y}' + p(x)\tilde{y} = g(x)y^a$$

with $a = -1$. So we make the change of variable

$$u = \tilde{y}^{1-a} = \tilde{y}^{1-(-1)} = \tilde{y}^2 \Rightarrow u' = 2\tilde{y}\tilde{y}'$$

$$u' = 2\tilde{y}\left(\frac{1}{2x}\tilde{y} + \frac{1}{2}\frac{1-x^2}{x}\tilde{y}^{-1}\right)$$

$$u' = \frac{1}{x}\tilde{y}^2 + \frac{1-x^2}{x}$$

$$u' = \frac{1}{x}u + \frac{1-x^2}{x}$$

$$u' - \frac{1}{x}u = \frac{1-x^2}{x}$$

This is a linear equation whose solution is

$$\begin{array}{rcl} h & = & \int -\frac{1}{x}dx = -\log|x| \\ e^h & = & e^{-\log|x|} = \frac{1}{x} \\ u & = & e^{-h}\left(\int e^h r dx + c^*\right) \\ & = & x\left(\int \frac{1}{x}\frac{1-x^2}{x}dx + c^*\right) \\ & = & x\left(-\frac{x^2+1}{x} + c^*\right) \\ & = & -x^2 - 1 + c^*x \end{array}$$

Undoing the change of variable

$$\tilde{y}^2 = -x^2 - 1 + c^*x$$

$$\tilde{y}^2 + x^2 - c^*x = -1$$

$$\tilde{y}^2 + x^2 - c^*x + \left(\frac{c^*}{2}\right)^2 = -1 + \left(\frac{c^*}{2}\right)^2$$

$$\tilde{y}^2 + \left(x - \frac{c^*}{2}\right)^2 = -1 + \left(\frac{c^*}{2}\right)^2$$

$$\tilde{y}^2 + (x - c^*)^2 = -1 + (c^*)^2$$

$$\boxed{(x + c^*)^2 + \tilde{y}^2 = (c^*)^2 - 1}$$

Kreyszig, 1.6.13

Carlos Oscar Sorzano, Aug. 31st, 2014

Temperature field. Let the isotherms (curves of constant temperature) in a body in the upper half-plane $y > 0$ be given by

$$4x^2 + 9y^2 = c.$$

. Find the orthogonal trajectories (the curves along which heat will flow in regions filled with heat-conducting material and free of heat sources or heat sinks).

Solution: Let us analyze first the curves

$$4x^2 + 9y^2 = c$$

$$\frac{4}{c}x^2 + \frac{9}{c}y^2 = 1$$

$$\left(\frac{x}{\frac{\sqrt{c}}{2}}\right)^2 + \left(\frac{y}{\frac{\sqrt{c}}{3}}\right)^2 = 1$$

So they are ellipses of semiaxes $\frac{\sqrt{c}}{2}$ and $\frac{\sqrt{c}}{3}$.

Their orthogonal trajectories can be determined by differentiating the family of curves:

$$8x + 18yy' = 0$$

$$4x + 9yy' = 0$$

$$y' = -\frac{4x}{9y} = f(x, y)$$

The orthogonal trajectories fulfill the differential equation

$$\tilde{y}' = -\frac{1}{f(x, \tilde{y})} = \frac{9\tilde{y}}{4x}$$

$$\frac{d\tilde{y}}{9\tilde{y}} = \frac{dx}{4x}$$

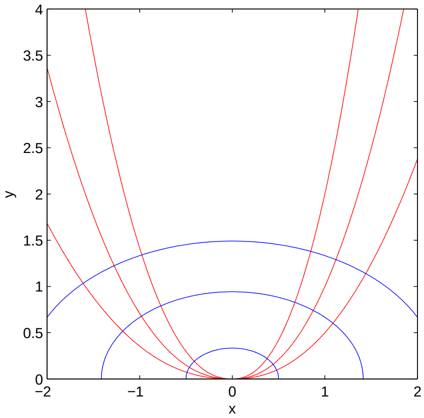
$$\frac{1}{9}\log|\tilde{y}| = \frac{1}{4}\log|x| + K$$

$$\log|\tilde{y}| = \frac{9}{4}\log|x| + K$$

$$\tilde{y} = Kx^{\frac{9}{4}} = Kx^{2.25}$$

In MATLAB:

```
close all
h=ezplot('y-x^2.25',[-2 2 0 4])
set(h,'Color','red')
hold on
h=ezplot('y-2*x^2.25',[-2 2 0 4])
set(h,'Color','red')
h=ezplot('y-0.5*x^2.25',[-2 2 0 4])
set(h,'Color','red')
h=ezplot('4*x^2+9*y^2=1',[-2 2 0 4])
set(h,'Color','blue')
h=ezplot('4*x^2+9*y^2=8',[-2 2 0 4])
set(h,'Color','blue')
h=ezplot('4*x^2+9*y^2=20',[-2 2 0 4])
set(h,'Color','blue')
axis square
title ''
```



Problema

Carlos Oscar Sorzano, Nov. 4th, 2014

It starts snowing in the morning and continues steadily throughout the day. A snow- plow that removes snow at a constant rate starts plowing at noon. It plows 2 km in the first hour, and 1 km in the second. What time did it start snowing?

Solution: Let us assume that the snowplow removes snow at a constant rate $\alpha[cm^3/h]$ and the snow falls at a fixed rate $k[cm^3/h]$. Assume that the width of the snowplow is equal to the road width $w[cm]$. Assume that it starts to snow at $t = -t_0$. Then, the height of the snow in the road must fulfill the differential equation

$$1[cm]w[cm]\frac{dh}{dt}[cm/h] = k[cm^3/h] \quad h(-t_0) = 0$$

whose solution is

$$dh = \frac{k}{w}dt$$

$$h = C + \frac{k}{w}t$$

The constant C is obtained by the initial condition

$$h(-t_0) = 0$$

$$C - \frac{k}{w}t_0 = 0 \Rightarrow C = \frac{kt_0}{w}$$

So the height becomes

$$h = \frac{k}{w}(t + t_0)[cm]$$

Let us call $x(t)$ the distance that the snowplow has gone since $t = 0$. The speed of the snowplow depends on the amount of snow that it can remove by unit of time

$$w[cm]h[cm]\frac{dx}{dt}[cm/h] = \alpha[cm^3/h]$$

$$\frac{dx}{dt} = \frac{\alpha}{wh} = \frac{\alpha}{k(t+t_0)}$$

$$dx = \frac{\alpha}{k}\frac{dt}{t+t_0}$$

$$x = \frac{\alpha}{k}\log|t+t_0| + C$$

We have the initial condition $x(0) = 0$ from which

$$0 = \frac{\alpha}{k}\log|t_0| + C \Rightarrow C = -\frac{\alpha}{k}\log|t_0|$$

Consequently, the distance gone by the snowplow is

$$x(t) = \frac{\alpha}{k}(\log|t+t_0| - \log|t_0|) = \frac{\alpha}{k}\log\left|\frac{t}{t_0}+1\right|$$

From the problem statement we know that $x(1) = 2000$ and $x(2) = 3000$, that is

$$\begin{aligned} x(2) = 3000 = \tfrac{\alpha}{k}\log\left|\tfrac{2}{t_0}+1\right| \\ x(1) = 2000 = \tfrac{\alpha}{k}\log\left|\tfrac{1}{t_0}+1\right| \end{aligned}$$

Dividing both equations

$$\frac{3}{2} = \frac{\log\left|\frac{2}{t_0}+1\right|}{\log\left|\frac{1}{t_0}+1\right|}$$

$$3\log\left|\frac{1}{t_0}+1\right| = 2\log\left|\frac{2}{t_0}+1\right|$$

$$\log\left|\left(\frac{1}{t_0}+1\right)^3\right| = \log\left|\left(\frac{2}{t_0}+1\right)^2\right|$$

$$\left(\frac{1}{t_0}+1\right)^3 = \left(\frac{2}{t_0}+1\right)^2$$

$$\frac{(1+t_0)^3}{t_0^3} = \frac{(2+t_0)^2}{t_0^2}$$

$$(1+t_0)^3 = t_0(2+t_0)^2$$

$$t_0^3 + 3t_0^2 + 3t_0 + 1 = t_0^3 + 4t_0^2 + 4t_0$$

$$t_0^2 + t_0 - 1 = 0 \Rightarrow t_0 = \begin{cases} \frac{-1-\sqrt{5}}{2} \\ \frac{\sqrt{5}-1}{2} \end{cases}$$

The only valid solution is

$$t_0 = \frac{\sqrt{5}-1}{2} = 0.618[h] = 37[min]$$

So it started snowing at 11h 23’ AM.

2 Chapter 2

Kreyszig, 2.1.1

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that
$$F(x,y',y'')=0$$
can be reduced to a first-order equation in $z=y'$.
Solution: If we do the change of variable
$$z=y' \Rightarrow z'=y''$$
Substituting in the original ODE, we have
$$\boxed{F(x,z,z')=0}$$
that is a first-order equation.

For example,
$$y''+\frac{1}{x}y'=\cosh(x)$$
can be transformed into
$$z'+\frac{1}{x}z=\cosh(x)$$

Kreyszig, 2.1.2

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that
$$F(y,y',y'')=0$$
can be reduced to a first-order equation in $z=y'$.
Solution: If we do the change of variable
$$z=y' \Rightarrow z'=\frac{dy'}{dy}\frac{dy}{dx}=\frac{dz}{dy}z=z_yz$$
Substituting in the original ODE, we have
$$\boxed{F(y,z,z_yz)=0}$$
that is a first-order equation.

For example,
$$y''+\frac{1}{y}y'+y^2=0$$
can be transformed into
$$z_yz+\frac{1}{y}z=-y^2$$

Kreyszig, 2.1.4

Carlos Oscar Sorzano, Nov. 2nd, 2014

Solve

$$2xy'' = 3y'$$

Solution: We make the change of variable

$$z = y'$$

$$z' = y''$$

The differential equation becomes

$$2xz' = 3z$$

$$\frac{z'}{z} = \frac{3}{2x}$$

$$\frac{dz}{z} = \frac{3dx}{2x}$$

Integrating

$$\log|z| = \frac{3}{2}\log|x| + C_1$$

$$\log|z| = \log|x^{\frac{3}{2}}| + C_1$$

$$z = C_1 x^{\frac{3}{2}}$$

Undoing the change of variable

$$y' = C_1 x^{\frac{3}{2}}$$

$$y = C_1 \int x^{\frac{3}{2}} dx + C_2$$

$$y = \frac{2}{5} C_1 x^{\frac{5}{2}} + C_2$$

After absorbing constants, the general solution can be rewritten as

$$y = C_1 x^{\frac{5}{2}} + C_2$$

Kreyszig, 2.1.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve $yy'' = 3(y')^2$

Solution: If we do the change of variable

$$z = y' \Rightarrow z' = \frac{dy'}{dy}\frac{dy}{dx} = \frac{dz}{dy}z = z_y z$$

Substituting in the original ODE, we have

$$y(z_y z) = 3z^2$$

$$z_y y = 3z$$

$$\frac{dz}{z} = \frac{3dy}{y}$$

$$\log|z| = 3\log|y| + C_1$$
$$z = C_1 y^3$$

Now we solve

$$y' = C_1 y^3$$
$$y^{-3}dy = C_1 dx$$
$$-\frac{1}{2y^2} = C_1 x + C_2$$
$$\boxed{C_1 x y^2 + C_2 y^2 = 1}$$

Kreyszig, 2.1.12

Carlos Oscar Sorzano, Aug. 31st, 2014

Hanging cable. It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving
$$y'' = k\sqrt{1+(y')^2}$$
where the constant k depends on the weight. This curve is called catenary (from Latin catena = the chain). Find and graph $y(x)$, assuming that and those fixed points are $(-1,0)$ and $(1,0)$ in a vertical xy-plane.
Solution: If we do the change of variable

$$z = y' \Rightarrow z' = y''$$

Substituting in the original ODE, we have

$$z' = k\sqrt{1+z^2}$$
$$\frac{dz}{\sqrt{1+z^2}} = kdx$$
$$\operatorname{asinh}(z) = kx + c_1$$
$$z = y' = \sinh(kx + c_1)$$

Since the catenary is symmetric with respect to the middle point, at this point we have no slope, that is

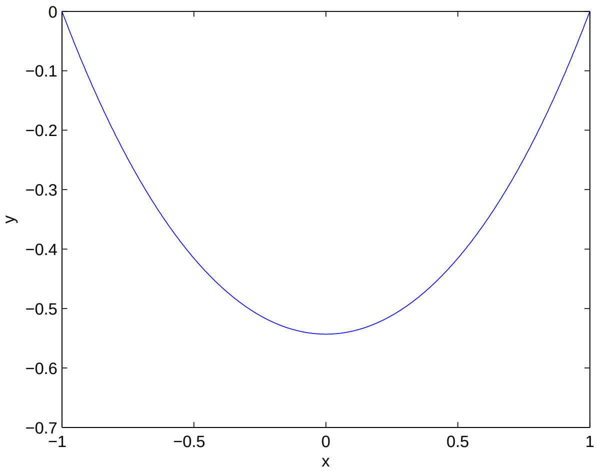
$$y'(0) = 0 = \sinh(c_1) \Rightarrow c_1 = 0$$

Now we solve the ODE
$$y' = \sinh(kx)$$
$$y = \int \sinh(kx)dx = \frac{1}{k}\cosh(kx) + c_2$$

Imposing the boundary condition

$$y(-1) = 0 = \frac{1}{k}\cosh(-k) + c_2 \Rightarrow c_2 = -\frac{1}{k}\cosh(-k) = -\frac{1}{k}\cosh(k)$$

So the final curve is
$$\boxed{y = \frac{1}{k}\cosh(kx) - \frac{1}{k}\cosh(k)}$$



Kreyszig, 2.1.13

Carlos Oscar Sorzano, Aug. 31st, 2014

Motion. If, in the motion of a small body on a straight line, the sum of velocity and acceleration equals a positive constant, how will the distance $y(t)$ depend on the initial velocity and position?
Solution: If the sum of velocity and acceleration equals a positive constant, then

$$y' + y'' = k$$

We make the change of variable

$$z = y' \Rightarrow z' = y''$$

Then the ODE becomes

$$z + z' = k$$

whose solution is given by

$$\begin{array}{rcl} h &=& \int 1 dx = t \\ z &=& e^{-h}(\int e^h r dt + c_1) = e^{-t}(\int e^t k dt + c_1) = e^{-t}(ke^t + c_1) = k + c_1 e^{-t} \end{array}$$

Now we solve

$$z = y' = k + c_1 e^{-t}$$

$$y = kt - c_1 e^{-t} + c_2$$

We now impose the initial conditions

$$y(0) = y_0 = -c_1 + c_2$$

$$y'(0) = v_0 = k + c_1 \Rightarrow c_1 = v_0 - k$$

$$c_2 = y_0 + c_1 = y_0 + v_0 - k$$

So the final dependence of motion on the initial conditions is

$$y = kt + (k - v_0)e^{-t} + y_0 + v_0 - k = \boxed{y_0 + kt + (k - v_0)(e^{-t} - 1)}$$

Kreyszig, 2.1.17

Carlos Oscar Sorzano, Aug. 31st, 2014

Verify that the functions $x^{\frac{3}{2}}$ and $x^{-\frac{1}{2}}$ are a basis of solutions of the ODE

$$4x^2y'' - 3y = 0$$

Find the particular solution satisfying $y(1) = -3$, $y'(1) = 0$.
Solution: Let us calculate the derivatives of the two given functions

$$\begin{aligned}
y_1 &= x^{\frac{3}{2}} \\
y_1' &= \tfrac{3}{2}x^{\frac{1}{2}} \\
y_1'' &= \tfrac{3}{2}\tfrac{1}{2}x^{-\frac{1}{2}} = \tfrac{3}{4}x^{-\frac{1}{2}} \\
y_2 &= x^{-\frac{1}{2}} \\
y_2' &= -\tfrac{1}{2}x^{-\frac{3}{2}} \\
y_2'' &= (-\tfrac{1}{2})(-\tfrac{3}{2})x^{-\frac{5}{2}} = \tfrac{3}{4}x^{-\frac{5}{2}}
\end{aligned}$$

We now substitute these two functions in the ODE to verify if they are solutions of it

$$\begin{aligned}
4x^2y_1'' - 3y_1 &= 4x^2(\tfrac{3}{4}x^{-\frac{1}{2}}) - 3x^{\frac{3}{2}} = 3x^{\frac{3}{2}} - 3x^{\frac{3}{2}} = 0 \\
4x^2y_2'' - 3y_2 &= 4x^2(\tfrac{3}{4}x^{-\frac{5}{2}}) - 3x^{-\frac{1}{2}} = 3x^{-\frac{1}{2}} - 3x^{-\frac{1}{2}} = 0
\end{aligned}$$

So they are two independent (one is not a multiple of the other) solutions of a second-order ODE, consequently, they are a basis of solutions. The general solution can be written as

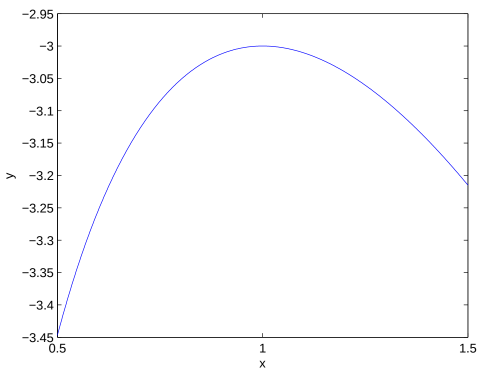
$$y = c_1y_1 + c_2y_2 = c_1x^{\frac{3}{2}} + c_2x^{-\frac{1}{2}}$$

The solution satisfying the initial values must fulfill

$$\left.\begin{aligned}
y_p(1) &= -3 = c_1 + c_2 \\
y_p'(1) &= 0 = \tfrac{3}{2}c_1 - \tfrac{1}{2}c_2
\end{aligned}\right\} \Rightarrow c_1 = -\frac{3}{4}, c_2 = -\frac{9}{4}$$

So

$$\boxed{y_p = -\frac{3}{4}x^{\frac{3}{2}} - \frac{9}{4}x^{-\frac{1}{2}}}$$



Kreyszig, 2.1.19

Álvaro Martín Ramos, Dec. 27th, 2014

Verify that the functions

$$e^{-x}\cos(x), e^{-x}\sin(x)$$

are a basis of the ODE

$$y'' + 2y' + 2y = 0$$

Find the particular solution satisfying y(0)=0, y'(0)=15.

Solution: Let us calculate the derivatives of the two given functions

$$\begin{aligned}
y_1 &= e^{-x}\cos(x)\\
y_1' &= -e^{-x}\cos(x) - e^{-x}\sin(x)\\
y_1'' &= [e^{-x}\cos(x) + e^{-x}\sin(x)] - [-e^{-x}\sin(x) + e^{-x}\cos(x)]\\
&= 2e^{-x}\sin(x)\\
y_2 &= e^{-x}\sin(x)\\
y_2' &= -e^{-x}\sin(x) + e^{-x}\cos(x)\\
y_2'' &= [e^{-x}\sin(x) - e^{-x}\cos(x)] + [-e^{-x}\cos(x) - e^{-x}\sin(x)]\\
&= -2e^{-x}\cos(x)
\end{aligned}$$

We now substitute these two functions in the ODE to verify if they are solutions of it

$$y_1'' + 2y_1' + 2y_1 = 0$$

$$2e^{-x}\sin(x) + 2[-e^{-x}\cos(x) - e^{-x}\sin(x)] + 2[e^{-x}\cos(x)] = 0$$

$$0 = 0$$

Similarly

$$y_2'' + 2y_2' + 2y_2 = 0$$

$$-2e^{-x}\cos(x) + 2[-e^{-x}\sin(x) + e^{-x}\cos(x)] + 2[e^{-x}\sin(x)]$$

$$0 = 0$$

So they are two independent(one is not multiple of the other) solutions of a second-order ODE, consequently, they are a basis of solutions. The general solution can be written as

$$y = c_1y_1 + c_2y_2 = c_1e^{-x}\cos(x) + c_2e^{-x}\sin(x)$$

The solution satisfying the initial values must fulfill

$$y(0) = 0 = c_1$$

$$y'(0) = 15 = c_1[-e^0\cos(0) - e^0\sin(0)] + c_2[-e^0\sin(0) + e^0\cos(0)] = -c_1 + c_2$$

$$-c_1 + c_2 = 15 \Rightarrow c_2 = 15$$

So

$$\boxed{y_p = 15e^{-x}\sin(x)}$$

Kreyszig, 2.2.11

Álvaro Martín Ramos, Dec. 27th, 2014

Solve the ODE

$$y'' - 4y' - 3y = 0$$

Solution: The characteristic equation is

$$4\lambda^2 - 4\lambda - 3 = 0$$

whose solutions are

$$\lambda = \frac{4 \pm \sqrt{16+48}}{8} = \frac{4 \pm 8}{8} \Rightarrow \lambda_1 = \frac{3}{2}, \lambda_2 = -\frac{1}{2}$$

The general solution is

$$\boxed{y = c_1 e^{\frac{3}{2}x} + c_2 e^{(-\frac{1}{2})x}}$$

Kreyszig, 2.2.16

Carlos Oscar Sorzano, Aug. 31st, 2014

Find an ODE whose basis of solutions are $e^{2.6x}$ and $e^{-4.3x}$.
Solution: We look for an ODE of the form

$$y'' + ay' + by = 0$$

If the exponential $e^{\lambda x}$ is to be a solution of the ODE, it must fulfill

$$P(\lambda) = \lambda^2 + a\lambda + b = 0$$

But we already know that $\lambda = 2.6$ and $\lambda = -4.3$ are two solutions, so the characteristic polynomial can be factorized as

$$P(\lambda) = (\lambda - 2.6)(\lambda + 4.3) = \lambda^2 + 1.7\lambda - 11.18$$

The corresponding ODE is

$$\boxed{y'' + 1.7y' - 11.18y = 0}$$

Kreyszig, 2.2.17

Carlos Oscar Sorzano, Aug. 31st, 2014

Find an ODE whose basis of solutions are $e^{\sqrt{5}x}$ and $xe^{\sqrt{5}x}$.
Solution: As in the Problem 2.2.16, we know that the characteristic polynomial can be factorized as

$$P(\lambda) = (\lambda - \sqrt{5})^2 = \lambda^2 - 2\sqrt{5}\lambda + 5$$

The corresponding ODE is

$$\boxed{y'' - 2\sqrt{5}y' + 5y = 0}$$

Kreyszig, 2.2.19

Álvaro Martín Ramos, Dec. 27th, 2014

Find an ODE whose basis of solutions are $e^{(-2+i)x}$ and $e^{(-2-i)x}$.
Solution: We look for an ODE of the form

$$y'' + ay' + by = 0$$

If those are solutions of the ODE, it must fulfill

$$P(\lambda) = \lambda^2 + a\lambda + b = 0$$

We can solve it

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}) = -\frac{a}{2} + i\frac{w}{2}$$
$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}) = -\frac{a}{2} - i\frac{w}{2}$$

Where

$$w = \sqrt{a^2 - 4b}$$

We now that the generic solutions of the differential equation are of the form

$$y_1 = e^{(-\frac{a}{2} + i\frac{w}{2})x}$$
$$y_2 = e^{(-\frac{a}{2} - i\frac{w}{2})x}$$

Our solutions are

$$e^{(-2+i)x}, e^{(-2-i)x}$$

So

$$-2 = -\frac{a}{2} \Rightarrow a = 4$$

And

$$\frac{w}{2} = 1 \Rightarrow \sqrt{16 - 4b} = 2 \Rightarrow b = 3$$

Therefore the corresponding ODE is

$$\boxed{y'' + 4y' + 3y = 0}$$

Kreyszig, 2.2.31

Carlos Oscar Sorzano, Aug. 31st, 2014

Are the functions e^{kx} and xe^{kx} linearly independent on any interval?
Solution: Let us call

$$y_1 = e^{kx}$$
$$y_2 = xe^{kx}$$

The two functions are linearly dependent if we can find two constants, not all of them zero, such that

$$c_1 y_1 + c_2 y_2 = 0$$

If c_1 is different from 0, then

$$\frac{y_1}{y_2} = -\frac{c_2}{c_1}$$

If c_2 is different from 0, then

$$\frac{y_2}{y_1} = -\frac{c_1}{c_2}$$

That is if they are linearly dependent, one function must be a multiple of the other or 0. The ratio

$$\frac{y_2}{y_1} = \frac{xe^{kx}}{e^{kx}} = x$$

is not constant, and consequently, the two functions are linearly independent.

Kreyszig, 2.2.33

Álvaro Martín Ramos, Dec. 27th, 2014

Are the functions x^2 and $x^2 ln(x)$ linearly independent on the interval x > 1 ?

Solution: The ratio

$$\frac{x^2}{x^2 ln(x)} = \frac{1}{ln(x)}$$

is a function of x and not a constant, consequently, the two functions are linearly independent. If they were linearly dependent, their ratio would be constant.

Kreyszig, 2.2.34

Carlos Oscar Sorzano, Aug. 31st, 2014

Are the functions $\log(x)$ and $\log(x^3)$ linearly independent on the interval $x > 1$?

Solution: The ratio

$$\frac{\log(x^3)}{\log(x)} = \frac{3\log(x)}{\log(x)} = 3$$

is constant, and consequently, the two functions are linearly dependent (one is a multiple of the other).

Kreyszig, 2.2.35

Carlos Oscar Sorzano, Aug. 31st, 2014

Are the functions $\sin(2x)$ and $\cos(x)\sin(x)$ linearly independent on the interval $x < 0$?

Solution: The ratio

$$\frac{\sin(2x)}{\cos(x)\sin(x)} = \frac{2\cos(x)\sin(x)}{\cos(x)\sin(x)} = 2$$

is constant, and consequently, the two functions are linearly dependent (one is a multiple of the other).

Kreyszig, 2.3.5

Carlos Oscar Sorzano, June 15th, 2015

Apply the operator $(D-2I)(D+3I)$ to the functions e^{2x}, xe^{2x}, and e^{-3x}. Show all steps in detail.

Solution:

$$\begin{aligned}
(D-2I)(D+3I)(e^{2x}) &= (D-2I)(2e^{2x}+3e^{2x}) \\
&= (D-2I)(5e^{2x}) \\
&= 10e^{2x}-10e^{2x} \\
&= 0 \\
(D-2I)(D+3I)(xe^{2x}) &= (D-2I)((1+2x)e^{2x}+3xe^{2x}) \\
&= (D-2I)((1+5x)e^{2x}) \\
&= (7+10x)e^{2x}-2(1+5x)e^{2x} \\
&= 5e^{2x} \\
(D-2I)(D+3I)(e^{-3x}) &= (D-2I)(-3e^{-3x}+3e^{-3x}) \\
&= (D-2I)(0) \\
&= 0
\end{aligned}$$

Kreyszig, 2.3.14

Carlos Oscar Sorzano, Aug. 31st, 2014

If $L = D^2 + aD + bI$ has distinct roots μ and λ, show that a particular solution is

$$y = \frac{e^{\mu x} - e^{\lambda x}}{\mu - \lambda}$$

Obtain from this a solution $xe^{\lambda x}$ by letting $\mu \to \lambda$ and applying L'Hôpital rule.
Solution: Since μ and λ are roots of the polynomial $s^2 + as + b$ and we know that

$$(D^2 + aD + bI)e^{\mu x} = 0$$

$$(D^2 + aD + bI)e^{\lambda x} = 0$$

Let us check whether the function $y = \frac{e^{\mu x}-e^{\lambda x}}{\mu-\lambda}$ is a solution of the ODE

$$Ly = 0$$

$$\begin{aligned}
(D^2+aD+bI)\left(\frac{e^{\mu x}-e^{\lambda x}}{\mu-\lambda}\right) &= \frac{1}{\mu-\lambda}(D^2+aD+bI)e^{\mu x} - \frac{1}{\mu-\lambda}(D^2+aD+bI)e^{\lambda x} \\
&= \frac{1}{\mu-\lambda}0 - \frac{1}{\mu-\lambda}0 \\
&= 0
\end{aligned}$$

So y is a solution.

Let us study the behaviour of y as $\mu \to \lambda$

$$\begin{aligned}
\lim_{\mu\to\lambda} \frac{e^{\mu x}-e^{\lambda x}}{\mu-\lambda} &= \lim_{\mu\to\lambda} \frac{\left(e^{(\mu-\lambda)x}-1\right)e^{\lambda x}}{\mu-\lambda} \\
&= e^{\lambda x} \lim_{\mu\to\lambda} \frac{e^{(\mu-\lambda)x}-1}{\mu-\lambda} \\
&= e^{\lambda x} \lim_{\mu\to\lambda} \frac{1+(\mu-\lambda)x-1}{\mu-\lambda} \\
&= e^{\lambda x} \lim_{\mu\to\lambda} \frac{(\mu-\lambda)x}{\mu-\lambda} \\
&= xe^{\lambda x}
\end{aligned}$$

Kreyszig, 2.4.3

Álvaro Martín Ramos, Dec. 27th, 2014

How does the frequency of the harmonic oscillation change if we (i) double the mass, (ii) take a spring of twice the modulus?
Solution: By the Newton's second law and Hooke's law we know that

$$-ky = my''$$

$$y'' + \frac{k}{m}y = 0$$

We can solve its characteristic equation

$$\lambda^2 + \frac{k}{m} = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

(i) If we double the mass

$$\lambda^2 + \frac{k}{2m} = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{k}{2m}} = \pm i\frac{1}{\sqrt{2}}\sqrt{\frac{k}{m}} = \pm i\frac{1}{\sqrt{2}}\omega_0$$

So the frequency will be lower by a factor $\frac{1}{\sqrt{2}}$.
(ii) If we take a spring of twice the modulus

$$\lambda^2 + 2\frac{k}{m}y = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{2k}{m}} = \pm i\sqrt{2}\sqrt{\frac{k}{m}} = \pm i\sqrt{2}w_0$$

So the frequency will be higher by a factor $\sqrt{2}$.

Kreyszig, 2.4.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Springs in parallel. What are the frequencies of vibration of a body of mass $m = 5[kg]$ (i) on a spring of modulus $k_1 = 20[N/m]$, (ii) on a spring of modulus $k_2 = 45[N/m]$, (iii) on the two springs in parallel?
Solution: For the cases (i) and (ii), with a single spring, the differential equation governing the system is

$$F = -ky = my'' \Rightarrow y'' + \frac{k}{m}y = 0$$

The frequency of vibration comes from the analysis of the characteristic polynomial of the ODE

$$\lambda^2 + \frac{k}{m} = 0 \Rightarrow \lambda = \pm i\omega_0 = \pm i\sqrt{\frac{k}{m}}$$

In the first case it is

$$\omega_{01} = \sqrt{\frac{k_1}{m}} = \sqrt{\frac{20[N/m]}{5[Kg]}} = 2[s^{-1}]$$

In the second case

$$\omega_{02} = \sqrt{\frac{k_2}{m}} = \sqrt{\frac{45[N/m]}{5[Kg]}} = 3[s^{-1}]$$

If we put the springs in parallel, the system would be described by

$$F_1 + F_2 = -k_1 y - k_2 y = my'' \Rightarrow y'' + \frac{k_1 + k_2}{m} y = 0$$

$$\boxed{\omega_{03} = \sqrt{\frac{k_1 + k_2}{m}}} = \sqrt{\frac{65[N/m]}{5[Kg]}} = 3.6[s^{-1}]$$

Kreyszig, 2.4.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Springs in series. What is the frequency of vibration if the two springs are in series instead of in parallel?
Solution: The force applied on the mass must fulfill

$$F = -ky = -k(y_1 + y_2)$$

On another side,

$$F = -k_1 y_1 = -k_2 y_2$$

Then we can write

$$y = y_1 + y_2$$
$$-\frac{F}{k} = -\frac{F}{k_1} - \frac{F}{k_2}$$
$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow k = \frac{k_1 k_2}{k_1 + k_2}$$

Then, we can calculate the frequency of oscillation as

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1 k_2}{(k_1 + k_2)m}}} = \sqrt{\frac{45 \cdot 20}{(45 + 20)5}} = 1.66[s^{-1}]$$

Kreyszig, 2.4.7

Carlos Oscar Sorzano, Aug. 31st, 2014

Pendulum. Find the frequency of oscillation of a pendulum of length L, neglecting air resistance and the weight of the rod, and assuming θ to be so small that $\sin(\theta)$ practically equals θ.
Solution: The movement of the pendulum is along an arch whose length is $L\Theta$. The acceleration is the second derivative of this variable $(L\Theta)''$, and Newton's second law of motion states

$$F = ma$$
$$-mg\sin(\theta) = m(L\theta)''$$
$$-g\sin(\theta) = (L\theta)''$$

For a small angle $\sin(\theta) \approx \theta$

$$-g\theta = (L\theta)''$$
$$\theta'' + \frac{g}{L}\theta = 0$$

The characteristic polynomial is

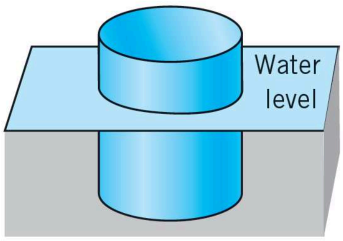
$$\lambda^2 + \frac{g}{L} = 0$$

$$\lambda = \pm i\omega_0 = \pm i\sqrt{\frac{g}{L}}$$

Kreyszig, 2.4.8

Carlos Oscar Sorzano, Nov. 2nd, 2014

Archimedian principle. This principle states that the buoyancy force equals the weight of the water displaced by the body (partly or totally submerged). The cylindrical buoy of diameter 60 cm in the following figure is floating in water with its axis vertical. When depressed downward in the water and released, it vibrates with period 2 sec. What is its weight?



Solution: Let y be the height of the cylinder that has been submerged. The force that the buoy experiences is

$$F = \rho(\pi r^2)y$$

where ρ is the specific weight of water ($\rho = 980[(cm/s^2)(g/cm^3)] = 980[g/(cm^2s^2)]$) and r is the radius of the cylinder (60 cm). By Newton's law:

$$my'' = -\rho(\pi r^2)y$$

$$my'' + \rho(\pi r^2)y = 0$$

The oscillation frequency comes from the roots of the characteristic equation

$$m\lambda^2 + \rho(\pi r^2) = 0$$

$$\lambda = \pm ir\sqrt{\frac{\rho\pi}{m}} = \pm i\omega_0 = \pm\frac{2\pi}{T}$$

where T is the oscillation period. Solving for T we have

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{r\sqrt{\frac{\rho\pi}{m}}} = \frac{2}{r}\sqrt{\frac{\pi m}{\rho}}$$

Finally, the weight of the cylinder is

$$m = \frac{\rho r^2 T^2}{4\pi}$$

In this particular case

$$m = \frac{\rho r^2 T^2}{4\pi} = \frac{980[g/(cm^2 s^2)]30^2[cm^2]2^2[s^2]}{4\pi} = 280.75[kg]$$

Kreyszig, 2.4.14

Carlos Oscar Sorzano, Aug. 31st, 2014

Shock absorber. What is the smallest value of the damping constant of a shock absorber in the suspension of a wheel of a car (consisting of a spring and an absorber) that will provide (theoretically) an oscillation free ride if the mass of the car is 2000 $[Kg]$ and the spring constant equals 4500 $[Kg/s^2]$?
Solution: The equation defining motion is

$$my'' = -ky - cy'$$

whose characteristic polynomial is

$$m\lambda^2 = -k - c\lambda$$

$$m\lambda^2 + c\lambda + k = 0$$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Critical damping is attained if

$$c^2 - 4km = 0 \Rightarrow c = 2\sqrt{km} = \sqrt{4500[Kg/s^2]2000[Kg]} = 3000[Kg/s]$$

If $c \geq 3000[Kg/s]$, there are no oscillations in the car.

Kreyszig, 2.4.18

Carlos Oscar Sorzano, Aug. 31st, 2014

Logarithmic decrement. Show that the ratio of two consecutive maximum amplitudes of a damped oscillation

$$y(t) = Ce^{-at}\cos(\omega_0 t - \delta)$$

is constant, and the natural logarithm of this ratio called the logarithmic decrement, equals

$$\Delta = \frac{2\pi a}{\omega_0}.$$

Find Δ for the solutions of $y'' + 2y' + 5 = 0$. .
Solution: Let us calculate the maxima of the oscillation curve

$$\frac{dy}{dt} = 0 \quad = \quad C\left(-ae^{-at}\cos(\omega_0 t - \delta) - e^{-at}\omega_0 \sin(\omega_0 t - \delta)\right)$$

which implies that

$$a\cos(\omega_0 t - \delta) + \omega_0 \sin(\omega_0 t - \delta) = 0$$

$$\tan(\omega_0 t - \delta) = -\frac{a}{\omega_0}$$

$$\omega_0 t - \delta = \text{atan}\left(-\frac{a}{\omega_0}\right) + \pi k$$

$$t = \frac{1}{\omega_0}\left(\delta - \text{atan}\left(\frac{a}{\omega_0}\right) + \pi k\right)$$

Let t_1 denote the time of a maximum and t_2 the time of the next maximum

$$\begin{aligned} t_1 &= \tfrac{1}{\omega_0}\left(\delta - \text{atan}\left(\tfrac{a}{\omega_0}\right) + \pi k_1\right) \\ t_2 &= \tfrac{1}{\omega_0}\left(\delta - \text{atan}\left(\tfrac{a}{\omega_0}\right) + \pi(k_1+2)\right) = t_1 + \tfrac{2\pi}{\omega_0} \end{aligned}$$

Let us evaluate the oscillation curve at these two time points

$$\begin{aligned} y(t_1) &= Ce^{-at_1}\cos(\omega_0 t_1 - \delta) \\ &= Ce^{-at_1}\cos\left(\omega_0 \tfrac{1}{\omega_0}\left(\delta - \text{atan}\left(\tfrac{a}{\omega_0}\right) + \pi k_1\right) - \delta\right) \\ &= Ce^{-at_1}\cos\left(-\text{atan}\left(\tfrac{a}{\omega_0}\right) + \pi k_1\right) \\ y(t_2) &= Ce^{-at_2}\cos(\omega_0 t_2 - \delta) \\ &= Ce^{-a\left(t_1+\frac{2\pi}{\omega_0}\right)}\cos\left(\omega_0(t_1 + \tfrac{2\pi}{\omega_0}) - \delta\right) \\ &= Ce^{-a\left(t_1+\frac{2\pi}{\omega_0}\right)}\cos(\omega_0 t_1 - \delta + 2\pi) \\ &= Ce^{-a\left(t_1+\frac{2\pi}{\omega_0}\right)}\cos(\omega_0 t_1 - \delta) \end{aligned}$$

Let us calculate now the ratio

$$\frac{y(t_1)}{y(t_2)} = \frac{Ce^{-at_1}\cos(\omega_0 t_1 - \delta)}{Ce^{-a\left(t_1+\frac{2\pi}{\omega_0}\right)}\cos(\omega_0 t_1 - \delta)} = e^{\frac{2\pi a}{\omega_0}}$$

The logarithm of this quantity is the logarithmic decrement

$$\Delta = \log\frac{y(t_1)}{y(t_2)} = \frac{2\pi a}{\omega_0}$$

The characteristic polynomial of the ODE

$$y'' + 2y' + 5 = 0$$

is

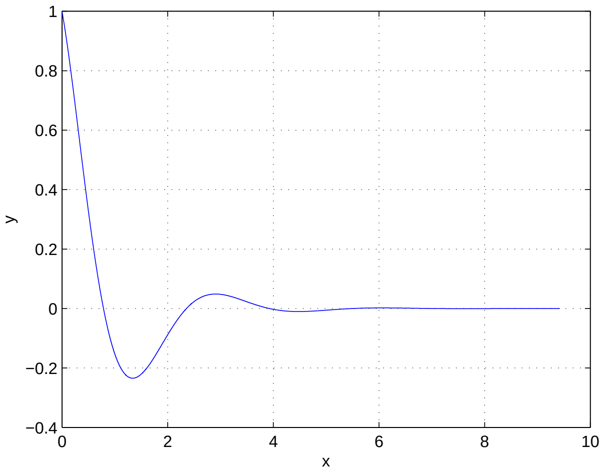
$$\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda = -1 \pm 2i = -a \pm i\omega_0$$

Consequently,

$$\Delta = \frac{2\pi(1)}{2} = \pi$$

So, from one maximum to the next, there is a factor

$$e^{-\Delta} = 0.043$$



Kreyszig, 2.5.11

Álvaro Martín Ramos, Dec. 27th, 2014

Solve the ODE

$$(x^2D^2 - 3xD + 10I)y = 0$$

Solution: We may rewrite the ODE as

$$x^2y'' - 3xy' + 10y = 0$$

Which is an equation of the form

$$x^2y'' + axy' + by = 0$$

The ODE is an Euler-Cauchy equation, so we try with a solution of the form

$$y = x^m$$

whose derivatives are

$$\begin{aligned} y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2} \end{aligned}$$

Substituting into the ODE we get

$$x^2(m(m-1)x^{m-2}) - 3x(mx^{m-1}) + 10x^m = 0$$

$$m(m-1) - 3m + 10 = 0$$

$$m^2 - 4m + 10 = 0 \Rightarrow m_1,_{,2} = 2 \pm i\sqrt{6}$$

$$\boxed{y = x^2(c_1 \cos\left(\sqrt{6}\log(x)\right) + c_2 \sin\left(\sqrt{6}\log(x)\right)}$$

Kreyszig, 2.6.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that the functions x^2 and x^3 are linearly independent calculating their ratio and their Wronskian.
Solution: The functions

$$y_1 = x^2$$
$$y_2 = x^3$$

are linearly independent because their ratio

$$\frac{y_1}{y_2} = \frac{x^2}{x^3} = \frac{1}{x}$$

is not constant. This independence is confirmed because their Wronskian

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \neq 0$$

is not 0 for all $x \neq 0$.

Kreyszig, 2.6.12

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the ODE whose basis of solutions are the functions x^2 and $x^2\log(x)$. Show the linear independence of the two functions and solve the initial value problem that satisfies $y(1) = 4$ and $y'(1) = 6$.
Solution: This basis is the basis of solutions of the Euler-Cauchy ODE with a double root at $m = 2$. So the Euler-Cauchy auxiliary equation

$$m^2 + (a-1)m + b = 0$$

must be equal to

$$(m-2)^2 = 0 = m^2 - 4m + 4$$

So $a = -3$ and $b = 4$. The corresponding ODE is

$$x^2y'' - 3xy' + 4y = 0$$

To show that the two functions are independent, we calculate their Wronskian

$$\begin{vmatrix} x^2 & x^2\log(x) \\ 2x & 2x\log(x) + x \end{vmatrix} = x^3 \begin{vmatrix} 1 & \log(x) \\ 2 & 2\log(x) + 1 \end{vmatrix} = x^3$$

The solution of the Initial Value Problem must be of the form

$$y_p = c_1x^2 + c_2x^2\log(x)$$
$$y_p(1) = 4 = c_1$$
$$y_p' = 2c_1x + 2c_2x\log(x) + c_2x$$
$$y_p'(1) = 6 = 2c_1 + c_2 = 8 + c_2 \Rightarrow c_2 = -2$$

So the solution sought is

$$y_p = x^2(4 - 2\log(x))$$

Kreyszig, 2.7.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the real general solution of

$$y'' + y' + (\pi^2 + \tfrac{1}{4})y = e^{-\frac{x}{2}}\sin(\pi x)$$

Solution: The solution of the homogeneous problem is given by the characteristic polynomial

$$\lambda^2 + \lambda + \pi^2 + \tfrac{1}{4} = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4\pi^2 - 1}}{2} = -\frac{1}{2} \pm i\pi$$

The real general solution of the homogeneous problem is

$$y_h = Ae^{-\frac{x}{2}}\cos(\pi x) + Be^{-\frac{x}{2}}\sin(\pi x)$$

Since the excitation signal $e^{-\frac{x}{2}}\sin(\pi x)$ corresponds to one of the basis, we try a particular function of the form

$$\begin{array}{rcl} y_p &=& K_1 xe^{-\frac{x}{2}}\cos(\pi x) + K_2 xe^{-\frac{x}{2}}\sin(\pi x) \\ y_p' &=& \frac{1}{2}e^{-\frac{x}{2}}\left[\cos(\pi x)(2\pi K_2 x - K_1(x-2)) - \sin(\pi x)(2\pi K_1 x + K_2(x-2))\right] \\ y_p'' &=& \frac{1}{4}e^{-\frac{x}{2}}\left[\sin(\pi x)\left(4\pi K_1(x-2) + K_2(-4\pi^2 x + x - 4)\right) + \right. \\ && \left. \cos(\pi x)\left(K_1(-4\pi^2 x + x - 4) - 4\pi K_2(x-2)\right)\right] \end{array}$$

We now substitute in the original equation

$$\begin{array}{rcl} y'' + y' + (\pi^2 + \frac{1}{4})y &=& 2\pi e^{-\frac{x}{2}}(K_2\cos(\pi x) - K_1\sin(\pi x)) \\ &=& e^{-\frac{x}{2}}\sin(\pi x) \end{array}$$

From where

$$K_2 = 0$$

$$-2\pi K_1 = 1 \Rightarrow K_1 = -\frac{1}{2\pi}$$

So the particular solution is of the form

$$y_p = -\frac{1}{2\pi}xe^{-\frac{x}{2}}\cos(\pi x)$$

And the general solution

$$y = y_p + y_h = \boxed{e^{-\frac{x}{2}}\left(\left(A - \frac{x}{2\pi}\right)\cos(\pi x) + B\sin(\pi x)\right)}$$

Kreyszig, 2.7.13

Carlos Oscar Sorzano, Jan. 15th, 2015

Find the real general solution of

$$8y'' - 6y' + y = 6\cosh(x) \quad y(0) = 0.2, y'(0) = 0.05$$

Solution: The solution of the homogeneous problem is given by the characteristic polynomial

$$8\lambda^2 - 6\lambda + 1 = 0 = 8\left(\lambda - \frac{1}{4}\right)\left(\lambda - \frac{1}{2}\right) \Rightarrow \lambda = \frac{1}{4}, \frac{1}{2}$$

The real general solution of the homogeneous problem is

$$y_h = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{4}}$$

The excitation function $6\cosh(x)$ does not belong to the space function of the homogeneous equation. We try a solution of the form

$$\begin{array}{rcl} y_p & = & A\cosh(x) + B\sinh(x) \\ y_p' & = & A\sinh(x) + B\cosh(x) \\ y_p'' & = & A\cosh(x) + B\sinh(x) \end{array}$$

Substituting into the differential equation

$$8(A\cosh(x)+B\sinh(x)) - 6(A\sinh(x)+B\cosh(x)) + A\cosh(x)+B\sinh(x) = 6\sinh(x)$$

$$(9A-6B)\cosh(x)+(9B-6A)\sinh(x) = 6\sinh(x) \Rightarrow \left\{ \begin{array}{l} 9A - 6B = 0 \\ 9B - 6A = 6 \end{array} \right. \Rightarrow A = \frac{4}{5}, B = \frac{6}{5}$$

The general solution is of the form

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{4}} + \frac{4}{5}\cosh(x) + \frac{6}{5}\sinh(x)$$

We need now to determine c_1 and c_2 using the initial values

$$\left.\begin{array}{rcl} y(0) & = & 0.2 = c_1 + c_2 + \frac{4}{5} \\ y'(0) & = & 0.05 = \frac{1}{2}c_1 + \frac{1}{4}c_2 + \frac{6}{5} \end{array}\right\} \Rightarrow c_1 = -4, c_2 = \frac{17}{5}$$

Finally, the solution of the IVP is

$$\boxed{y = -4e^{\frac{x}{2}} + \frac{17}{5}e^{\frac{x}{4}} + \frac{4}{5}\cosh(x) + \frac{6}{5}\sinh(x)}$$

Kreyszig, 2.8.13

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the transient motion of the mass-spring system modeled by the ODE

$$(D^2 + I)y = \cos(\omega t) \quad \omega \neq 1$$

Solution: The characteristic equation associated to this ODE is

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

So, the homogeneous response is of the form

$$y_h = A\cos(t) + B\sin(t)$$

Note that the external excitation does not have the same frequency as the internal natural frequency. For that reason, for the particular response to the external excitation we look a solution of the form

$$y_p = K_1\cos(\omega t) + K_2\sin(\omega t)$$
$$y_p' = -K_1\omega\sin(\omega t) + K_2\omega\cos(\omega t)$$
$$y_p'' = -K_1\omega^2\cos(\omega t) - K_2\omega^2\sin(\omega t)$$

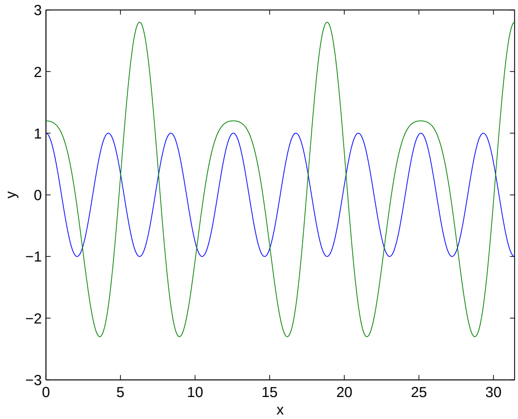
The ODE becomes

$$K_1(1-\omega^2)\cos(\omega t) + K_2(1-\omega^2)\sin(\omega t) = \cos(\omega t) \Rightarrow K_1 = \frac{1}{1-\omega^2}, K_2 = 0$$

So the general solution is

$$\boxed{y = y_h + y_p = A\cos(t) + B\sin(t) + \frac{1}{1-\omega^2}\cos(\omega t)}$$

The graph below shows this function for $\omega = 1.5$, $A = B = 1$



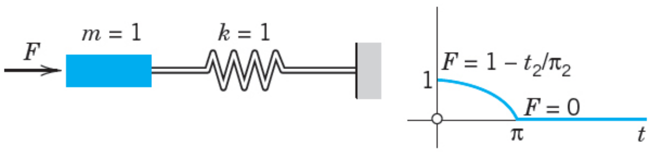
Kreyszig, 2.8.24

Carlos Oscar Sorzano, Jan. 15th, 2015

Gun barrel. Solve

$$y'' + y = F(t)$$

where $F(t) = \begin{cases} 1 - \frac{t^2}{\pi^2} & 0 \le t \le \pi \\ 0 & \text{otherwise} \end{cases}$ and $y(0) = 0$, $y'(0) = 0$. This models an undamped system on which a force F acts during some interval of time (see figure below), for instance, the force on a gun barrel when a shell is fired, the barrel being braked by heavy springs (and then damped by a dashpot, which we disregard for simplicity). Hint: At π both y and y' must be continuous.



Solution: The general solution of the homogeneous equation is given by the roots of the characteristic equation

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$y_h = c_1 \cos(t) + c_2 \sin(t)$$

In the interval $0 \le t \le \pi$ we look for a particular solution of the form

$$\begin{aligned} y_p &= A + Bt + Ct^2 \\ y_p' &= B + 2Ct \\ y_p'' &= 2C \end{aligned}$$

Substituting into the ODE

$$2C + (A + Bt + Ct^2) = 1 - \frac{t^2}{\pi^2}$$

$$(2C + A) + Bt + Ct^2 = 1 - \frac{1}{\pi^2}t^2$$

$$\left.\begin{array}{l} 2C + A = 1 \\ B = 0 \\ C = -\frac{1}{\pi^2} \end{array}\right\} \Rightarrow A = 1 + \frac{2}{\pi^2}, B = 0, C = -\frac{1}{\pi^2}$$

The general solution in this interval is of the form

$$y = c_1 \cos(t) + c_2 \sin(t) + 1 + \frac{2}{\pi^2} - \frac{t^2}{\pi^2}$$

To determine c_1 and c_2 we impose the initial conditions

$$\begin{aligned} y(0) &= 0 = c_1 + 1 + \tfrac{2}{\pi^2} \Rightarrow c_1 = -\left(1 + \tfrac{2}{\pi^2}\right) \\ y'(0) &= 0 = c_2 \end{aligned}$$

Finally, the solution in this interval is

$$y = \left(1 + \frac{2}{\pi^2}\right)(1 - \cos(t)) - \frac{t^2}{\pi^2}$$

Note that at $t = \pi$ we have

$$\begin{array}{l} y(\pi) = \left(1 + \frac{2}{\pi^2}\right)(1 - \cos(\pi)) - \frac{\pi^2}{\pi^2} = 1 + \frac{4}{\pi^2} \\ y'(\pi) = \left(1 + \frac{2}{\pi^2}\right)\sin(\pi) - \frac{2\pi}{\pi^2} = -\frac{2}{\pi} \end{array}$$

In the interval $t > 0$ there is no external force, so the solution is given only by the homogeneous solution. At $t = \pi$ the solution, and its derivative, must be continuous, so we have the solution

$$y = c_1 \cos(t) + c_2 \sin(t)$$

with the initial values

$$\begin{aligned} y(\pi) &= 1 + \tfrac{4}{\pi^2} = c_1 \cos(\pi) + c_2 \sin(\pi) \Rightarrow c_1 = -\left(1 + \tfrac{4}{\pi^2}\right) \\ y'(\pi) &= -\tfrac{2}{\pi} = -c_1 \sin(\pi) + c_2 \cos(\pi) \Rightarrow c_2 = \tfrac{2}{\pi} \end{aligned}$$

That is the solution in this interval is

$$y = -\left(1 + \frac{4}{\pi^2}\right)\cos(t) + \frac{2}{\pi}\sin(t)$$

Note that this solution is oscillatory and never vanishes because we have disregarded damping.

Finally we can write the solution to the initial problem as

$$\boxed{y(t) = \begin{cases} \left(1 + \frac{2}{\pi^2}\right)(1 - \cos(t)) - \frac{t^2}{\pi^2} & 0 \le t \le \pi \\ -\left(1 + \frac{4}{\pi^2}\right)\cos(t) + \frac{2}{\pi}\sin(t) & \text{otherwise} \end{cases}}$$

Kreyszig, 2.9.1

Carlos Oscar Sorzano, Aug. 31st, 2014

Model the RC circuit of the figure below. Find the current due to a constant E

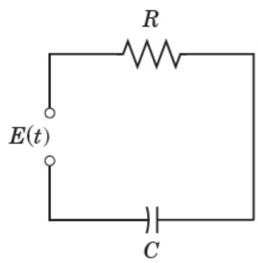


Fig. 64. *RC*-circuit

Solution: To model the circuit we sum the drops of voltage along the RC loop

$$E(t) - iR - \frac{1}{C}Q = 0$$

$$E(t) - iR - \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0$$

Differentiating

$$E' - i'R - \frac{1}{C}i = 0$$

$$i' + \frac{1}{RC}i = \frac{1}{R}E'$$

If is constant, $E = E_0$, then $E' = 0$ and the solution is given by the homogeneous equation whose characteristic equation is

$$\lambda + \frac{1}{RC} = 0 \Rightarrow \lambda = -\frac{1}{RC}$$

$$i(t) = Ae^{-\frac{t}{RC}}$$

If at $t = 0$ we have $i(0)$, then

$$i(0) = A$$

Finally,

$$\boxed{i(t) = i(0)e^{-\frac{t}{RC}}}$$

Kreyszig, 2.10.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve

$$(D^2 + 6D + 9I)y = 16\frac{e^{-3x}}{x^2+1}$$

by variation of parameters.

Solution: Let us find first the solution to the homogeneous problem. We need the roots of the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda = -3, -3$$

So the homogeneous solution is

$$y_h = c_1 y_1 + c_2 y_2 = c_1 e^{-3x} + c_2 x e^{-3x}$$

The Wronskian of the y_1 and y_2 functions is

$$W = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{vmatrix} = \left(e^{-3x}\right)^2 \begin{vmatrix} 1 & x \\ -3 & -3x+1 \end{vmatrix} = e^{-6x}$$

So the particular solution to the non-homogeneous problem is given by

$$\begin{array}{rcl} y_p & = & -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx \\ & = & -e^{-3x} \int \frac{xe^{-3x} 16 \frac{e^{-3x}}{x^2+1}}{e^{-6x}} dx + xe^{-3x} \int \frac{e^{-3x} 16 \frac{e^{-3x}}{x^2+1}}{e^{-6x}} dx \\ & = & -16e^{-3x} \int \frac{x}{x^2+1} dx + 16xe^{-3x} \int \frac{1}{x^2+1} dx \\ & = & -8e^{-3x} \log(x^2+1) + 16xe^{-3x} \mathrm{atan}(x^2+1) \end{array}$$

The general solution is

$$y = y_h + y_p = \boxed{(c_1 - 8\log(x^2+1))e^{-3x} + (c_2 + 16\mathrm{atan}(x^2+1))xe^{-3x}}$$

3 Chapter 3

Kreyszig, 3.1.1

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that the functions 1, x, x^2, x^3 are solutions of

$$y^{iv} = 0$$

and form a basis on any interval.

Solution: Let us calculate the fourth derivative of all these functions

y_i	1	x	x^2	x^3
y_i'	0	1	$2x$	$3x^2$
y_i''	0	0	2	$6x$
y_i'''	0	0	0	6
y_i^{iv}	0	0	0	0

So, the proposed functions are solutions of the ODE. To see if they are linearly independent, we calculate their Wronskian

$$W = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12$$

Since they are 4 independent solutions of a 4th order ODE, they are a basis of solutions.

Kreyszig, 3.1.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that the functions 1, $e^{-x}\cos(2x)$, $e^{-x}\sin(2x)$ are solutions of

$$y''' + 2y'' + 5y' = 0$$

and form a basis on any interval.

Solution: Let us write the ODE as

$$(D^3 + 2D^2 + 5D)y = 0$$

$$D(D^2 + 2D + 5)y = 0$$

$$D((D+1)^2 + 2^2)y = 0$$

The function $y_1 = 1$ is a solution of the first factor

$$Dy = 0$$

while the functions $y_2 = e^{-x}\cos(2x)$ and $y_3 = e^{-x}\sin(2x)$ are solutions of the second

$$((D+1)^2 + 2^2)y = 0$$

So, the proposed functions are solutions of the ODE. To see if they are linearly independent, we calculate their Wronskian

$$W = \begin{vmatrix} 1 & e^{-x}\cos(2x) & e^{-x}\sin(2x) \\ 0 & -e^{-x}(\cos(x) + \sin(x)) & e^{-x}(\cos(x) - \sin(x)) \\ 0 & 2e^{-x}\sin(x) & -2e^{-x}\cos(x) \end{vmatrix} = 2e^{-2x}$$

Kreyszig, 3.1.10

Carlos Oscar Sorzano, Jan. 15th, 2015

Are the functions e^{2x}, xe^{2x} and x^2e^{2x} linearly dependent or independent in the interval $x \geq 0$?

Solution: Let us call $f_1(x) = e^{2x}$, $f_2(x) = xe^{2x}$ and $f_3(x) = x^2e^{2x}$. For checking the linear dependence or not of the three functions we calculate the Wronskian of the three functions

$$\begin{aligned} W(x) &= \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ \frac{df_1(x)}{dx} & \frac{df_2(x)}{dx} & \frac{df_3(x)}{dx} \\ \frac{d^2f_1(x)}{dx^2} & \frac{d^2f_2(x)}{dx^2} & \frac{d^2f_3(x)}{dx^2} \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} & x^2e^{2x} \\ 2e^{2x} & (1+2x)e^{2x} & 2(1+x)xe^{2x} \\ 4e^{2x} & 4(1+x)e^{2x} & 2(2x^2+4x+1)e^{2x} \end{vmatrix} \\ &= e^{6x}\begin{vmatrix} 1 & x & x^2 \\ 2 & 1+2x & 2(1+x)x \\ 4 & 4(1+x) & 2(2x^2+4x+1) \end{vmatrix} = 2e^{6x} \end{aligned}$$

Since $W(x) > 0$ for $x \geq 0$, then the three functions f_1, f_2 and f_3 are linearly independent in this interval.

Kreyszig, 3.2.5

Álvaro Martín Ramos, Jan. 4th, 2015

Solve

$$(D^4 + 10D^2 + 9I)y = 0$$

Solution: The characteristic polynomial of the ODE is

$$\lambda^4 + 10\lambda^2 + 9 = 0 = (\lambda^2)^2 + (10\lambda^2) + 9$$

$$\lambda = \pm i3, \pm i$$

So the general solution is

$$\boxed{y = A\cos(x) + B\sin(x) + C\cos(3x) + D\sin(3x)}$$

Kreyszig, 3.2.6

Carlos Oscar Sorzano, Nov. 14th, 2014

Solve the differential equation

$$(D^5 + 8D^3 + 16D)y = 0$$

Solution: Let us factorize the differential operator

$$(D^5 + 8D^3 + 16D) = D(D^4 + 8D^2 + 16) = D(D^2 + 4)^2$$

The characteristic equation is

$$\lambda(\lambda^2 + 4)^2 = 0$$

$$\lambda(\lambda - 2i)^2(\lambda + 2i)^2 = 0$$

The general solution of the differential equation is

$$\boxed{y = c_1 + (c_2 + c_3x)\cos(2x) + (c_4 + c_5x)\sin(2x)}$$

Kreyszig, 3.2.7

Carlos Oscar Sorzano, Nov. 14th, 2014

Solve the IVP

$$y''' + 3.2y'' + 4.81y' = 0 \quad y(0) = 3.4, y'(0) = -4.6, y''(0) = 9.91$$

Solution: The characteristic equation is

$$\lambda^3 + 3.2\lambda^2 + 4.81\lambda = 0$$

$$\lambda(\lambda^2 + 3.2\lambda + 4.81) = 0$$

$$\lambda = 0, -1.6 \pm 1.5i$$

The general solution of the differential equation is

$$y = c_1 + e^{-1.6x}(c_2 \cos(1.5x) + c_3 \sin(1.5x))$$

Let us calculate the first and second derivatives of the solution we have

$$y' = e^{-1.6x}((-1.5c_2 - 1.6c_3)\sin(1.5x) + (1.5c_3 - 1.6c_2)\cos(1.5x))$$

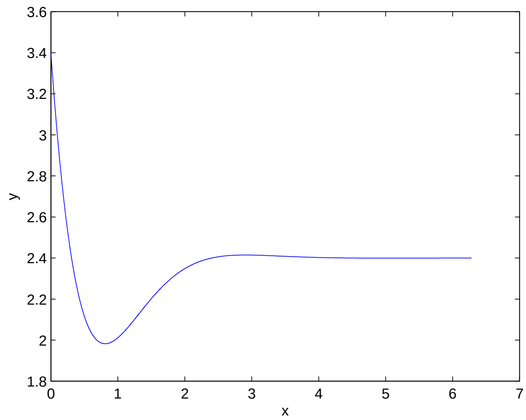
$$y'' = e^{-1.6x}((4.8c_2 + 0.31c_3)\sin(1.5x) + (0.31c_2 - 4.8c_3)\cos(1.5x))$$

Particularizing at $x = 0$

$$\left.\begin{array}{rcrcl} y(0) &=& 3.4 &=& c_1 + c_2 \\ y'(0) &=& -4.6 &=& 1.5c_3 - 1.6c_2 \\ y''(0) &=& 9.91 &=& 0.31c_2 - 4.8c_3 \end{array}\right\} \Rightarrow c_1 = 2.4, c_2 = 1, c_3 = -2$$

So the particular solution is

$$\boxed{y = 2.4 + e^{-1.6x}(\cos(1.5x) - 2\sin(1.5x))}$$



Kreyszig, 3.2.14

Carlos Oscar Sorzano, Aug. 31st, 2014

Reduction of order. If a solution of a linear, constant-coefficient ODE is known, y_1, we can reduce its order by assuming that

$$y = uy_1$$

1. Extend the method to a variable-coefficient ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

Assuming a solution y_1 to be known, show that another solution is

$$y_2 = uy_1$$

with

$$u = \int z(x)dx$$

and z obtained by solving

$$y_1z'' + (3y_1' + p_2y_1)z' + (3y_1'' + 2p_2y_1' + p_1y_1)z = 0$$

2. Reduce

$$x^3y''' - 3x^2y'' + (6 - x^2)xy' - (6 - x^2)y = 0$$

using $y_1 = x$ (perhaps obtainable by inspection).

Solution:

1. Let us assume that

$$y = uy_1$$

then

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + u'y_1' + u'y_1' + uy_1'' \\ &= u''y_1 + 2u'y_1' + uy_1'' \\ y''' &= u'''y_1 + u''y_1' + 2u''y_1' + 2u'y_1'' + u'y_1'' + uy_1''' \\ &= u'''y_1 + 3u''y_1' + 3u'y_1'' + uy_1''' \end{aligned}$$

Substituting in the ODE

$$y''' + p_2y'' + p_1y' + p_0y = 0$$

$$(u'''y_1 + 3u''y_1' + 3u'y_1'' + uy_1''') + p_2\,(u''y_1 + 2u'y_1' + uy_1'') + p_1(u'y_1 + uy_1') + p_0uy_1 = 0$$

$$(uy_1''') + p_2\,(uy_1'') + p_1(u'y_1 + uy_1') + p_0uy_1 = 0$$

$$y_1u''' + (3y_1' + p_2y_1)u'' + (3y_1'' + 2p_2y_1' + p_1y_1)u' + (y_1''' + p_2y_1'' + p_1y_1' + p_0y_1)u = 0$$

Since y_1 is a solution of the ODE, we have

$$y_1''' + p_2y_1'' + p_1y_1' + p_0y_1 = 0$$

Defining

$$z = u'$$

we can write the ODE as

$$\boxed{y_1z'' + (3y_1' + p_2y_1)z' + (3y_1'' + 2p_2y_1' + p_1y_1)z = 0}$$

as required by the problem.

2. Let us divide by x^3

$$y''' - \frac{3}{x}y'' + \left(\frac{6}{x^2} - 1\right)y' - \left(\frac{6}{x^3} - \frac{1}{x}\right)y = 0$$

We can now apply the formula derived in this exercise, in particular

$$(x)z'' + \left(3(1) + \left(-\frac{3}{x}\right)x\right)z' + \left(3(0) + 2\left(-\frac{3}{x}\right)(1) + \left(\frac{6}{x^2} - 1\right)x\right)z = 0$$

$$xz'' - xz = 0$$

$$z'' - z = 0$$

whose characteristic polynomial is

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

So the solution is

$$z = c_1e^x + c_2e^{-x}$$

$$u = \int z dx = \int (c_1e^x + c_2e^{-x})dx = c_1e^x + c_2e^{-x}$$

and the solution sought

$$y = uy_1 = (c_1e^x + c_2e^{-x})x = c_1xe^x + c_2xe^{-x}$$

Finally, the general solution is

$$\boxed{y = c_1xe^x + c_2xe^{-x} + c_3x}$$

Kreyszig, 3.3.5

Álvaro Martín Ramos, Jan. 4th, 2015

Solve

$$(x^3D^3 + x^2D^2 - 2xD + 2I)y = x^{-2}$$

Solution: We can rewrite the ODE as

$$x^3y''' + x^2y'' - 2xy' + 2y = x^{-2}$$

The homogeneous ODE is an Euler-Cauchy equation, so we try a solution of the form

$$\begin{aligned} y &= x^m \\ y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2} \\ y''' &= m(m-2)(m-1)x^{m-3} \end{aligned}$$

Substituting in the equation

$$x^3m(m-2)(m-1)x^{m-3} + x^2m(m-1)x^{m-2} - 2xmx^{m-1} + 2x^m = 0$$

$$(m(m-2)(m-1) + m(m-1) - 2m + 2)x^m = 0$$

$$m^3 - 2m^2 - m + 2 = 0 = (m-2)(m-1)(m+1)$$

So $m = 2, 1, -1$ are the roots of the characteristic polynomial. The general solution of the homogeneous problem is

$$y_h = c_1 x + c_2 x^{-1} + c_3 x^2$$

For finding a particular solution of the non-homogeneous problem we use the method of variation parameters whose solution is

$$y_p = \sum_{k=1}^{3} y_k \int \frac{W_k}{W} r dx$$

Where y_k are the 3 homogeneous solutions and W and W_k are the following matrices

$$\begin{aligned}
W &= \begin{bmatrix} x & x^{-1} & x^2 \\ 1 & -x^{-2} & 2x \\ 0 & 2x^{-3} & 2 \end{bmatrix} = \tfrac{2x^2-8}{x} \\
W_1 &= \begin{bmatrix} 0 & x^{-1} & x^2 \\ 0 & -x^{-2} & 2x \\ 1 & 2x^{-3} & 2 \end{bmatrix} = 3 \\
W_2 &= \begin{bmatrix} x & 0 & x^2 \\ 1 & 0 & 2x \\ 0 & 1 & 2 \end{bmatrix} = x^2 - 2x^2 \\
W_3 &= \begin{bmatrix} x & x^{-1} & 0 \\ 1 & -x^{-2} & 0 \\ 0 & 2x^{-3} & 1 \end{bmatrix} = \tfrac{-2}{x}
\end{aligned}$$

We write the ODE in the standard form

$$x^3 y''' + x^2 y'' - 2xy' + 2y = x^{-2} \Rightarrow y''' + \frac{1}{x} y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = x^{-5}$$

We now calculate the integrals

$$\int \frac{W_1}{W} r dx = \int \frac{3x}{2x^2 - 8} x^{-5} dx = \frac{-3x^3 \tanh^{-1}(\frac{x}{2}) + 6x^2 + 8}{64x^3}$$

$$\int \frac{W_2}{W} r dx = \int \frac{x^2 - 2x^2}{2x^2 - 8} x^{-5} dx = \frac{x \tanh^{-1}(\frac{x}{2} - 2)}{16x}$$

$$\int \frac{W_3}{W} r dx = \int \frac{-2}{2x^2 - 8} x^{-5} dx = \frac{-1}{16} - \frac{1}{32x^2} - \frac{1}{128} \log(x^2 - 4) + \frac{\log(x)}{64}$$

Finally, the particular solution is

$$\begin{aligned}
y_p &= x \tfrac{-3x^3 \tanh^{-1}(\frac{x}{2}) + 6x^2 + 8}{64x^3} + x^{-1} \tfrac{x \tanh^{-1}(\frac{x}{2} - 2)}{16x} + x^2 (\tfrac{-1}{16} - \tfrac{1}{32x^2} - \tfrac{1}{128} \log(x^2 - 4) + \tfrac{\log(x)}{64}) \\
&= -\tfrac{1}{12x^2}
\end{aligned}$$

Finally, the general solution is of the form

$$\boxed{y = c_1 x + c_2 x^{-1} + c_3 x^2 - \frac{1}{12x^2}}$$

Kreyszig, 3.3.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve $(D^3 + 4D)y = \sin(x)$.
Solution: The homogeneous problem has a characteristic polynomial

$$\lambda^3 + 4\lambda = 0 = \lambda(\lambda^2 + 4) \Rightarrow \lambda = 0, \pm 2i$$

So the homogeneous solution is given by

$$y_h = c_1 + c_2 \cos(2x) + c_3 \sin(2x)$$

To find a particular solution for non-homogeneous problem we try a function of the form

$$y_p = K_1 \cos(x) + K_2 \sin(x)$$
$$y_p' = -K_1 \sin(x) + K_2 \cos(x)$$
$$y_p'' = -K_1 \cos(x) - K_2 \sin(x)$$
$$y_p''' = K_1 \sin(x) - K_2 \cos(x)$$

Substituting in the ODE

$$(K_1 \sin(x) - K_2 \cos(x)) + 4(-K_1 \sin(x) + K_2 \cos(x)) = \sin(x)$$

$$-3K_1 \sin(x) + 3K_2 \cos(x) = \sin(x)$$

$$K_1 = -\frac{1}{3}, K_2 = 0$$

Finally, the general solution is

$$\boxed{y = c_1 + c_2 \cos(2x) + c_3 \sin(2x) - \frac{1}{3} \sin(x)}$$

Kreyszig, 3.3.7

Álvaro Martín Ramos, Jan. 4th, 2015

Solve

$$(D^3 - 9D^2 + 27D - 27I)y = 27 \sin(3x)$$

Solution: The homogeneous problem has a characteristic equation

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0 = (\lambda - 3)^3 = 0 \Rightarrow \lambda = 3(3 \text{ times})$$

So the homogeneous solution is given by

$$y_h = (c_1 + c_2 x + c_3 x^2) e^{3x}$$

To find a particular solution for non-homogeneous problem we try a function of the form

$$\begin{aligned} y_p &= K\cos(3x) + M\sin(3x) \\ y_p' &= -3K\sin(3x) + 3M\cos(3x) \\ y_p'' &= -9K\cos(3x) - 9M\sin(3x) \\ y_p''' &= 27K\sin(3x) - 27M\cos(3x) \end{aligned}$$

Substituting in the ODE

$$(27K\sin(3x) - 27M\cos(3x)) - 9(-9K\cos(3x) - 9M\sin(3x)) + 27(-3K\sin(3x) + 3M\cos(3x)) - 27(K\cos(3x) + M\sin(3x)) = 27\sin(3x)$$

$$27K\sin(3x) - 27M\cos(3x) + 81K\cos(3x) + 81M\sin(3x) - 81K\sin(3x) + 81M\cos(3x) - 27K\cos(3x) - 27M\sin(3x) = 27\sin(3x)$$

Dividing the equation by 27

$$K\sin(3x) - M\cos(3x) + 3K\cos(3x) + 3M\sin(3x) - 3K\sin(3x) + 3M\cos(3x) - K\cos(3x) - M\sin(3x) = \sin(3x)$$

$$(-2K + 4M)\sin(3x) + (2M + 2K)\cos(3x) = \sin(3x)$$

$$-2K + 4M = 1$$

$$2M + 2K = 0$$

$$M = \frac{1}{4}, K = -\frac{1}{4}$$

Finally, the general solution is

$$\boxed{y = (c_1 + c_2x + c_3x^2)e^{3x} - \frac{1}{4}(\cos(3x) - \sin(3x))}$$

Kreyszig, 3.3.8

Carlos Oscar Sorzano, June 15th 2015

Solve the IVP

$$y^{iv} - 5y'' + 4y = 10e^{-3x}$$

with $y(0) = 1, y'(0) = y''(0) = y'''(0) = 0$.

Solution: The solution of the homogeneous equation comes from the solution of the characteristic equation

$$\lambda^4 - 5\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = 4, 1$$

$$(\lambda^2 - 4)(\lambda^2 - 1) = 0 \Rightarrow \lambda = \pm 2, \pm 1$$

So the homogeneous solution is of the form

$$y_h = K_1e^{-2x} + K_2e^{-x} + K_3e^{x} + K_4e^{2x}$$

For the particular solution we look for a function of the form

$$\begin{aligned} y_p &= Ke^{-3x} \\ y_p' &= -3Ke^{-3x} \\ y_p'' &= 3^2Ke^{-3x} \\ y_p''' &= -3^3Ke^{-3x} \\ y_p^{iv} &= 3^4Ke^{-3x} \end{aligned}$$

Substituting in the differential equation

$$3^4Ke^{-3x} - 5(3^2Ke^{-3x}) + 4Ke^{-3x} = 10e^{-3x}$$

$$(3^4 - 5 \cdot 3^2 + 4)K = 10 \Rightarrow K = \frac{1}{4}$$

The general solution of the non-homogeneous equation is

$$y = y_h + y_p = K_1 e^{-2x} + K_2 e^{-x} + K_3 e^{x} + K_4 e^{2x} + \frac{1}{4} e^{-3x}$$

To solve the IVP we calculate the derivatives of the general solution

$$\begin{array}{rcl} y' & = & -2K_1 e^{-2x} - K_2 e^{-x} + K_3 e^{x} + 2K_4 e^{2x} - \frac{3}{4} e^{-3x} \\ y'' & = & 2^2 K_1 e^{-2x} + K_2 e^{-x} + K_3 e^{x} + 2^2 K_4 e^{2x} + \frac{3^2}{4} e^{-3x} \\ y''' & = & -2^3 K_1 e^{-2x} - K_2 e^{-x} + K_3 e^{x} + 2^3 K_4 e^{2x} - \frac{3^3}{4} e^{-3x} \end{array}$$

and impose the initial value conditions

$$\begin{array}{rcl} y(0) = 1 & = & K_1 + K_2 + K_3 + K_4 + \frac{1}{4} \\ y'(0) = 0 & = & -2K_1 - K_2 + K_3 + 2K_4 - \frac{3}{4} \\ y''(0) = 0 & = & 2^2 K_1 + K_2 + K_3 + 2^2 K_4 + \frac{3^2}{4} \\ y'''(0) = 0 & = & -2^3 K_1 - K_2 + K_3 + 2^3 K_4 - \frac{3^3}{4} \end{array}$$

The solution of this equation system is

$$K_1 = -1, K_2 = \frac{3}{2}, K_3 = \frac{1}{4}, K_4 = 0$$

Finally, the solution of the IVP is

$$\boxed{y = -e^{-2x} + \frac{3}{2} e^{-x} + \frac{1}{4} e^{x} + \frac{1}{4} e^{-3x}}$$

Kreyszig, 3.3.10

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve

$$x^3 y''' + xy' - y = x^2 \quad y(1) = 1, y'(1) = 3, y''(1) = 14$$

Solution: The homogeneous ODE is an Euler-Cauchy equation, so we try with a solution of the form

$$\begin{array}{rcl} y & = & x^m \\ y' & = & mx^{m-1} \\ y'' & = & m(m-1)x^{m-2} \\ y''' & = & m(m-1)(m-2)x^{m-3} \end{array}$$

Substituting in the equation

$$x^3(m(m-1)(m-2)x^{m-3}) + x(mx^{m-1}) - x^m = 0$$

$$m(m-1)(m-2) + m - 1 = 0$$

$$(m-1)(m(m-2) + 1) = 0$$

$$(m-1)(m-1)^2 = 0$$

So $m = 1$ is a triple root of the characteristic polynomial. The general solution of the homogeneous problem is

$$y_h = c_1 x + c_2 x \log(x) + c_3 x \log^2(x)$$

For finding a particular solution of the non-homogeneous problem we use the method of variation of parameters whose solution is

$$y_p = \sum_{k=1}^{3} y_k \int \frac{W_k}{W} r dx$$

where y_k are the 3 homogeneous solutions and W and W_k are the following matrices

$$\begin{aligned}
W &= \begin{vmatrix} x & x\log(x) & x\log^2(x) \\ 1 & 1+\log(x) & 2\log(x)+\log^2(x) \\ 0 & \frac{1}{x} & \frac{2}{x}+\frac{2}{x}\log(x) \end{vmatrix} = 2 \\
W_1 &= \begin{vmatrix} 0 & x\log(x) & x\log^2(x) \\ 0 & 1+\log(x) & 2\log(x)+\log^2(x) \\ 1 & \frac{1}{x} & \frac{2}{x}+\frac{2}{x}\log(x) \end{vmatrix} = x\log^2(x) \\
W_2 &= \begin{vmatrix} x & 0 & x\log^2(x) \\ 1 & 0 & 2\log(x)+\log^2(x) \\ 0 & 1 & \frac{2}{x}+\frac{2}{x}\log(x) \end{vmatrix} = -2x\log(x) \\
W_3 &= \begin{vmatrix} x & x\log(x) & 0 \\ 1 & 1+\log(x) & 0 \\ 0 & \frac{1}{x} & 1 \end{vmatrix} = x
\end{aligned}$$

We write the ODE in the standard form

$$x^3 y''' + xy' - y = x^2 \Rightarrow y''' + \frac{1}{x^2}y' - \frac{1}{x^3}y = \frac{1}{x}$$

We now calculate the integrals (with $r = \frac{1}{x}$)

$$\begin{aligned}
\int \frac{W_1}{W} r dx &= \int \frac{x\log^2(x)}{2}\frac{1}{x}dx = \frac{x(\log^2(x) - 2\log(x) + 2)}{2} \\
\int \frac{W_2}{W} r dx &= \int \frac{-2x\log(x)}{2}\frac{1}{x}dx = -x(\log(x) - 1) \\
\int \frac{W_3}{W} r dx &= \int \frac{x}{2}\frac{1}{x}dx = \frac{x}{2}
\end{aligned}$$

Finally, the particular solution is

$$\begin{aligned}
y_p &= y_1 \int \frac{W_1}{W} r dx + y_2 \int \frac{W_2}{W} r dx + y_3 \int \frac{W_3}{W} r dx \\
&= x\frac{x(\log^2(x) - 2\log(x) + 2)}{2} + x\log(x)\left(-x(\log(x) - 1)\right) + x\log^2(x)\frac{x}{2} \\
&= x^2
\end{aligned}$$

The general solution is of the form

$$y = c_1 x + c_2 x \log(x) + c_3 x \log^2(x) + x^2$$

Imposing the initial conditions

$$\begin{aligned}
y(x) &= c_1 x + c_2 x \log(x) + c_3 x \log^2(x) + x^2 \Rightarrow y(1) = 1 = c_1 + 1 \Rightarrow c_1 = 0 \\
y'(x) &= c_1 + c_2(1 + \log(x)) + c_3(2\log(x) + \log^2(x)) + 2x \Rightarrow y'(1) = 3 = c_2 \\
y''(x) &= c_2 \frac{1}{x} + c_3\left(\frac{2}{x} + \frac{2}{x}\log(x)\right) + 2 \Rightarrow y'(1) = 14 = c_2 + 2c_3 \Rightarrow c_3 = \frac{11}{2}
\end{aligned}$$

So the particular solution sought is

$$\boxed{y = 3x\log(x) + \frac{11}{2}x\log^2(x) + x^2}$$

Problema

Carlos Oscar Sorzano, June 15th, 2015

Let S be the fraction of a population susceptible of getting a diphtheria and I the fraction of that population infected by diphtheria. An ill person can disseminate the disease while he is not recovered. Assume that the number of contacts between susceptible and infected people occurs at a rate α. The daily fraction of susceptible population that is vaccinated is β. Assume that diphtheria is a disease that can be passed only once. Assume that diphtheria has a daily death rate δ_1 and a daily recovery rate δ_2. Assume also that the population size is stable with daily birth and death rates δ. Propose a disease dissemination model.

Solution: Let us call $NS(t)$ the instant proportion of non-susceptible people. The sum of proportions must be 1

$$S(t) + I(t) + NS(t) = 1$$

Additionally the proportions follow the equation system

$$\begin{array}{rcl} S'(t) & = & -\alpha S(t)I(t) + \delta - \delta S(t) - \beta S(t) \\ I'(t) & = & \alpha S(t)I(t) - \delta_1 I(t) - \delta_2 I(t) - \delta I(t) \\ NS'(t) & = & \beta S(t) + \delta_2 I(t) - \delta NS(t) \end{array}$$

The term $\alpha S(t)I(t)$ accounts for the proportion of susceptible people that gets infected every day. The term δ in $S'(t)$ accounts for the daily birth rate. The terms $\delta S(t)$, $\delta I(t)$, $\delta NS(t)$ account for the daily rate of deaths non-related to diphteria. The term $\beta S(t)$ is the proportion of people that becomes non-susceptible by vaccination. The term $\delta_1 I(t)$ accounts for the daily rate of deaths caused by diphtheria, and finally $\delta_2 I(t)$ accounts for the daily rate of people that recovers from diphtheria and becomes non-susceptible.

We may eliminate one of the variables. For instance, we solve for S in the first equation

$$S(t) = 1 - I(t) - NS(t)$$

and substitute in the equation system

$$\begin{array}{rcl} I'(t) & = & \alpha(1 - I(t) - NS(t))I(t) - \delta_1 I(t) - \delta_2 I(t) - \delta I(t) \\ NS'(t) & = & \beta(1 - I(t) - NS(t)) + \delta_2 I(t) - \delta NS(t) \end{array}$$

Grouping terms

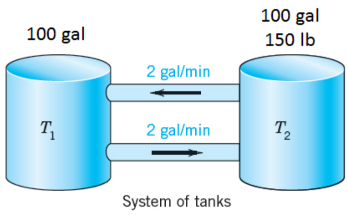
$$\boxed{\begin{array}{rcl} I'(t) & = & (\alpha(1 - I(t) - NS(t)) - \delta_1 - \delta_2 - \delta)I(t) \\ NS'(t) & = & \beta - (\delta + \beta)NS(t) + (\delta_2 - \beta)I(t) \end{array}}$$

4 Chapter 4

Kreyszig, 4.1.1

Carlos Oscar Sorzano, Aug. 31st, 2014

Find out, without calculation, whether doubling the flow rate in the following example has the same effect as halfing the tank sizes.



Solution: Original case:

$$y_1' = \text{inflow-outflow} = \frac{y_2}{100}\left[\frac{lb}{gal}\right]2\left[\frac{gal}{min}\right] - \frac{y_1}{100}\left[\frac{lb}{gal}\right]2\left[\frac{gal}{min}\right]$$

$$y_2' = \text{inflow-outflow} = \frac{y_1}{100}\left[\frac{lb}{gal}\right]2\left[\frac{gal}{min}\right] - \frac{y_2}{100}\left[\frac{lb}{gal}\right]2\left[\frac{gal}{min}\right]$$

$$\left.\begin{array}{r} y_1' = -0.02y_1 + 0.02y_2 \\ y_2' = 0.02y_1 - 0.02y_2 \end{array}\right\} \Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \mathbf{y}' = A\mathbf{y}$$

Doubling the flow rate:

$$y_1' = \text{inflow-outflow} = \frac{y_2}{100}\left[\frac{lb}{gal}\right]4\left[\frac{gal}{min}\right] - \frac{y_1}{100}\left[\frac{lb}{gal}\right]4\left[\frac{gal}{min}\right]$$

$$y_2' = \text{inflow-outflow} = \frac{y_1}{100}\left[\frac{lb}{gal}\right]4\left[\frac{gal}{min}\right] - \frac{y_2}{100}\left[\frac{lb}{gal}\right]4\left[\frac{gal}{min}\right]$$

$$\left.\begin{array}{r} y_1' = -0.04y_1 + 0.04y_2 \\ y_2' = 0.04y_1 - 0.04y_2 \end{array}\right\} \Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -0.04 & 0.04 \\ 0.04 & -0.04 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \mathbf{y}' = A_1\mathbf{y}$$

Halfing the tank sizes:

$$y_1' = \text{inflow-outflow} = \frac{y_2}{50}\left[\frac{lb}{gal}\right]2\left[\frac{gal}{min}\right] - \frac{y_1}{50}\left[\frac{lb}{gal}\right]2\left[\frac{gal}{min}\right]$$

$$y_2' = \text{inflow-outflow} = \frac{y_1}{50}\left[\frac{lb}{gal}\right]2\left[\frac{gal}{min}\right] - \frac{y_2}{50}\left[\frac{lb}{gal}\right]2\left[\frac{gal}{min}\right]$$

$$\left.\begin{array}{r} y_1' = -0.04y_1 + 0.04y_2 \\ y_2' = 0.04y_1 - 0.04y_2 \end{array}\right\} \Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -0.04 & 0.04 \\ 0.04 & -0.04 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \mathbf{y}' = A_2\mathbf{y}$$

Since $A_1 = A_2$, the effect on the amount of salt in both tanks is the same if we double the flow rate or halve the tank size.

Kreyszig, 4.1.11

Álvaro Martín Ramos, Jan. 4th, 2015

Solve
$$4y'' - 15y' - 4y = 0$$
Solution: To convert the ODE into an ODE system we do the following changes of variables
$$y = y_1$$
$$y_1' = y_2$$
So that the original ODE can be written as
$$4y_2' - 15y_2 - 4y_1 = 0 \Rightarrow y_2' = \frac{15}{4}y_2 + y_1$$
Together the system ODE is
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \frac{15}{4} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
That is of the form
$$\mathbf{y}' = A\mathbf{y}$$
The eigenvalues and eigenvectors of A are:
$$\lambda_1 = 4, \mathbf{v}_1 = (1, 4)^T$$
$$\lambda_2 = -\tfrac{1}{4}, \mathbf{v}_2 = \left(1, -\tfrac{1}{4}\right)^T$$
The general solution of the ODe system is
$$\mathbf{y} = c_1 \mathbf{v_1} e^{\lambda_1 x} + c_2 \mathbf{v_2} e^{\lambda_2 x} = \boxed{c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4x} + c_2 \begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix} e^{-\frac{x}{4}}}$$

Kreyszig, 4.1.12

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve
$$y''' + 2y'' - y' - 2y = 0$$
by solving it directly and by reducing it to an ODE system.
Solution: The characteristic equation of the ODE is
$$\lambda^3 + 2\lambda^2 - \lambda + 2 = 0 \Rightarrow \lambda = -2, -1, 1$$
So that the general solution is
$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x}$$
To convert the ODE into an ODE system we do the following changes of variables
$$\begin{array}{rcl} y_1 & = & y \\ y_2 & = & y_1' = y' \\ y_3 & = & y_2' = y'' \end{array}$$
So that the original ODE can be written as
$$y_3' + 2y_3 - y_2 - 2y_1 = 0$$

$$y_3' = -2y_3 + y_2 + 2y_1$$

Altogether the ODE system is

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\mathbf{y}' = A\mathbf{y}$$

The eigenvalues and eigenvectors of A are

$$\begin{array}{ll} \lambda_1 = -2 & \mathbf{v}_1 = (1, -2, 4)^T \\ \lambda_2 = -1 & \mathbf{v}_2 = (1, -1, 1)^T \\ \lambda_3 = 1 & \mathbf{v}_3 = (1, 1, 1)^T \end{array}$$

The general solution of the ODE system is

$$\begin{aligned} \mathbf{y} &= c_1\mathbf{v_1}e^{\lambda_1 x} + c_2\mathbf{v_2}e^{\lambda_2 x} + c_3\mathbf{v_3}e^{\lambda_3 x} \\ &= c_1 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2x} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-x} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{x} \\ &= \begin{pmatrix} c_1e^{-2x} + c_2e^{-x} + c_3e^{x} \\ -2c_1e^{-2x} - c_2e^{-x} + c_3e^{x} \\ 4c_1e^{-2x} + c_2e^{-x} + c_3e^{x} \end{pmatrix} \end{aligned}$$

Finally, remind that $y = y_1$, so we are mostly interested in its first component that is

$$y = c_1e^{-2x} + c_2e^{-x} + c_3e^{x}$$

That is, the same result as we obtained by the direct method.

Kreyszig, 4.3.1

Carlos Oscar Sorzano, Nov. 14th, 2014

Give the general solution of the equation system

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= 3y_1 - y_2 \end{aligned}$$

Solution: Let us write the equation system as

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The characteristic polynomial of the system matrix is

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 3 = (\lambda-2)(\lambda+2) = 0$$

The eigenvector of $\lambda_1 = 2$ comes from the equation system

$$(A - 2I)\mathbf{x} = \mathbf{0}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

whose eigenvector is $\mathbf{x}_1 = (1, 1)$.

The eigenvector of $\lambda_2 = -2$ comes from

$$(A + 2I)\mathbf{x} = \mathbf{0}$$

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

whose eigenvector is $\mathbf{x}_2 = (-1, 3)$.

Finally, the solution of the differential equation system is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-2t} = \boxed{\begin{pmatrix} c_1 e^{2t} - c_2 e^{-2t} \\ c_1 e^{2t} + 3c_2 e^{-2t} \end{pmatrix}}$$

Kreyszig, 4.3.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Find a general solution of the ODE system

$$\begin{aligned} y_1' &= 2y_1 - 2y_2 \\ y_2' &= 2y_1 + 2y_2 \end{aligned}$$

Solution: We can write the ODE system as

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\mathbf{y}' = A\mathbf{y}$$

The eigenvalues and eigenvectors of A are

$$\begin{aligned} \lambda_1 = 2 + 2i \quad & \mathbf{v}_1 = (i, 1)^T \\ \lambda_2 = 2 - 2i \quad & \mathbf{v}_2 = (-i, 1)^T \end{aligned}$$

The general solution of the ODE system is

$$\begin{aligned} \mathbf{y} &= c_1\mathbf{v_1}e^{\lambda_1 x} + c_2\mathbf{v_2}e^{\lambda_2 x} \\ &= c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+2i)x} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(2-2i)x} \end{aligned}$$

If we want the solution to be real, we must perform a change of basis. Instead of the basis functions

$$\begin{aligned} \mathbf{y}_1 &= \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(2+2i)x} \\ \mathbf{y}_2 &= \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(2-2i)x} = y_1^* \end{aligned}$$

we define the functions

$$\begin{aligned} \tilde{\mathbf{y}}_1 &= \tfrac{\mathbf{y}_1 + \mathbf{y}_2}{2} = \mathrm{Re}\{\mathbf{y}_1\} = \mathrm{Re}\left\{\begin{pmatrix} ie^{(2+2i)x} \\ e^{(2+2i)x} \end{pmatrix}\right\} = \begin{pmatrix} -e^{2x}\sin(2x) \\ e^{2x}\cos(2x) \end{pmatrix} \\ \tilde{\mathbf{y}}_2 &= \tfrac{\mathbf{y}_1 - \mathbf{y}_2}{2i} = \mathrm{Im}\{\mathbf{y}_1\} = \begin{pmatrix} e^{2x}\cos(2x) \\ e^{2x}\sin(2x) \end{pmatrix} \end{aligned}$$

The general solution can be written as

$$y = c_1\tilde{\mathbf{y}}_1 + c_2\tilde{\mathbf{y}}_2 = c_1 \begin{pmatrix} -e^{2x}\sin(2x) \\ e^{2x}\cos(2x) \end{pmatrix} + c_2 \begin{pmatrix} e^{2x}\cos(2x) \\ e^{2x}\sin(2x) \end{pmatrix} = \boxed{e^{2x}\begin{pmatrix} -c_1\sin(2x) + c_2\cos(2x) \\ c_1\cos(2x) + c_2\sin(2x) \end{pmatrix}}$$

Kreyszig, 4.3.7

Carlos Oscar Sorzano, Aug. 31st, 2014

Find a general solution of the ODE system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 + y_3 \\ y_3' &= -y_2 \end{aligned}$$

Solution: We can write the ODE system as

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\mathbf{y}' = A\mathbf{y}$$

The eigenvalues and eigenvectors of A are

$$\begin{array}{ll} \lambda_1 = 0 & \mathbf{v}_1 = (1,0,1)^T \\ \lambda_2 = \sqrt{2}i & \mathbf{v}_2 = (-i,\sqrt{2},i)^T \\ \lambda_3 = -\sqrt{2}i & \mathbf{v}_2 = (i,\sqrt{2},-i)^T \end{array}$$

The general solution of the ODE system is

$$\begin{aligned} \mathbf{y} &= c_1\mathbf{v_1}e^{\lambda_1 x} + c_2\mathbf{v_2}e^{\lambda_2 x} + c_3\mathbf{v_3}e^{\lambda_3 x} \\ &= c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix} e^{i\sqrt{2}x} + c_3\begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix} e^{-i\sqrt{2}x} \end{aligned}$$

If we want the solution to be real, we must perform a change of basis. Instead of the basis functions

$$\begin{aligned} \mathbf{y}_2 &= \begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix} e^{i\sqrt{2}x} \\ \mathbf{y}_3 &= \begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix} e^{-i\sqrt{2}x} = y_2^* \end{aligned}$$

we define the functions

$$\begin{aligned} \tilde{\mathbf{y}}_2 &= \tfrac{\mathbf{y}_2+\mathbf{y}_3}{2} = \mathrm{Re}\{\mathbf{y}_2\} = \mathrm{Re}\left\{\begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix} e^{i\sqrt{2}x}\right\} = \begin{pmatrix} \sin(\sqrt{2}x) \\ \sqrt{2}\cos(\sqrt{2}x) \\ -\sin(\sqrt{2}x) \end{pmatrix} \\ \tilde{\mathbf{y}}_3 &= \tfrac{\mathbf{y}_2-\mathbf{y}_3}{2i} = \mathrm{Im}\{\mathbf{y}_2\} = \begin{pmatrix} -\cos(\sqrt{2}x) \\ \sqrt{2}\sin(\sqrt{2}x) \\ \cos(\sqrt{2}x) \end{pmatrix} \end{aligned}$$

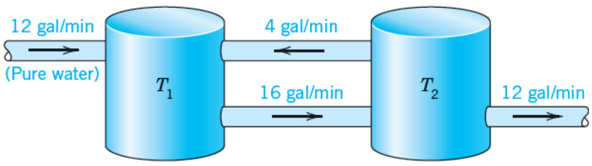
The general solution can be written as

$$\begin{aligned} y &= c_1\mathbf{y_1} + c_2\tilde{\mathbf{y}}_2 + c_3\tilde{\mathbf{y}}_3 \\ &= c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}\sin(\sqrt{2}x)\\ \sqrt{2}\cos(\sqrt{2}x)\\ -\sin(\sqrt{2}x)\end{pmatrix} + c_3\begin{pmatrix}-\cos(\sqrt{2}x)\\ \sqrt{2}\sin(\sqrt{2}x)\\ \cos(\sqrt{2}x)\end{pmatrix} \\ &= \boxed{\begin{pmatrix}c_1 + c_2\sin(\sqrt{2}x) - c_3\cos(\sqrt{2}x)\\ c_2\sqrt{2}\cos(\sqrt{2}x) + c_3\sqrt{2}\sin(\sqrt{2}x)\\ c_1 - c_2\sin(\sqrt{2}x) + c_3\cos(\sqrt{2}x)\end{pmatrix}} \end{aligned}$$

Kreyszig, 4.3.18

Carlos Oscar Sorzano, Aug. 31st, 2014

Each of the two tanks contains 200 gal of water, in which initially 100 lb (Tank T_1) and 200 lb (Tank T_2) of fertilizer are dissolved. The inflow, circulation, and outflow are shown in the figure below. The mixture is kept uniform by stirring. Find the fertilizer contents $y_1(t)$ in T_1 and $y_2(t)$ in T_2.



Solution: We can model the system with the following differential equations:

$$\begin{aligned} y_1' &= -\frac{y_1}{200}16 + \frac{y_2}{200}4 + 0\cdot 12 \\ y_2' &= \frac{y_1}{200}16 - \frac{y_2}{200}(4+12) \end{aligned}$$

Equivalently

$$\begin{pmatrix}y_1'\\y_2'\end{pmatrix} = \begin{pmatrix}-\frac{16}{200} & \frac{4}{200}\\ \frac{16}{200} & -\frac{16}{200}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

The eigenvalues and eigenvectors of this matrix are

$$\begin{aligned} \lambda_1 = -\tfrac{3}{25}, &\quad \mathbf{v}_1 = (-\tfrac{1}{2}, 1)^T \\ \lambda_2 = -\tfrac{1}{25}, &\quad \mathbf{v}_2 = (\tfrac{1}{2}, 1)^T \end{aligned}$$

The general solution of the ODE system is

$$\mathbf{y} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1\begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix}e^{-\frac{3}{25}t} + c_2\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}e^{-\frac{1}{25}t}$$

As stated in the problem at $t=0$ we have

$$\mathbf{y}(0) = \begin{pmatrix}100\\200\end{pmatrix} = c_1\begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix} + c_2\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix} \Rightarrow c_1 = 0, c_2 = 200$$

So the solution sought is

$$\boxed{\mathbf{y} = 200\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}e^{-\frac{t}{25}}}$$

Kreyszig, 4.4.1

Álvaro Martín Ramos, Jan. 4th, 2015

Determine the type and stability of the critical point of

$$y_1' = y_1$$

$$y_2' = 2y_2$$

Then find a real general solution.
Solution: The proposed ODE system is equivalent to

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The eigenvalues of the matrix are given by

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2 = 0$$

$$p = 3, (> 0)$$

$$q = 2 (> 0)$$

$$\Delta = p^2 - 4q = 1 (> 0)$$

So it is an unstable improper node. The general solution is

$$\boxed{y_1 = c_1 e^t, y_2 = c_2 e^{2t}}$$

Kreyszig, 4.4.3

Carlos Oscar Sorzano, Aug. 31st, 2014

Determine the type and stability of the critical point of

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -9y_1 \end{aligned}$$

Then find a real general solution and sketch or graph some of the trajectories in the phase plane.
Solution: The proposed ODE system is equivalent to

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -9 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

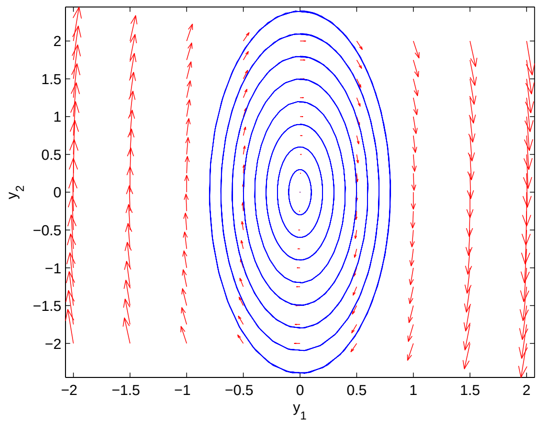
Critical points are points at which $\mathbf{y}' = \mathbf{0}$, in this case the only critical point is

$$\mathbf{y} = \mathbf{0}$$

whose eigenvalues and eigenvectors are

$$\begin{aligned} \lambda_1 &= 3i, \quad \mathbf{v}_1 = (-\tfrac{1}{3}i, 1) \\ \lambda_2 &= -3i, \quad \mathbf{v}_2 = (\tfrac{1}{3}i, 1) \end{aligned}$$

This corresponds to a center as can be clearly seen in the figure below



To find a real solution we construct the functions

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{v}_1 e^{\lambda_1 x} = \begin{pmatrix} -\frac{1}{3}i \\ 1 \end{pmatrix} e^{i3x} \\
\mathbf{y}_2 &= \mathbf{v}_2 e^{\lambda_2 x} = \begin{pmatrix} \frac{1}{3}i \\ 1 \end{pmatrix} e^{-i3x} \\
\tilde{\mathbf{y}}_1 &= \frac{\mathbf{y}_1+\mathbf{y}_2}{2} = \mathrm{Re}\{\mathbf{y}_1\} = \begin{pmatrix} \frac{1}{3}\sin(3x) \\ \cos(3x) \end{pmatrix} \\
\tilde{\mathbf{y}}_2 &= \frac{\mathbf{y}_1-\mathbf{y}_2}{2i} = \mathrm{Im}\{\mathbf{y}_1\} = \begin{pmatrix} -\frac{1}{3}\cos(3x) \\ \sin(3x) \end{pmatrix}
\end{aligned}$$

The general real solution is given by

$$\boxed{\mathbf{y} = c_1\tilde{\mathbf{y}}_1 + c_2\tilde{\mathbf{y}}_2 = \begin{pmatrix} c_1\frac{1}{3}\sin(3x) - c_2\frac{1}{3}\cos(3x) \\ c_1\cos(3x) + c_2\sin(3x) \end{pmatrix}}$$

Kreyszig, 4.4.7

Álvaro Martín Ramos, Jan. 4th, 2015

Determine the type and stability of the critical point of

$$y_1' = y_1 + 2y_2$$

$$y_2' = 2y_1 + y_2$$

Then find a real general solution.
Solution: The proposed ODE system is equivalent to

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The characteristic equation of the matrix is

$$\lambda^2 - 2\lambda - 3 = 0$$

$$p = 2, (> 0)$$

$$q = -3 (< 0)$$

So it is a saddle point, always unstable. The eigenvalues and eigenvectors of the system matrix are

$$\lambda_1 = 3, \mathbf{v}_1 = (1,1)^T$$
$$\lambda_2 = -1, \mathbf{v}_2 = (1,-1)^T$$

The general solution is given by

$$\mathbf{y} = c_1 \mathbf{v}_1 e^{\lambda_1 x} + c_2 \mathbf{v}_2 e^{\lambda_2 x} = \boxed{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3x} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x}}$$

Kreyszig, 4.4.14

Carlos Oscar Sorzano, Aug. 31st, 2014

Transformation of parameters. What happens to the critical point of

$$\begin{aligned} y_1' &= y_1 \\ y_2' &= 2y_2 \end{aligned}$$

if you introduce $\tau = -t$ as the new independent variable? trajectories in the phase plane.

Solution:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Its critical point is $\mathbf{y} = \mathbf{0}$ and the eigenvalues of the matrix used to calculate the derivative are 1 and 2, that is, it is an unstable node (because $p = \lambda_1 + \lambda_2 > 0$.

If we do the change of variable $\tau = -t$, then

$$\frac{dy_i}{d\tau} = \frac{dy_i}{dt}\frac{dt}{d\tau} = -\frac{dy_i}{dt}$$

So the equation system becomes

$$\begin{pmatrix} \frac{dy_1}{d\tau} \\ \frac{dy_2}{d\tau} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

That is the direction of motion changes, and the two eigenvalues become negative. Then we have $p = \lambda_1 + \lambda_2 < 0$ and $q = \lambda_1 \lambda_2 > 0$, consequently a stable node.

Kreyszig, 4.4.17

Carlos Oscar Sorzano, Aug. 31st, 2014

Perturbation. The system

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{y}$$

has a center as its critical point. Replace each a_{ij} by $a_{ij} + b$. Find values of b such that you get (a) a saddle point, (b) a stable and attractive node, (c) a stable and attractive spiral, (d) an unstable spiral, (e) an unstable node.

Solution: The perturbed system is

$$\mathbf{y}' = \begin{pmatrix} b & 1+b \\ -4+b & b \end{pmatrix} \mathbf{y}$$

The characteristic polynomial is

$$|A - \lambda I| = (b-\lambda)^2 - (1+b)(-4+b) = \lambda^2 - 2b\lambda + 4 + 3b$$

This polynomial is of the form

$$\lambda^2 - p\lambda + q$$

so

$$p = 2b$$

$$q = 4 + 3b$$

Saddle point: to get a saddle point we need

$$q < 0 \Rightarrow 4 + 3b < 0 \Rightarrow b < -\frac{4}{3}$$

Stable and attractive node: to get a stable and attractive node we need

$$p < 0, q = 0 \text{ (stable and attractive) and } q > 0, \Delta = p^2 - 4q \geq 0 \text{ (node)}$$

$$2b < 0, 4 + 3b = 0 \Rightarrow -\frac{4}{3} = b$$

$$(2b)^2 - 4(4+3b) \geq 0 \Rightarrow b^2 - 3b - 4 \geq 0 \Rightarrow b \in (-\infty, -1] \cap [4, \infty)$$

The intersection of both sets gives $b = -\frac{4}{3}$.

Stable and attractive spiral: to get a stable and attractive spiral we need

$$p < 0, q = 0 \text{ (stable and attractive) and } \Delta = p^2 - 4q < 0, p \neq 0 \text{ (spiral)}$$

$$2b < 0, 4 + 3b = 0, 2b \neq 0 \Rightarrow -\frac{4}{3} = b$$

$$(2b)^2 - 4(4+3b) < 0 \Rightarrow b^2 - 3b - 4 < 0 \Rightarrow -1 < b < 4$$

Since $\{-\frac{4}{3}\} \cap (-1, 4) = \emptyset$, there is no b satisfying all conditions.

Unstable spiral: to get an unstable spiral we need

$$p > 0 \text{ or } q < 0 \text{ (unstable) and } \Delta = p^2 - 4q < 0, p \neq 0 \text{ (spiral)}$$

$$2b > 0 \text{ or } 4 + 3b < 0 \Rightarrow b \in (0, \infty) \cup (-\infty, -\tfrac{4}{3}) = (-\infty, -\tfrac{4}{3}) \cup (0, \infty)$$

$$(2b)^2 - 4(4+3b) < 0 \Rightarrow b^2 - 3b - 4 < 0 \Rightarrow -1 < b < 4$$

Finally,

$$\big((-\infty, -\tfrac{4}{3}) \cup (0, \infty)\big) \cap (-1, 4) = (0, 4)$$

Unstable node: to get an unstable node we need

$$p > 0 \text{ or } q < 0 \text{ (unstable) and } q > 0, \Delta = p^2 - 4q \geq 0 \text{ (node)}$$

$$2b > 0 \text{ or } 4 + 3b < 0 \Rightarrow b \in (0, \infty) \cup (-\infty, -\tfrac{4}{3}) = (-\infty, -\tfrac{4}{3}) \cup (0, \infty)$$

$$4 + 3b > 0, p^2 - 4q \geq 0 \Rightarrow b \in (-\tfrac{4}{3}, \infty) \cap ((-\infty, -1] \cup [4, \infty)) = (-\tfrac{4}{3}, -1] \cup [4, \infty)$$

Finally

$$\big((-\infty, -\tfrac{4}{3}) \cup (0, \infty)\big) \cap \big((-\tfrac{4}{3}, -1] \cup [4, \infty)\big) = [4, \infty)$$

Kreyszig, 4.5.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the location and all critical points by linearization of the ODE

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 + \tfrac{1}{2}y_1^2 \end{aligned}$$

Solution: The ODE system can be rewritten as

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -y_1 + \frac{1}{2}y_1^2 \end{pmatrix}$$

Critical points are solutions of the equation system

$$\begin{pmatrix} y_2 \\ -y_1 + \frac{1}{2}y_1^2 \end{pmatrix} = \mathbf{0} \Rightarrow y_1 = 0, 2; y_2 = 0$$

Case $(y_1, y_2) = (0, 0)$:
If we linearize around the point $(0, 0)$ we get

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}\right|_{\mathbf{y}=\mathbf{0}} = \left.\begin{pmatrix} 0 & 1 \\ -1 + y_1 & 0 \end{pmatrix}\right|_{\tilde{\mathbf{y}}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

And the equation system behaves in the vicinity of $(0, 0)$ as

$$\mathbf{y}' = A\mathbf{y}$$

The charactertistic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

So, $p = 0$, $q = 1$ and $\Delta = p^2 - 4q = -4$. Consequently, $(0, 0)$ is a stable $(p \leq 0, q > 0)$ center $(p = 0, \Delta < 0)$.

Case $(y_1, y_2) = (2, 0)$:
Let us make the change of variables

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} y_1 - 2 \\ y_2 \end{pmatrix}$$

Then

$$\tilde{\mathbf{y}}' = \begin{pmatrix} \tilde{y}_2 \\ -(\tilde{y}_1 + 2) + \frac{1}{2}(\tilde{y}_1 + 2)^2 \end{pmatrix} = \begin{pmatrix} \tilde{y}_2 \\ \tilde{y}_1 + \frac{1}{2}\tilde{y}_1^2 \end{pmatrix}$$

We now linearize around the point $(\tilde{y}_1, \tilde{y}_2) = (0, 0)$

$$\tilde{A} = \left.\begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} \\ \frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2} \end{pmatrix}\right|_{\tilde{\mathbf{y}}=\mathbf{0}} = \left.\begin{pmatrix} 0 & 1 \\ 1 + \tilde{y}_1 & 0 \end{pmatrix}\right|_{\tilde{\mathbf{y}}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now, the characteristic polynomial is

$$|\tilde{A} - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

So, $p = 0$, $q = -1$ and $\Delta = p^2 - 4q = 4$. Consequently, $(2, 0)$ is an unstable ($q < 0$) saddle point ($q < 0$).

Kreyszig, 4.5.9

Carlos Oscar Sorzano, Jan. 15th, 2015

Find the location and type of all critical points by first converting the ODE to a system and then linearizing it.

$$y'' - 9y + y^3 = 0$$

Solution: Let us define

$$\begin{aligned} y_1 &= y \\ y_2 &= y_1' \end{aligned}$$

Then we may rewrite the ODE as

$$y_2' - 9y_1 + y_1^3 = 0$$

or the ODE system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 9y_1 - y_1^3 = y_1(3 - y_1)(3 + y_1) \end{aligned}$$

There are three critical points at $y = y_1 = 0, 3, -3$, $y_2 = 0$. Let us linearize the ODE at the three points. For doing so, let us rewrite the ODE system as a vector differential equation:

$$\mathbf{y}' = \mathbf{F}(\mathbf{y})$$

Case $y = 0$:

$$\mathbf{y}' = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix}\Bigg|_{y_1=0, y_2=0} \mathbf{y} = \begin{pmatrix} 0 & 1 \\ 9 - 3y_1^2 & 0 \end{pmatrix}\Bigg|_{y_1=0, y_2=0} \mathbf{y} = \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix} \mathbf{y}$$

The eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = -3$. Since the two eigenvalues are real and of opposite sign, the critical point is a saddle point.

Case $y = 3$:

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ 9 - 3y_1^2 & 0 \end{pmatrix}\Bigg|_{y_1=3, y_2=0} \mathbf{y} = \begin{pmatrix} 0 & 1 \\ -18 & 0 \end{pmatrix} \mathbf{y}$$

whose eigenvalues are $\lambda_1 = \sqrt{18}i$ and $\lambda_2 = -\sqrt{18}i$. Since the two eigenvalues are pure imaginary, the critical point is a center.

Case $y = -3$:

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ 9 - 3y_1^2 & 0 \end{pmatrix}\Bigg|_{y_1=-3, y_2=0} \mathbf{y} = \begin{pmatrix} 0 & 1 \\ -18 & 0 \end{pmatrix} \mathbf{y}$$

Again, the critical point is a center.

Kreyszig, 4.5.11

Carlos Oscar Sorzano, June 15th, 2015

Find the location and type of all critical points by first converting the ODE to a system and then linearizing it.

$$y'' + \cos(y) = 0$$

Solution: Let us define

$$\begin{aligned} y_1 &= y \\ y_2 &= y_1' \end{aligned}$$

Then we may rewrite the ODE as

$$y_2' + \cos(y_1) = 0$$

or the ODE system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\cos(y_1) \end{aligned}$$

There are three critical points at $y = y_1 = \frac{\pi}{2} + n\pi$, $y_2 = 0$. Let us linearize the ODE at the two different kind of points. For doing so, let us rewrite the ODE system as a vector differential equation:

$$\mathbf{y}' = \mathbf{F}(\mathbf{y})$$

Case $y = \frac{\pi}{2}$:

$$\mathbf{y}' = \left.\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix}\right|_{y_1=\frac{\pi}{2}, y_2=0} \mathbf{y} = \left.\begin{pmatrix} 0 & 1 \\ \sin(y_1) & 0 \end{pmatrix}\right|_{y_1=\frac{\pi}{2}, y_2=0} \mathbf{y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{y}$$

The eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\lambda_1 = -1, \lambda_2 = 1$. Since the eigenvalues are real and of opposite sign the critical point is a saddle point.

Case $y = \frac{3\pi}{2}$:

$$\mathbf{y}' = \left.\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix}\right|_{y_1=\frac{3\pi}{2}, y_2=0} \mathbf{y} = \left.\begin{pmatrix} 0 & 1 \\ \sin(y_1) & 0 \end{pmatrix}\right|_{y_1=\frac{3\pi}{2}, y_2=0} \mathbf{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}$$

The eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ are $\lambda_1 = i, \lambda_2 = -i$. The eigenvalues are pure imaginary and, consequently, the critical point is a center.

Kreyszig, 4.6.3

Álvaro Martín Ramos, Jan. 4th, 2015

Find a general solution of

$$y_1' = y_2 + e^{3t}$$

$$y_2' = y_1 - 3e^{3t}$$

Solution: Let us write the ODE system as

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t}$$

The eigenvalues and eigenvectors of the system matrix are

$$\lambda_1 = 1, \mathbf{v}_1 = (1, -1)^T$$

$$\lambda_2 = -1, \mathbf{v}_2 = (1, 1)^T$$

So the general solution of the homogeneous problem is

$$\mathbf{y}_h = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} e^t & e^{-t} \\ -e^t & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Y\mathbf{c}$$

For the particular solution we now that

$$\mathbf{y}_p = Y\mathbf{u}$$

Where

$$\mathbf{u}' = Y^{-1}\mathbf{g}$$

So

$$Y^{-1} = \frac{1}{2} \begin{pmatrix} e^{-t} & -e^{-t} \\ e^t & e^t \end{pmatrix}$$

$$\mathbf{u}' = \frac{1}{2} \begin{pmatrix} e^{-t} & -e^{-t} \\ e^t & e^t \end{pmatrix} \begin{pmatrix} e^{3t} \\ -3e^{3t} \end{pmatrix} = \begin{pmatrix} -e^{2t} \\ 2e^{4t} \end{pmatrix}$$

$$\mathbf{u} = \int \begin{pmatrix} -e^{2t} \\ 2e^{4t} \end{pmatrix} dt = \begin{pmatrix} \frac{-e^{2t}}{2} \\ \frac{e^{4t}}{2} \end{pmatrix}$$

$$\mathbf{y_p} = Y\mathbf{u} = \begin{pmatrix} e^t & e^{-t} \\ -e^t & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{-e^{2t}}{2} \\ \frac{e^{4t}}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix}$$

Finally,

$$\boxed{\mathbf{y} = \mathbf{y_h} + \mathbf{y_p} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix}}$$

Kreyszig, 4.6.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Find a general solution of

$$y_1' = 4y_1 + y_2 + 0.6t$$
$$y_2' = 2y_1 + 3y_2 - 2.5t$$

Solution: Let us write the ODE system as

$$\mathbf{y}' = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0.6t \\ -2.5t \end{pmatrix}$$

The eigenvalues and eigenvectors of the system matrix are

$$\lambda_1 = 5, \mathbf{v}_1 = (1, 1)^T$$
$$\lambda_2 = 2, \mathbf{v}_2 = (-1, 2)^T$$

So the general solution of the homogeneous problem is

$$y_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{2t}$$

For the particular solution, we try a solution of the type

$$\mathbf{y} = \mathbf{k}_0 + \mathbf{k}_1 t$$

Substituting into the ODE we get

$$\mathbf{k}_1 = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} (\mathbf{k}_0 + \mathbf{k}_1 t) + \begin{pmatrix} 0.6 \\ -2.5 \end{pmatrix} t$$

From where

$$\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \mathbf{k}_1 + \begin{pmatrix} 0.6 \\ -2.5 \end{pmatrix} = 0 \Rightarrow \mathbf{k}_1 = \begin{pmatrix} -0.43 \\ 1.12 \end{pmatrix}$$

and

$$\mathbf{k}_1 = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \mathbf{k}_0 \Rightarrow \mathbf{k}_0 = \begin{pmatrix} -0.241 \\ 0.534 \end{pmatrix}$$

So, the general solution is

$$\boxed{y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{2t} + \begin{pmatrix} -0.241 \\ 0.534 \end{pmatrix} + \begin{pmatrix} -0.43 \\ 1.12 \end{pmatrix} t}$$

5 Chapter 5

Kreyszig, 5.1.7

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE

$$y' = -2xy$$

using the power series method.

Solution: Let us expand the solution of the ODE as

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

Then

$$y' = \sum_{m=1}^{\infty} a_m m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + ...$$

Let us write the ODE as

$$y' + 2xy = 0$$

and substitute the two series

$$\left(\sum_{m=1}^{\infty} a_m m x^{m-1} \right) + 2x \left(\sum_{m=0}^{\infty} a_m x^m \right) = 0$$

$$\left(a_1 + 2a_2x + \sum_{m=3}^{\infty} a_m m x^{m-1}\right) + 2a_0x + \left(\sum_{m=1}^{\infty} 2a_m x^{m+1}\right) = 0$$

Let us do the change of variable $m' = m - 2$ in the first sum

$$\left(a_1 + 2a_2x + \sum_{m'=1}^{\infty} a_{m'+2}(m'+2)x^{m'+1}\right) + 2a_0x + \left(\sum_{m=1}^{\infty} 2a_m x^{m+1}\right) = 0$$

$$a_1 + 2(a_0 + a_2)x + \sum_{m'=1}^{\infty} a_{m'+2}(m'+2)x^{m'+1} + \sum_{m=1}^{\infty} 2a_m x^{m+1} = 0$$

$$a_1 + 2(a_0 + a_2)x + \sum_{m=1}^{\infty} ((m+2)a_{m+2} + 2a_m)x^{m+1} = 0$$

Since the whole series is 0, all its terms must be 0

$$a_1 = 0$$

$$a_0 + a_2 = 0 \Rightarrow a_2 = -a_0$$

Let us analyze now the odd terms

$$m = 1 \Rightarrow 3a_3 + 2a_1 = 0 \Rightarrow a_3 = 0$$

$$m = 3 \Rightarrow 5a_5 + 2a_3 = 0 \Rightarrow a_5 = 0$$

$$...$$

So all odd terms are null. Let us analyze now the even terms

$$m = 2 \Rightarrow 4a_4 + 2a_2 = 0 \Rightarrow a_4 = -\frac{2}{4}a_2 = \frac{1}{2}a_0$$

$$m = 4 \Rightarrow 6a_6 + 2a_4 = 0 \Rightarrow a_6 = -\frac{2}{6}a_4 = -\frac{1}{3}\frac{1}{2}a_0$$

$$m = 6 \Rightarrow 8a_8 + 2a_6 = 0 \Rightarrow a_8 = -\frac{2}{8}a_6 = \frac{1}{4}\frac{1}{3}\frac{1}{2}a_0$$

$$...$$

And in general, for m even, we have

$$a_m = (-1)^{\frac{m}{2}} \frac{1}{\frac{m}{2}!} a_0$$

The general solution is then

$$\boxed{y = a_0\left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - ...\right)}$$

It can be easily checked that this is the Taylor expansion of

$$y = a_0 e^{-x^2}$$

Kreyszig, 5.1.11

Álvaro Martín Ramos, Jan. 4th, 2015

Solve the ODE

$$y'' - y' - x^2 y = 0$$

using the power series method

Solution: Let us expand the solution of the ODE as

$$y = \sum_{m=0}^{\infty} a_m x^m$$

Then

$$\begin{aligned} y' &= \sum_{m=1}^{\infty} a_m m x^{m-1} \\ y'' &= \sum_{m=2}^{\infty} a_m m(m-1) x^{m-2} \end{aligned}$$

Substituting the series in the ODE

$$\left(\sum_{m=2}^{\infty} a_m m(m-1)x^{m-2}\right) - \left(\sum_{m=1}^{\infty} a_m m x^{m-1}\right) + x^2 \left(\sum_{m=0}^{\infty} a_m x^m\right) = 0$$

Let us do the change of variable $m' = m+1$ in the second sum and $m' = m+4$ in the third sum

$$\sum_{m=2}^{\infty} a_m m(m-1)x^{m-2} - \sum_{m'=2}^{\infty} a_{m'-1}(m'-1)x^{m'-2} + \sum_{m'=4}^{\infty} a_{m'-4}x^{m-2} = 0$$

$$2a_2 + 6a_3 x - (a_1 + 2a_2 x) + \sum_{m=4}^{\infty} \left((m-1)m a_m - (m-1)a_{m-1} + a_{m-4}\right) x^{m-2} = 0$$

The whole series is 0, all terms must be 0

$$2a_2 - a_1 = 0 \Rightarrow a_2 = \frac{a_1}{2}$$

$$6a_3 - 2a_2 = 0 \Rightarrow a_3 = \frac{a_2}{3} = \frac{a_1}{3!}$$

When m=4

$$12a_4 - 3a_3 + a_0 = 0 \Rightarrow a_4 = \frac{3a_3 - a_0}{12} = \frac{a_1}{4!} - \frac{a_0}{12}$$

When m=5

$$20a_5 - 4a_4 + a_1 = 0 \Rightarrow a_5 = \frac{4a_4 - a_1}{20} = \frac{a_4}{5} - \frac{a_1}{20} =$$

$$= \frac{a_1}{5!} - \frac{a_0}{60} - \frac{a_1}{20} = -\frac{a_0}{60} + \frac{a_1}{5!} - \frac{6a_1}{5!} = -\frac{a_0}{60} - \frac{5a_1}{5!} = -\frac{a_0}{60} - \frac{a_1}{4!}$$

The general solution is then

$$\boxed{y = a_0\left(1 - \frac{1}{12}x^4 - \frac{1}{60}x^5 ...\right) + a_1\left(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 ...\right)}$$

Kreyszig, 5.1.20

Carlos Oscar Sorzano, Aug. 31st, 2014

In numerics we use partial sums of power series. To get a feel for the accuracy for various x, experiment with $\sin(x)$. Graph partial sums of the Maclaurin series of an increasing number of terms, describing qualitatively the "breakaway points" of these graphs from the graph of $\sin(x)$.

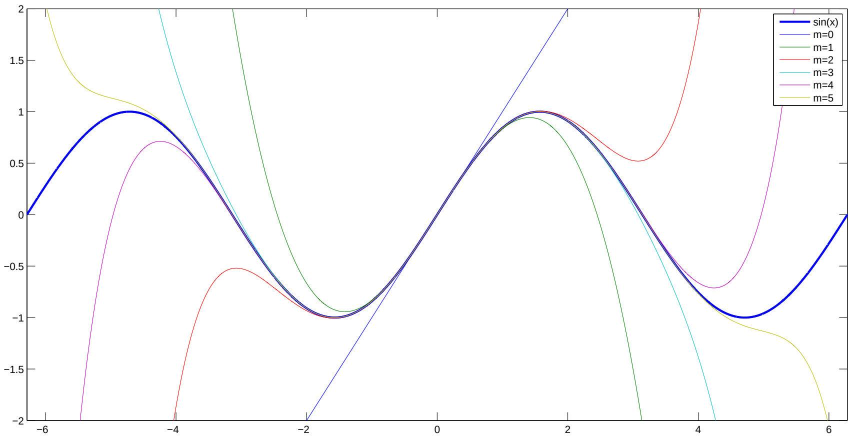
Solution: We know that the MacLaurin series of $\sin(x)$ is

$$\sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

We may program this in MATLAB as follows `x=[-2*pi:0.001:2*pi]`

```
M=5;
yp=zeros(M+1,length(x));
for m=0:M
yp(m+1,:)=(-1)^m/factorial(2*m+1)*x.^(2*m+1);
if m>0
yp(m+1,:)=yp(m+1,:)+yp(m,:);
end
end

plot(x,sin(x),'LineWidth',2)
axis([-2*pi 2*pi -2 2])
hold on
plot(x,yp)
legend('sin(x)','m=0','m=1','m=2','m=3','m=4','m=5')
```



Kreyszig, 5.2.2

Show that

$$y_2 = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - ...$$

with $n = 1$ becomes

$$y_2 = P_1 = x$$

and

$$y_1 = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - ...$$

with $n = 1$ becomes

$$y_1 = 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - ... = 1 - \frac{1}{2}x\log(x)$$

Solution: Let's start first with y_2. For a general n, y_2 is

$$y_2 = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - ...$$

In particular for $n = 1$, it becomes

$$y_2 = x - \frac{(0)(3)}{3!}x^3 + \frac{(-2)(0)(3)(5)}{5!}x^5 - ... = x = P_1(x)$$

as stated by the problem.

y_1 is for any n

$$y_1 = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - ...$$

that can be written as

$$y_1 = a_0 + a_2x^2 + a_4x^4 + ...$$

In general, we have the recursion

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)}a_m$$

which for $n = 1$ becomes

$$a_{m+2} = -\frac{(1-m)(m+2)}{(m+2)(m+1)}a_m = \frac{(m-1)}{(m+1)}a_m$$

In this way, we note that

$$\begin{aligned}
a_0 &= 1 \\
a_2 &= \tfrac{-1}{1}a_0 = -1 \\
a_4 &= \tfrac{1}{3}a_2 = -\tfrac{1}{3} \\
a_6 &= \tfrac{3}{5}a_4 = -\tfrac{3}{5}\tfrac{1}{3} = -\tfrac{1}{5} \\
a_8 &= \tfrac{5}{7}a_6 = -\tfrac{5}{7}\tfrac{1}{5} = -\tfrac{1}{7}
\end{aligned}$$

and, in general,

$$a_m = -\frac{1}{m-1}a_0$$

Then, we can wwrite y_1 as

$$y_1 = 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \frac{1}{7}x^8 - ...$$

We know that the McLaurin series of $\frac{1}{2}\log\frac{1+x}{1-x}$ in the interval $-1 < x < 1$ is

$$\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ...$$

If we now calculate

$$\begin{aligned} 1 - \tfrac{1}{2}x\log\tfrac{1+x}{1-x} &= 1 - x\left(x + \tfrac{x^3}{3} + \tfrac{x^5}{5} + \tfrac{x^7}{7} + ...\right) \\ &= 1 - x^2 - \tfrac{x^4}{3} - \tfrac{x^6}{5} - \tfrac{x^8}{7} - ... \end{aligned}$$

which is equal to y_1 as stated by the problem.

Kreyszig, 5.2.11

Carlos Oscar Sorzano, Dec. 19th, 2014

Find a solution of

$$(a^2 - x^2)y'' - 2xy' + n(n+1)y = 0$$

by reduction to a Legendre equation.

Solution: Let us perform the change of variable

$$u = \frac{x}{a}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\frac{1}{a}$$

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{du}\frac{1}{a}\right)\frac{du}{dx} = \frac{d^2y}{du^2}\frac{1}{a^2}$$

Substituting into the differential equation, we get

$$(a^2 - a^2u^2)\frac{d^2y}{du^2}\frac{1}{a^2} - 2(au)\frac{dy}{du}\frac{1}{a} + n(n+1)y = 0$$

$$(1 - u^2)\frac{d^2y}{du^2} - 2u\frac{dy}{du} + n(n+1)y = 0$$

whose general solution is

$$y(u) = c_1y_1(u) + c_2y_2(u)$$

or what is the same

$$y(x) = c_1y_1\left(\frac{x}{a}\right) + c_2y_2\left(\frac{x}{a}\right)$$

Kreyszig, 5.3.2

Solve

$$(x+2)^2y'' + (x+2)y' - y = 0$$

by the Frobenius method.

Solution: We can make the change of variable $z = x + 2$. Under this change the equation can be written as

$$z^2\ddot{y} + z\dot{y} - y = 0$$

$$\ddot{y} + \frac{1}{z}\dot{y} - \frac{1}{z^2}y = 0$$

We can apply the Frobenius method to this problem because it is of the form

$$\ddot{y} + \frac{b(z)}{z}\dot{y} + \frac{c(z)}{z^2}y = 0$$

being $b(z) = 1$ and $c(z) = -1$ analytical functions at $z = 0$. We look for a solution of the form

$$y = z^r \sum_{m=0}^{\infty} a_m z^m$$

Its derivatives are

$$\begin{aligned} \dot{y} &= \tfrac{dy}{dz} = z^{r-1}\sum_{m=0}^{\infty}(m+r)a_m z^m \\ \ddot{y} &= \tfrac{d^2y}{dz^2} = z^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m z^m \end{aligned}$$

Substituting into the differential equation:

$$z^2\left(z^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m z^m\right) + z\left(z^{r-1}\sum_{m=0}^{\infty}(m+r)a_m z^m\right) - \left(z^r\sum_{m=0}^{\infty}a_m z^m\right) = 0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m z^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m z^{m+r} - \sum_{m=0}^{\infty}a_m z^{m+r} = 0$$

$$\sum_{m=0}^{\infty}((m+r)(m+r-1) + (m+r) - 1)a_m z^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r+1)(m+r-1)a_m z^{m+r} = 0$$

The indicial equation comes from the coefficient of lowest degree, i.e., $m = 0$

$$(r+1)(r-1) = 0$$

whose solutions are

$$r_1 = 1, r_2 = -1$$

<u>Case $r_1 = 1$</u>

Substituting $r = 1$ in the differential equation we get

$$\sum_{m=0}^{\infty}(m+2)ma_m z^{m+1} = 0 = 2\cdot 1a_1z^2 + 3\cdot 2a_2z^3 + 4\cdot 3a_3z^4 + ...$$

which implies $a_1 = a_2 = a_3 = ... = 0$. So, any function of the form

$$y = z^{r_1} \sum_{m=0}^{\infty} a_m z^m = z(a_0)$$

is a solution of the differential equation. In particular, we may choose any constant a_0, for instance, $a_0 = 1$, to obtain a basis function

$$y_1 = z$$

Case $r_2 = -1$

Since the difference between r_2 and r_1 is an integer value

$$r_2 - r_1 = -1 - 1 = -2$$

we must look for a solution of the form

$$\begin{aligned} y_2 &= k y_1 \log(z) + z^{r_2} \sum_{m=0}^{\infty} a_m z^m \\ &= kz \log(z) + z^{-1} \sum_{m=0}^{\infty} a_m z^m \end{aligned}$$

Let us first calculate

$$\dot{y}_2 = k(\log(z) + 1) + z^{-2} \sum_{m=0}^{\infty} (m-1) a_m z^m$$

$$\ddot{y}_2 = kz^{-1} + z^{-3} \sum_{m=0}^{\infty} (m-1)(m-2) a_m z^m$$

We now substitute into the differential equation

$$\begin{gathered} z^2 \left(kz^{-1} + z^{-3} \sum_{m=0}^{\infty} (m-1)(m-2) a_m z^m \right) + z \left(k(\log(z) + 1) + z^{-2} \sum_{m=0}^{\infty} (m-1) a_m z^m \right) \\ - \left(kz \log(z) + z^{-1} \sum_{m=0}^{\infty} a_m z^m \right) = 0 \\ \left(kz + \sum_{m=0}^{\infty} (m-1)(m-2) a_m z^{m-1} \right) + \left(kz \log(z) + kz + \sum_{m=0}^{\infty} (m-1) a_m z^{m-1} \right) \\ - \left(kz \log(z) + \sum_{m=0}^{\infty} a_m z^{m-1} \right) = 0 \\ 2kz + \sum_{m=0}^{\infty} \left[(m-1)(m-2) + (m-1) - 1 \right] a_m z^{m-1} = 0 \\ 2kz + \sum_{m=0}^{\infty} m(m-2) a_m z^{m-1} = 0 \\ (-1) a_1 + 2kz + \sum_{m=3}^{\infty} m(m-2) a_m z^{m-1} = 0 \end{gathered}$$

So, $a_1 = k = a_3 = a_4 = a_5 = ... = 0$. a_0 and a_2 are free so any solution of the kind

$$y_2 = z^{-1}(a_0 + a_2 z^2) = a_0 z^{-1} + a_2 z$$

is a solution of the differential equation. Actually, we already knew that z was a solution, so the only novelty brought by this solution is (with $a_0 = 1$).

$$y_2 = z^{-1}$$

Any solution of the differential equation is of the form

$$\boxed{y} = c_1 y_1 + c_2 y_2 = c_1 z + \frac{c_2}{z} = \boxed{c_1(x+2) + \frac{c_2}{x+2}}$$

Kreyszig, 5.3.4

Carlos Oscar Sorzano, June 15th, 2015

Solve

$$xy'' + y = 0$$

by the Frobenius method.

Solution: Dividing by x we have

$$y'' + \frac{1}{x}y = 0$$

We can apply the Frobenius method to this problem because it is of the form

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

being $b(x) = 0$ and $c(x) = x$ analytical functions at $x = 0$. We look for a solution of the form

$$y = x^r \sum_{m=0}^{\infty} a_m x^m$$

Its derivatives are

$$\begin{aligned} y' &= \tfrac{dy}{dx} = x^{r-1} \sum_{m=0}^{\infty} (m+r)a_m x^m \\ y'' &= \tfrac{d^2y}{dx^2} = x^{r-2} \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m \end{aligned}$$

Substituting into the differential equation:

$$x\left(x^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m\right) + \left(x^r \sum_{m=0}^{\infty} a_m x^m\right) = 0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

If we take out the first term from the first summation, we get

$$r(r-1)a_0 x^{r-1} + \sum_{m=1}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

We now make a change of variable to make the first summation to start at $m = 0$

$$r(r-1)a_0 x^{r-1} + \sum_{m=0}^{\infty}(m+r+1)(m+r)a_{m+1} x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$r(r-1)a_0x^{r-1}+\sum_{m=0}^{\infty}((m+r+1)(m+r)a_{m+1}+a_m)x^{m+r}=0$$

The indicial equation comes from the coefficient of lowest degree, i.e., the first one

$$r(r-1)=0$$

whose solutions are

$$r_1=0, r_2=1$$

Case $r_1 = 0$
Substituting $r=0$ in the differential equation we get

$$\sum_{m=0}^{\infty}((m+1)ma_{m+1}+a_m)x^m=0$$

Note that this is equal to

$$a_0+\sum_{m=1}^{\infty}((m+1)ma_{m+1}+a_m)x^m=0$$

which implies $a_0 = 0$ and (for $m \geq 1$)

$$(m+1)ma_{m+1}+a_m=0 \Rightarrow a_{m+1}=-\frac{a_m}{m(m+1)}$$

The first terms are

$$\begin{aligned} a_2 &= -\tfrac{a_1}{1\cdot 2}=-\tfrac{1}{1\cdot 2}a_1 \\ a_3 &= -\tfrac{a_2}{2\cdot 3}=\tfrac{1}{1\cdot 2\cdot 2\cdot 3}a_1 \\ a_4 &= -\tfrac{a_3}{3\cdot 4}=-\tfrac{1}{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4}a_1 \end{aligned}$$

We observe that the follow the general term (for $m \geq 1$)

$$a_m=\frac{(-1)^{m+1}}{m!(m-1)!}a_1$$

So, any function of the form

$$y=x^{r_1}\sum_{m=0}^{\infty}a_mx^m=a_1\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m!(m-1)!}x^m$$

is a solution of the differential equation. In particular, we may choose any constant a_1, for instance, $a_1 = 1$, to obtain a basis function

$$y_1=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m!(m-1)!}x^m=x-\frac{1}{2!1!}x^2+\frac{1}{3!2!}x^3-\frac{1}{4!3!}x^4+...$$

Case $r_2 = 1$
Since the difference between r_2 and r_1 is an integer value, we must look for a solution of the form

$$\begin{aligned} y_2 &= ky_1\log(x)+x^{r_2}\sum_{m=0}^{\infty}a_mx^m \\ &= k\left(\sum_{m=1}^{\infty}\tfrac{(-1)^{m+1}}{m!(m-1)!}x^m\right)\log(x)+x\sum_{m=0}^{\infty}a_mx^m \end{aligned}$$

Let us first calculate the derivatives of y_2

$$y_2' = k\left(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m!(m-1)!}(1+m\log(x))x^m\right)+\sum_{m=0}^{\infty}(m+1)a_m x^m$$

$$y_2'' = k\left(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m!(m-1)!}m(2+m\log(x))x^{m-1}\right)+\sum_{m=0}^{\infty}(m+1)ma_m x^{m-1}$$

We now substitute into the equation

$$xy_2''+y_2=0$$

$$k\left(\sum_{m=1}^{\infty}\tfrac{(-1)^{m+1}}{m!(m-1)!}m(2+m\log(x))x^m\right)+\sum_{m=0}^{\infty}(m+1)ma_m x^m+$$
$$k\left(\sum_{m=1}^{\infty}\tfrac{(-1)^{m+1}}{m!(m-1)!}x^m\right)\log(x)+x\sum_{m=0}^{\infty}a_m x^m=0$$

$$\sum_{m=1}^{\infty}\tfrac{(-1)^{m+1}}{m!(m-1)!}2kmx^m+\sum_{m=1}^{\infty}\tfrac{(-1)^{m+1}}{m!(m-1)!}m^2k\log(x)x^m+\sum_{m=0}^{\infty}(m+1)ma_m x^m+$$
$$\sum_{m=1}^{\infty}\tfrac{(-1)^{m+1}}{m!(m-1)!}k\log(x)x^m+x\sum_{m=0}^{\infty}a_m x^m=0$$

$$\sum_{m=1}^{\infty}\tfrac{(-1)^{m+1}}{m!(m-1)!}k\log(x)(1+m^2)x^m+$$
$$\sum_{m=0}^{\infty}(m+1)ma_m x^m+\sum_{m=1}^{\infty}\tfrac{(-1)^{m+1}}{m!(m-1)!}2kmx^m+x\sum_{m=0}^{\infty}a_m x^m=0$$

From the first row of previous equation we learn that $k=0$, because all terms in $x^m\log(x)$ must go (they are equal to 0 in the right-hand side). Then, the previous equation simplifies to

$$\sum_{m=0}^{\infty}(m+1)ma_m x^m+\sum_{m=0}^{\infty}a_m x^{m+1}=0$$

$$\sum_{m=1}^{\infty}(m+1)ma_m x^m+\sum_{m=0}^{\infty}a_m x^{m+1}=0$$

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+1}x^{m+1}+\sum_{m=0}^{\infty}a_m x^{m+1}=0$$

$$\sum_{m=0}^{\infty}((m+2)(m+1)a_{m+1}+a_m)x^{m+1}=0$$

From where

$$a_{m+1}=-\frac{a_m}{(m+1)(m+2)}$$

The first terms are

$$\begin{aligned}a_1&=-\tfrac{a_0}{1\cdot 2}=-\tfrac{1}{1\cdot 2}a_0\\ a_2&=-\tfrac{a_1}{2\cdot 3}=\tfrac{1}{1\cdot 2\cdot 2\cdot 3}a_0\\ a_3&=-\tfrac{a_2}{3\cdot 4}=-\tfrac{1}{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4}a_0\end{aligned}$$

The general term is

$$a_m=\frac{(-1)^m}{m!(m+1)!}a_0$$

In this way, we see that any function of the form

$$y_2 = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} x^m$$

is solution of the differential equation. In paticular, for $a_0 = 1$, we get

$$y_2 = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} x^m = 1 - \frac{1}{1!2!} x + \frac{1}{2!3!} x^2 - \frac{1}{3!4!} x^3 + ...$$

General solution: Finally, the general solution of the ODE is

$$y = K_1 y_1 + K_2 y_2 = \boxed{K_1 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!(m-1)!} x^m + K_2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} x^m}$$

By making the change of variable $z = \sqrt{x}$ in the original differential equation

$$xy'' + y = 0$$

The equation is transformed into a Bessel's equation whose general solution is

$$y = K_A \sqrt{x} J_1(2\sqrt{x}) + K_B \sqrt{x} Y_1(2\sqrt{x})$$

That is, both solutions (the series expansion and the Bessel's solution) are equivalent (i.e., given K_1 and K_2, one can always find K_A and K_B that gives the same function; and viceversa).

Kreyszig, 5.4.3

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve

$$xy'' + y' + \frac{1}{4} y = 0$$

by making the change of variable $z = \sqrt{x}$.

Solution: If $z = \sqrt{x}$, then

$$\begin{array}{rcl} z' & = & \frac{1}{2\sqrt{x}} = \frac{1}{2} z^{-1} \\ y' & = & \frac{dy}{dz} z' = \frac{dy}{dz} \frac{1}{2} z^{-1} \\ y'' & = & \frac{dy'}{dz} \frac{dz}{dx} = \frac{1}{2} \left(-z^{-2} \frac{dy}{dz} + z^{-1} \frac{d^2y}{dz^2} \right) \frac{1}{2} z^{-1} \\ & = & \frac{1}{4} \left(z^{-2} \frac{d^2y}{dz^2} - z^{-3} \frac{dy}{dz} \right) \end{array}$$

With these, we can rewrite the ODE as

$$z^2 \frac{1}{4} \left(z^{-2} \frac{d^2y}{dz^2} - z^{-3} \frac{dy}{dz} \right) + \frac{1}{2} z^{-1} \frac{dy}{dz} + \frac{1}{4} y = 0$$

$$\frac{1}{4} \frac{d^2y}{dz^2} - \frac{1}{4} z^{-1} \frac{dy}{dz} + \frac{1}{2} z^{-1} \frac{dy}{dz} + \frac{1}{4} y = 0$$

$$\frac{1}{4} \frac{d^2y}{dz^2} + \frac{1}{4} z^{-1} \frac{dy}{dz} + \frac{1}{4} y = 0$$

Multiplying by $4z^2$, we get

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + z^2y = 0$$

which is Bessel's equation with $\nu = 0$. Since ν is an integer, there is no solution of form

$$y = c_1J_\nu(z) + c_2J_{-\nu}(z)$$

Its general solution needs Bessel's functions of the second kind (that will be seen in next section). However, for the sake of completeness we already point out that the general solution is

$$y = c_1J_0(z) + c_2Y_0(z) = \boxed{c_1J_0(\sqrt{x}) + c_2Y_0(\sqrt{x})}$$

Kreyszig, 5.4.5

Carlos Oscar Sorzano, Dec. 19th, 2014

Solve

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0$$

by making the change of variables $\lambda x = z$.

Solution: Let us write the different elements we need from the change of variables

$$\begin{array}{rcl} \frac{dz}{dx} & = & \lambda \\ y' & = & \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \dot{y}\lambda \\ y'' & = & \frac{d^2y}{dx^2} = \frac{d}{dz}\left(\dot{y}\lambda\right)\frac{dz}{dx} = (\lambda\ddot{y})\lambda = \lambda^2\ddot{y} \end{array}$$

Substituting in the differential equation

$$\frac{z^2}{\lambda^2}(\lambda^2\ddot{y}) + \frac{z}{\lambda}(\lambda\dot{y}) + (z^2 - \nu^2)y = 0$$

$$z^2\ddot{y} + z\dot{y} + (z^2 - \nu^2)y = 0$$

which is a Bessel's equation of parameter ν. Its general solution is ($\nu \notin \mathbb{Z}$)

$$\boxed{y} = c_1J_\nu(z) + c_2J_{-\nu}(z) = \boxed{c_1J_\nu(\lambda x) + c_2J_{-\nu}(\lambda x)}$$

Kreyszig, 5.4.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve

$$x^2y'' + \tfrac{1}{4}(x + \tfrac{3}{4})y = 0$$

by making the change of variable $y = u\sqrt{x}, z = \sqrt{x}$.

Solution: If $z = \sqrt{x}$, then

$$z' = \frac{1}{2\sqrt{x}} = \tfrac{1}{2}z^{-1}$$

On the other side

$$\begin{array}{rcl} y &=& u\sqrt{x} = uz \\ y' &=& \frac{dy}{dz}\frac{dz}{dx} = \left(\frac{du}{dz}z + u\right)\frac{1}{2}z^{-1} \\ y'' &=& \frac{dy'}{dz}\frac{dz}{dx} \\ &=& \frac{1}{2}\left[z^{-1}\left(\frac{d^2u}{dz^2}z + \frac{du}{dz} + \frac{du}{dz}\right) + (-z^{-2})\left(\frac{du}{dz}z + u\right)\right]\frac{1}{2}z^{-1} \\ &=& \frac{1}{4}z^{-2}\left(\frac{d^2u}{dz^2}z + 2\frac{du}{dz}\right) - \frac{1}{4}z^{-3}\left(\frac{du}{dz}z + u\right) \\ &=& \frac{1}{4}z^{-1}\frac{d^2u}{dz^2} + \frac{1}{2}z^{-2}\frac{du}{dz} - \frac{1}{4}z^{-2}\frac{du}{dz} - \frac{1}{4}z^{-3}u \\ &=& \frac{1}{4}z^{-1}\frac{d^2u}{dz^2} + \frac{1}{4}z^{-2}\frac{du}{dz} - \frac{1}{4}z^{-3}u \end{array}$$

So, we can rewrite the ODE as

$$z^4\left(\tfrac{1}{4}z^{-1}\frac{d^2u}{dz^2} + \tfrac{1}{4}z^{-2}\frac{du}{dz} - \tfrac{1}{4}z^{-3}u\right) + \tfrac{1}{4}(z^2 + \tfrac{3}{4})uz = 0$$

$$\tfrac{1}{4}z^3\frac{d^2u}{dz^2} + \tfrac{1}{4}z^2\frac{du}{dz} - \tfrac{1}{4}zu + \tfrac{1}{4}z^3u + \tfrac{3}{16}uz = 0$$

$$\tfrac{1}{4}z^3\frac{d^2u}{dz^2} + \tfrac{1}{4}z^2\frac{du}{dz} + \tfrac{1}{4}z^3u - \tfrac{1}{16}uz = 0$$

Multiplying the whole equation by $4z^{-1}$, we get

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + z^2u - \tfrac{1}{4}u = 0$$

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left(z^2 - \tfrac{1}{4}\right)u = 0$$

That is Bessel's equation with $\nu = \frac{1}{2}$. Since ν is not an integer value, its general solution can be written as

$$u = c_1J_{\frac{1}{2}}(z) + c_2J_{-\frac{1}{2}}(z) = c_1J_{\frac{1}{2}}(\sqrt{x}) + c_2J_{-\frac{1}{2}}(\sqrt{x})$$

Finally, we undo the change of variable

$$y = uz = \boxed{\left(c_1J_{\frac{1}{2}}(\sqrt{x}) + c_2J_{-\frac{1}{2}}(\sqrt{x})\right)\sqrt{x}}$$

Kreyszig, 5.4.10

Carlos Oscar Sorzano, Jan. 13th, 2015

Solve

$$x^2y'' + (1-2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0$$

by making the change of variables $z = x^\nu$.

Solution: Let us perform the change of variables in two steps. We first make the change of variable

$$\begin{array}{rcl} z &=& x^\nu \Rightarrow x = z^{\frac{1}{\nu}} \\ \frac{dz}{dx} &=& \nu x^{\nu-1} = \nu z^{\frac{\nu-1}{\nu}} \\ y' &=& \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \dot{y}(\nu z^{\frac{\nu-1}{\nu}}) \\ y'' &=& \frac{d^2y}{dx^2} = \frac{d}{dz}\left(\dot{y}(\nu z^{\frac{\nu-1}{\nu}})\right)\frac{dz}{dx} \\ &=& \nu\left[\ddot{y}z^{\frac{\nu-1}{\nu}} + \dot{y}\frac{\nu-1}{\nu}z^{-\frac{1}{\nu}}\right](\nu z^{\frac{\nu-1}{\nu}}) \\ &=& \ddot{y}\nu^2z^{\frac{2\nu-2}{\nu}} + \dot{y}\nu(\nu-1)z^{\frac{\nu-2}{\nu}} \end{array}$$

Substituting into the ODE we get

$$z^{\frac{2}{\nu}}(\ddot{y}\nu^2 z^{\frac{2\nu-2}{\nu}} + \dot{y}\nu(\nu-1)z^{\frac{\nu-2}{\nu}}) + (1-2\nu)z^{\frac{1}{\nu}}(\dot{y}\nu z^{\frac{\nu-1}{\nu}}) + \nu^2(z^2+1-\nu^2)y = 0$$
$$(\ddot{y}\nu^2 z^2 + \dot{y}\nu(\nu-1)z) + (1-2\nu)(\dot{y}\nu z) + \nu^2(z^2+1-\nu^2)y = 0$$
$$\ddot{y}\nu^2 z^2 + \dot{y}\nu^2 z + \nu^2(z^2+1-\nu^2)y = 0$$
$$z^2\ddot{y} + z\dot{y} + (z^2 - (\nu^2-1))y = 0$$

This is Bessel's equation if $\nu^2 - 1 > 0$, in that case the general solution is given (if $\nu^2 \notin \mathbb{R}^2$) by

$$\boxed{y} = c_1 J_{\sqrt{\nu^2-1}}(z) + c_2 J_{-\sqrt{\nu^2-1}}(z) = \boxed{c_1 J_{\sqrt{\nu^2-1}}(x^\nu) + c_2 J_{-\sqrt{\nu^2-1}}(x^\nu)}$$

Kreyszig, 5.5.1

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve
$$x^2y'' + xy' + (x^2-16)y = 0$$

Solution: This is Bessel's equation with $\nu = 4$. Since ν is an integer value, we have to write the general solution making use of Bessel's functions of second kind:
$$y = c_1 J_4(x) + c_2 Y_4(x)$$

Kreyszig, 5.5.2

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve
$$xy'' + 5y' + xy = 0$$

by making the change of variable $y = \frac{u}{x^2}$.
Solution: If $y = ux^{-2}$, then

$$\begin{array}{rcl} y' &=& \frac{du}{dx}x^{-2} + u(-2x^{-3}) \\ y'' &=& \frac{d^2u}{dx^2}x^{-2} + \frac{du}{dx}(-2x^{-3}) + \frac{du}{dx}(-2x^{-3}) + u(6x^{-4}) \\ &=& x^{-2}\frac{d^2u}{dx^2} - 4x^{-3}\frac{du}{dx} + 6x^{-4}u \end{array}$$

Substituting in the ODE we get

$$x\left(x^{-2}\frac{d^2u}{dx^2} - 4x^{-3}\frac{du}{dx} + 6x^{-4}u\right) + 5\left(\frac{du}{dx}x^{-2} + u(-2x^{-3})\right) + xux^{-2} = 0$$
$$x^{-1}\frac{d^2u}{dx^2} - 4x^{-2}\frac{du}{dx} + 6x^{-3}u + 5x^{-2}\frac{du}{dx} - 10x^{-3}u + x^{-1}u = 0$$
$$x^{-1}\frac{d^2u}{dx^2} + x^{-2}\frac{du}{dx} + (x^{-1} - 4x^{-3})u = 0$$

Multiplying the equation by x^3

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2-4)u = 0$$

which is Bessel's equation with $\nu = 2$. Since ν is an integer value, the general solution is given by

$$u = c_1 J_2(x) + c_2 Y_2(x)$$

Undoing the change of variable

$$y = ux^{-2} = \boxed{c_1 x^{-2} J_2(x) + c_2 x^{-2} Y_2(x)}$$

6 Chapter 6

Kreyszig, 6.1.4

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Laplace transform of $\cos^2(\omega t)$.

Solution:

$$\begin{aligned}
\mathcal{L}\{\cos^2(\omega t)\} &= \int\limits_0^\infty \cos^2(\omega t) e^{-st} dt \\
&= \int\limits_0^\infty \tfrac{1+\cos(2\omega t)}{2} e^{-st} dt \\
&= \tfrac{1}{2} \int\limits_0^\infty e^{-st} dt + \tfrac{1}{2} \int\limits_0^\infty \cos(2\omega t) e^{-st} dt \\
&= \tfrac{1}{2} \left[\tfrac{1}{-s} e^{-st} \right]_0^\infty + \tfrac{1}{2} \int\limits_0^\infty \cos(2\omega t) e^{-st} dt \\
&= \tfrac{1}{2}\tfrac{1}{s} + \tfrac{1}{2} \int\limits_0^\infty \cos(2\omega t) e^{-st} dt \quad [\mathrm{Re}\{s\} < 0]
\end{aligned}$$

Now we make use of the Laplace transform

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

to get

$$\boxed{\mathcal{L}\{\cos^2(\omega t)\} = \frac{1}{2s} + \frac{1}{2} \frac{s}{s^2 + (2\omega)^2} \quad [\mathrm{Re}\{s\} < 0]}$$

Kreyszig, 6.1.20

Carlos Oscar Sorzano, Aug. 31st, 2014

Non-existence. Show that a function like e^{t^2} does not fulfill the condition

$$\left| e^{t^2} \right| \leq M e^{kt}$$

Solution: For $t > 0$ we have $e^{t^2} > 0$ so that $\left| e^{t^2} \right| = e^{t^2}$. Let us show that for any M and k, we can find t such that

$$e^{t^2} > M e^{kt} = e^{\log(M)} e^{kt}$$

Taking logarithms

$$t^2 > \log(M) + kt$$

$$t^2 - kt - \log(M) > 0$$

Let us find the point at which the curve crosses 0

$$t^2 - kt - \log(M) = 0 \Rightarrow t = \frac{k \pm \sqrt{k^2 + 4\log(M)}}{2}$$

That is for $t > \frac{k+\sqrt{k^2+4\log(M)}}{2}$ we have that

$$e^{t^2} > Me^{kt}$$

Kreyszig, 6.1.22

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}$$

Conclude from this that the conditions for existence are sufficient but not necessary for the existence of the Laplace transform.

Solution:

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int\limits_0^\infty \frac{1}{\sqrt{t}} e^{-st} = \int\limits_0^\infty t^{-\frac{1}{2}} e^{-st} dt$$

Let us make the change of variable

$$\tau = st \Rightarrow t = \frac{\tau}{s}, dt = \frac{d\tau}{s}$$

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} &= \int\limits_0^\infty \left(\frac{\tau}{s}\right)^{-\frac{1}{2}} e^{-\tau} \frac{d\tau}{s} = \int\limits_0^\infty \tau^{-\frac{1}{2}} s^{\frac{1}{2}} e^{-\tau} s^{-1} d\tau \\ &= s^{-\frac{1}{2}} \int\limits_0^\infty \tau^{-\frac{1}{2}} e^{-\tau} d\tau = s^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) = s^{-\frac{1}{2}} \sqrt{\pi} = \sqrt{\frac{\pi}{s}}\end{aligned}$$

So, there exists the Laplace transform of $\frac{1}{\sqrt{t}}$ although it is not well defined at $t = 0$.

Kreyszig, 6.1.26

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the inverse Laplace transform of $\frac{5s+1}{s^2-25}$

Solution:

$$\mathcal{L}^{-1}\left\{\frac{5s+1}{s^2-25}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2-25}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-25}\right\} = 5\cosh(5t) + \frac{1}{5}\sinh(5t)$$

where we have made used of the Laplace transforms

$$\begin{aligned}\mathcal{L}\left\{\frac{s}{s^2-a^2}\right\} &= \cosh(at) \\ \mathcal{L}\left\{\frac{a}{s^2-a^2}\right\} &= \sinh(at)\end{aligned}$$

Kreyszig, 6.1.29

Álvaro Martín Ramos, Jan. 11th, 2015

Find the inverse Laplace transform of

$$\frac{12}{s^4} - \frac{228}{s^6}$$

Solution:

$$\mathcal{L}^{-1}\{\frac{12}{s^4} - \frac{228}{s^6}\} = 2\mathcal{L}^{-1}\{\frac{3!}{s^4}\} - \frac{228}{5!}\mathcal{L}^{-1}\{\frac{5!}{s^6}\} = 2t^3 - 1.9t^5$$

Kreyszig, 6.1.30

Álvaro Martín Ramos, Jan. 11th, 2015

Find the inverse Laplace transform of

$$\frac{4s+32}{s^2-16}$$

Solution:

$$\mathcal{L}^{-1}\{\frac{4s+32}{s^2-16}\} = 4\mathcal{L}^{-1}\{\frac{s}{s^2-16}\} + 8\mathcal{L}^{-1}\{\frac{4}{s^2-16}\} = 4\cosh(4t) + 8\sinh(4t)$$

Kreyszig, 6.1.33

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Laplace transform of t^2e^{-3t}
Solution: We know that

$$\begin{aligned} \mathcal{L}\left\{t^2\right\} &= \tfrac{2!}{s^3} \\ \mathcal{L}\left\{e^{at}f(t)\right\} &= F(s-a) \end{aligned}$$

Both together we have that

$$\mathcal{L}\left\{t^2e^{-3t}\right\} = \tfrac{2!}{(s+3)^3}$$

Kreyszig, 6.1.39

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the inverse Laplace transform of $\frac{21}{(s+\sqrt{2})^4}$
Solution: We know that

$$\begin{aligned} \mathcal{L}\left\{t^n\right\} &= \tfrac{n!}{s^{n+1}} \\ \mathcal{L}\left\{e^{at}f(t)\right\} &= F(s-a) \end{aligned}$$

Then we have the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\tfrac{21}{s^4}\right\} = \tfrac{21}{3!}\mathcal{L}^{-1}\left\{\tfrac{3!}{s^4}\right\} = \tfrac{7}{2}t^3$$

and

$$\mathcal{L}^{-1}\left\{\tfrac{21}{(s+\sqrt{2})^4}\right\} = \tfrac{7}{2}t^3e^{-\sqrt{2}t}$$

Kreyszig, 6.2.3

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the Initial Value Problem

$$y'' - y' - 6y = 0 \quad y(0) = 11, y'(0) = 28$$

Solution: We know that

$$\begin{array}{rcl}\mathcal{L}\{y\} & = & Y \\ \mathcal{L}\{y'\} & = & sY - y(0) \\ \mathcal{L}\{y''\} & = & s^2Y - sy(0) - y'(0)\end{array}$$

Then, we can write the ODE as

$$(s^2Y - 11s - 28) - (sY - 11) - 6Y = 0$$

$$(s^2 - s - 6)Y - 11s - 17 = 0$$

$$Y = \frac{11s + 17}{s^2 - s - 6} = \frac{11s + 17}{(s-3)(s+2)} = \frac{1}{s+2} + \frac{10}{s-3}$$

Its inverse Laplace transform is

$$y = e^{-2t} + 10e^{3t}$$

which is the particular solution of the IVP satisfying the initial conditions.

Kreyszig, 6.2.12

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the Initial Value Problem

$$y'' - 2y' - 3y = 0 \quad y(4) = -3, y'(4) = 17$$

Solution: Let us define

$$\tilde{y}(\tilde{t}) = y(\tilde{t} + 4) \Leftrightarrow y(t) = \tilde{y}(t - 4)$$

Note that the relationship between the two time variables is

$$\tilde{t} = t - 4$$

Then

$$\begin{array}{rcl}y'(t) & = & \tilde{y}'(\tilde{t}) \\ y''(t) & = & \tilde{y}''(\tilde{t})\end{array}$$

Then we can rewrite the IVP as

$$\tilde{y}'' - 2\tilde{y}' - 3\tilde{y} = 0 \quad \tilde{y}(0) = -3, \tilde{y}'(0) = 17$$

We know that

$$\begin{array}{rcl}\mathcal{L}\{\tilde{y}\} & = & \tilde{Y} \\ \mathcal{L}\{\tilde{y}'\} & = & s\tilde{Y} - \tilde{y}(0) \\ \mathcal{L}\{\tilde{y}''\} & = & s^2\tilde{Y} - s\tilde{y}(0) - \tilde{y}'(0)\end{array}$$

Then, we can write the ODE as

$$(s^2\tilde{Y} + 3s - 17) - 2(s\tilde{Y} + 3) - 3\tilde{Y} = 0$$

$$(s^2 - 2s - 3)\tilde{Y} + 3s - 23 = 0$$

$$\tilde{Y} = \frac{-3s + 23}{s^2 - 2s - 3} = \frac{-3s + 23}{(s-3)(s+1)} = \frac{7}{2}\frac{1}{s-3} - \frac{13}{2}\frac{1}{s+1}$$

Its inverse Laplace transform is

$$\tilde{y}(\tilde{t}) = \frac{7}{2}e^{3\tilde{t}} - \frac{13}{2}e^{-\tilde{t}}$$

which is the particular solution of the IVP satisfying the initial conditions. If we undo now the time shift, we get

$$\boxed{y(t) = \tilde{y}(t-4) = \frac{7}{2}e^{3(t-4)} - \frac{13}{2}e^{-(t-4)}}$$

Kreyszig, 6.2.15

Álvaro Martín Ramos, Jan. 11th, 2015

Solve the Initial Value Problem

$$y'' + 3y' - 4y = 6e^{2t-3} \quad y(1.5) = 4, y'(1.5) = 5$$

Solution: We make a change of variable

$$\tilde{t} = t - 1.5 \Rightarrow t = \tilde{t} + 1.5$$

Then

$$y'(t) = \tilde{y}'(\tilde{t})$$

$$y''(t) = \tilde{y}''(\tilde{t})$$

Then, we can rewrite the IVP as

$$\tilde{y}''(\tilde{t}) + 3\tilde{y}'(\tilde{t}) - 4\tilde{y}(\tilde{t}) = 6e^{2\tilde{t}} \quad \tilde{y}(0) = 4, \tilde{y}'(0) = 5$$

Making the Laplace transform of both sides we get

$$(s^2\tilde{Y} - 4s - 5) + 3(s\tilde{Y} - 4) - 4\tilde{Y} = \frac{6}{s-2}$$

$$(s^2 + 3s - 4)\tilde{Y} - 4s - 17 = \frac{6}{s-2}$$

$$(s+4)(s-1)\tilde{Y} = \frac{6}{s-2} + 4s + 17$$

$$\tilde{Y} = \frac{3}{s-1} + \frac{1}{s-2}$$

Its inverse Laplace transform is

$$\tilde{y}(\tilde{t}) = 3e^{\tilde{t}} + e^{2\tilde{t}}$$

which is the particular solution of the IVP satisfying the initial conditions. If we undo now the time shift,we get

$$\boxed{y(t) = \tilde{y}(t-1.5) = 3e^{t-1.5} + e^{2(t-1.5)}}$$

Kreyszig, 6.2.16

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Laplace transform of $t\cos(4t)$
Solution: Let us define $f = t\cos(at)$ Let us differentiate f

$$\begin{array}{l} f' = \cos(at) - at\sin(at) \\ f'' = -a\sin(at) - a\sin(at) - a^2t\cos(at) = -2a\sin(at) - a^2t\cos(at) \end{array}$$

If we now take the Laplace transform of f'' we get

$$\mathcal{L}\{f''\} = -2a\frac{a}{s^2+a^2} - a^2\mathcal{L}\{t\cos(at)\} = -\frac{2a^2}{s^2+a^2} - a^2F$$

On the other side we know that

$$\mathcal{L}\{f''\} = s^2F - sf(0) - f'(0)$$

Substituting $f(0) = 0$, $f'(0) = 1$ we get

$$\mathcal{L}\{f''\} = s^2F - 1$$

Equating both expressions for $\mathcal{L}\{f''\}$ we get

$$-\frac{2a^2}{s^2+a^2} - a^2F = s^2F - 1$$

Solving for F

$$F = \frac{s^2-a^2}{(s^2+a^2)^2}$$

In particular, for $a = 4$ (as in the problem statement, we get

$$\boxed{\mathcal{L}\{t\cos(4t)\} = \frac{s^2-4^2}{(s^2+4^2)^2}}$$

Kreyszig, 6.2.24

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the inverse Laplace transform of $\frac{20}{s^3-2\pi s^2}$
Solution: We may factorize F as

$$F = \frac{1}{s^2}\frac{20}{s-2\pi}$$

The inverse Laplace transform of $\frac{20}{s-2\pi}$ is $20e^{2\pi t}$. The factor $\frac{1}{s^2}$ translates into a double time integral. Let's do it one by one:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{20}{s-2\pi}\right\} = \int_0^t 20e^{2\pi\tau}d\tau = 20\frac{e^{2\pi t}-1}{2\pi}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2}\frac{20}{s-2\pi}\right\}} = \int_0^t \frac{20}{2\pi}(e^{2\pi\tau}-1)d\tau = \boxed{\frac{20}{2\pi}\left(-t+\frac{e^{2\pi t}-1}{2\pi}\right)}$$

Kreyszig, 6.3.3

Álvaro Martín Ramos, Jan. 11th, 2015

Find the Laplace transform of

$$t-2(t>2)$$

Solution: Let us write the function to transform as

$$f(t) = (t-2)u(t-2)$$

$$\mathcal{L}\{(t-2)u(t-2)\} = e^{-2s}\mathcal{L}\{t\} = \boxed{\frac{e^{-2s}}{s^2}}$$

Kreyszig, 6.3.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Laplace transform of $e^t \quad \left(0 < t < \frac{\pi}{2}\right)$
Solution: Let us write the function to transform as

$$f(t) = e^t\left(u(t) - u\left(t-\tfrac{\pi}{2}\right)\right) = e^t u(t) - e^t u\left(t-\tfrac{\pi}{2}\right)$$

Let us transform each term separately

$$\mathcal{L}\{e^t u(t)\} = \frac{1}{s-1}$$

$$\mathcal{L}\left\{e^t u\left(t-\tfrac{\pi}{2}\right)\right\} = \mathcal{L}\left\{e^{t-\frac{\pi}{2}}e^{\frac{\pi}{2}}u\left(t-\tfrac{\pi}{2}\right)\right\} = e^{\frac{\pi}{2}}\mathcal{L}\left\{e^{t-\frac{\pi}{2}}u\left(t-\tfrac{\pi}{2}\right)\right\} = e^{\frac{\pi}{2}}e^{-\frac{\pi}{2}s}\frac{1}{s-1}$$

Altogether

$$\mathcal{L}\{f\} = \frac{1}{s-1} - e^{\frac{\pi}{2}}e^{-\frac{\pi}{2}s}\frac{1}{s-1} = \boxed{\frac{1}{s-1}\left(1-e^{-\frac{\pi}{2}(s-1)}\right)}$$

Kreyszig, 6.3.8

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Laplace transform of $t^2 \quad (1 < t < 2)$
Solution: We can rewrite the function to be transformed as

$$f = t^2(u(t-1)-u(t-2)) = t^2u(t-1) - t^2u(t-2)$$

Now, we transform each term separately

$$\mathcal{L}\{t^2u(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\} = e^{-s}\mathcal{L}\{t^2+2t+1\} = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$$

$$\mathcal{L}\{t^2u(t-2)\} = e^{-2s}\mathcal{L}\{(t+2)^2\} = e^{-2s}\mathcal{L}\{t^2+4t+4\} = e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$$

The Laplace transform of f is

$$\boxed{\mathcal{L}\{f\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)}$$

Kreyszig, 6.3.13

Álvaro Martín Ramos, Jan. 11th, 2015

Find the inverse Laplace transform of

$$\frac{6(1 - e^{-\pi s})}{s^2 + 9}$$

Solution:

$$\mathcal{L}^{-1}\left\{\frac{6(1-e^{-\pi s})}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{6e^{-\pi s}}{s^2+9}\right\} = 2\sin(3t) - 2\sin(3(t-\pi))$$

Kreyszig, 6.3.17

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the inverse Laplace transform of $(1 + e^{-2\pi(s+1)})\frac{s+1}{(s+1)^2+1}$

Solution: Let us find first the inverse Laplace transform of the function

$$G(s) = (1 + e^{-2\pi s})\frac{s}{s^2+1} = \frac{s}{s^2+1} + \frac{s}{s^2+1}e^{-2\pi s}$$

The inverse Laplace transform of this function is

$$g(t) = \cos(t)u(t) + \cos(t-2\pi)u(t-2\pi) = \cos(t)u(t) + \cos(t)u(t-2\pi) = \cos(t)(u(t) - u(t-2\pi))$$

However, we are interested in

$$F(s) = G(s+1)$$

whose inverse Laplace transform is

$$f(t) = g(t)e^{-t} = \boxed{e^{-t}\cos(t)(u(t) - u(t-2\pi))}$$

Kreyszig, 6.3.19

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the Initial Value Problem

$$y'' + 6y' + 8y = (e^{-3t} - e^{-5t})u(t) \quad y(0) = 0, y'(0) = 0$$

Solution: Let us take the Laplace transform of the whole equation. Since $y(0) = 0$ and $y'(0) = 0$, we have

$$\mathcal{L}\{y'\} = sY$$

$$\mathcal{L}\{y''\} = s^2Y$$

Then the ODE becomes

$$s^2Y + 6sY + 8Y = \frac{1}{s+3} - \frac{1}{s+5}$$

$$(s^2 + 6s + 8)Y = \frac{s+5-(s+3)}{(s+3)(s+5)}$$

$$(s+4)(s+2)Y = \frac{2}{(s+3)(s+5)}$$

$$Y = \frac{2}{(s+2)(s+3)(s+4)(s+5)}$$

$$Y = \frac{\frac{1}{3}}{s+2} - \frac{1}{s+3} + \frac{1}{s+4} - \frac{\frac{1}{3}}{s+5}$$

Its inverse Laplace transform is

$$\boxed{y(t) = \left(\frac{1}{3}e^{-2t} - e^{-3t} + e^{-4t} - \frac{1}{3}e^{-5t}\right) u(t)}$$

Kreyszig, 6.4.3

Carlos Oscar Sorzano, Jan. 15th, 2015

Sketch the solution of the IVP

$$y'' + 4y = \delta(t-\pi) \quad y(0) = 8, y'(0) = 0$$

Solution: Let us take the Laplace transform of the differential equation

$$(s^2Y(s) - sy(0) - y'(0)) + 4Y(s) = e^{-s\pi}$$

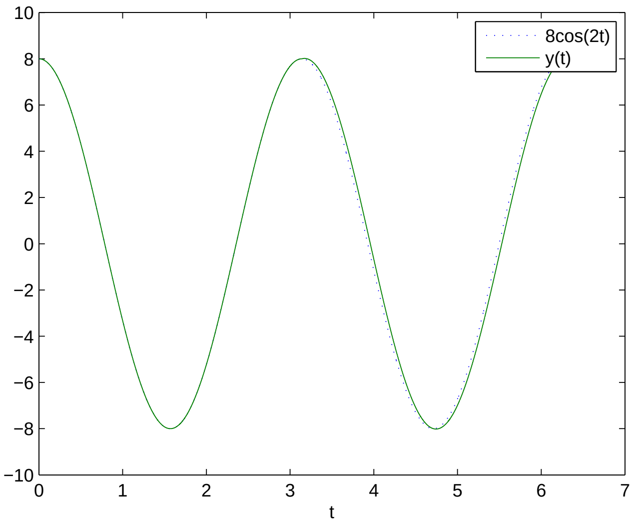
$$(s^2 + 4)Y(s) = 8s + e^{-s\pi}$$

$$Y(s) = \frac{8s}{s^2+4} + \frac{1}{s^2+4}e^{-s\pi}$$

$$Y(s) = \frac{8s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}e^{-s\pi}$$

Now we take the inverse Laplace transform

$$\boxed{y(t) = 8\cos(2t) + \frac{1}{2}\sin(2(t-\pi))u(t-\pi)}$$



Kreyszig, 6.4.10

Carlos Oscar Sorzano, June 15th, 2015

Sketch the solution of the IVP

$$y'' + 5y' + 6y = \delta\left(t - \frac{\pi}{2}\right) + u(t-\pi)\cos(t) \quad y(0) = 0, y'(0) = 0$$

Solution: We first note that

$$\cos(t) = -\cos(t - \pi)$$

Then, we can write the differential equation as

$$y'' + 5y' + 6y = \delta\left(t - \frac{\pi}{2}\right) - u(t-\pi)\cos(t-\pi)$$

Let us take the Laplace transform of the differential equation

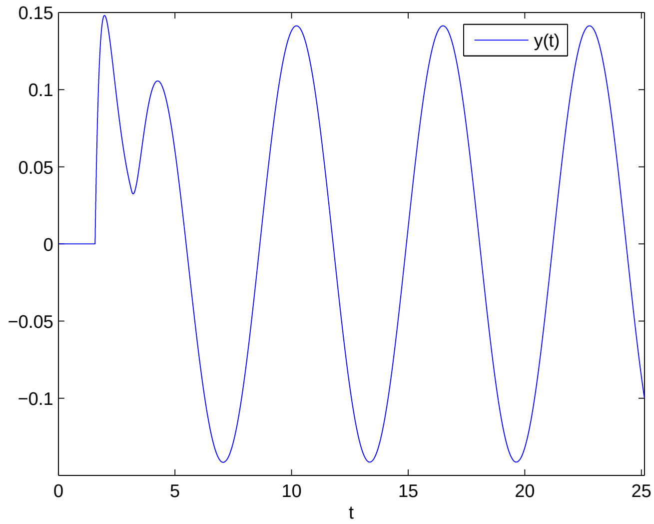
$$s^2Y(s) + 5sY(s) + 6Y(s) = e^{-\frac{\pi}{2}s} + e^{-\pi s}\frac{s}{s^2+1^2}$$

$$Y(s) = e^{-\frac{\pi}{2}s}\frac{1}{s^2+5s+6} + e^{-\pi s}\frac{s}{s^2+1^2}\frac{1}{s^2+5s+6}$$

$$Y(s) = e^{-\frac{\pi}{2}s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right) + e^{-\pi s}\left(\frac{1}{10}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1} - \frac{2}{5}\frac{1}{s+2} + \frac{3}{10}\frac{1}{s+3}\right)$$

Now we take the inverse Laplace transform

$$\boxed{\begin{array}{rcl} y(t) & = & \left(e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right)u(t-\pi/2) + \\ & & \left(\frac{\cos(t-\pi)+\sin(t-\pi)}{10} - \frac{2}{5}e^{-2(t-\pi)} + \frac{3}{10}e^{-3(t-\pi)}\right)u(t-\pi) \end{array}}$$



Kreyszig, 6.5.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the convolution of $e^{at} \star e^{bt}$ with $a \neq b$
Solution: Let us define $f(t) = e^{at}$ and $g(t) = e^{bt}$. Their convolution can be calculated thanks to the Laplace transform as

$$\begin{aligned}\mathcal{L}\{f(t) \star g(t)\} &= F(s)G(s) \\ &= \tfrac{1}{s-a}\tfrac{1}{s-b} \\ &= \tfrac{1}{a-b}\tfrac{1}{s-a} + \tfrac{1}{b-a}\tfrac{1}{s-b} \\ &= \tfrac{1}{a-b}\left(\tfrac{1}{s-a} - \tfrac{1}{s-b}\right)\end{aligned}$$

The inverse Laplace transform of this expression is

$$\boxed{f(t) \star g(t) = \frac{1}{a-b}(e^{at} - e^{bt})}$$

Kreyszig, 6.5.12

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the integral equation

$$y(t) + \int_0^t y(\tau)\cosh(t-\tau)d\tau = t + e^t$$

Solution: If we take the Laplace transform of this equation we get

$$Y + Y\frac{s}{s^2-1} = \frac{1}{s^2} + \frac{1}{s-1}$$

$$Y\left(1+\frac{s}{s^2-1}\right)=\frac{1}{s^2}+\frac{1}{s-1}$$
$$Y=\frac{\frac{1}{s^2}+\frac{1}{s-1}}{1+\frac{s}{s^2-1}}$$
$$Y=\frac{\frac{s-1+s^2}{s^2(s-1)}}{\frac{s^2-1+s}{s^2-1}}$$
$$Y=\frac{s+1}{s^2}$$

Now, we have the following inverse Laplace transforms

$$\begin{aligned}
\mathcal{L}\left\{s+1\right\} &= e^{-t} \\
\mathcal{L}\left\{\frac{s+1}{s}\right\} &= \int_0^t e^{-\tau}d\tau = 1-e^{-t} \\
\mathcal{L}\left\{\frac{s+1}{s^2}\right\} &= \int_0^t (1-e^{-\tau})d\tau = t+e^{-t}-1 = t-\sinh(t)+\cosh(t)-1
\end{aligned}$$

Finally,

$$\boxed{y=t+e^{-t}-1}$$

Kreyszig, 6.5.18

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the inverse Laplace transform of $\frac{1}{(s-a)^2}$
Solution: Let us write

$$F=\frac{1}{s-a}\frac{1}{s-a}$$

So its inverse transform is

$$f(t)=e^{-at}\star e^{-at}$$

Let us calculate the convolution

$$\begin{aligned}
e^{-at}\star e^{-at} &= \int_0^t e^{-a\tau}e^{-a(t-\tau)}d\tau \\
&= e^{-at}\int_0^t d\tau = te^{-at}
\end{aligned}$$

Finally

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\}=te^{-at}}$$

Kreyszig, 6.6.2

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Laplace transform of $3t\sinh(4t)$
Solution: We know the Laplace transform

$$\mathcal{L}\{3\sinh(4t)\}=3\frac{4}{s^2-4^2}=F(s)$$

Then, the required Laplace transform can be calculated as

$$\mathcal{L}\{3t\sinh(4t)\} = -F'(s) = -\frac{d}{ds}\left(3\frac{4}{s^2-4^2}\right) = \boxed{3\frac{8s}{(s^2-4^2)^2}}$$

Kreyszig, 6.6.3

Álvaro Martín Ramos, Jan. 11th, 2015

Find the Laplace transform of

$$\frac{1}{2}te^{-3t}$$

Solution: We know the Laplace transform of

$$\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$$

Then, the required Laplace transform can be calculated as

$$\mathcal{L}\{\frac{1}{2}te^{-3t}\} = \frac{1}{2}\mathcal{L}\{te^{-3t}\} = \frac{1}{2}(-F'(s)) = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s+3}\right) = \boxed{\frac{1}{2(s+3)^2}}$$

Kreyszig, 6.6.20

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the inverse Laplace transform of $\log\frac{s+a}{s+b}$

Solution: Let us define

$$F(s) = \log\frac{s+a}{s+b} = \log(s+a) - \log(s+b)$$

Let us calculate its derivative

$$G(s) = F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

Its inverse Laplace transform is

$$g(t) = e^{-at} - e^{-bt}$$

But we know that

$$g(t) = \mathcal{L}^{-1}\{F'(s)\} = tf(t)$$

From which

$$f(t) = \frac{g(t)}{t} = \boxed{\frac{e^{-at} - e^{-bt}}{t}}$$

Kreyszig, 6.7.2

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve the ODE system

$$\begin{array}{l} y_1' + y_2 = 0 \\ y_1 + y_2' = 2\cos(t) \end{array}$$

with $y_1(0) = 1$, $y_2(0) = 0$.
Solution: If we take the Laplace transform of both equations, we get

$$\begin{aligned} (sY_1 - y_1(0)) + Y_2 &= 0 \\ Y_1 + (sY_2 - y_2(0)) &= 2\tfrac{s}{s^2+1} \end{aligned}$$

Taking into account the initial values

$$\begin{aligned} sY_1 - 1 + Y_2 &= 0 \\ Y_1 + sY_2 &= 2\tfrac{s}{s^2+1} \end{aligned}$$

which can be rewritten as

$$\begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2\frac{s}{s^2+1} \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} &= \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2\frac{s}{s^2+1} \end{pmatrix} \\ &= \tfrac{1}{s^2-1} \begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix} \begin{pmatrix} -1 \\ 2\frac{s}{s^2+1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{s}{s^2+1} \\ \frac{1}{s^2+1} \end{pmatrix} \end{aligned}$$

Taking the inverse Laplace transform

$$\boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}}$$

Kreyszig, 6.7.3

Carlos Oscar Sorzano, Dec. 19th, 2014

Solve the ODE system

$$\begin{aligned} y_1' &= -y_1 + 4y_2 \\ y_2' &= 3y_1 - 4y_2 \end{aligned}$$

with $y_1(0) = 3$, $y_2(0) = 4$.
Solution: If we take the Laplace transform of both equations, we get

$$\begin{aligned} sY_1 - y_1(0) &= -Y_1 + 4Y_2 \\ sY_2 - y_2(0) &= 3Y_1 - 4Y_2 \end{aligned}$$

Substituting the initial values

$$\begin{aligned} sY_1 - 3 &= -Y_1 + 4Y_2 \\ sY_2 - 4 &= 3Y_1 - 4Y_2 \end{aligned}$$

which can be rewritten as

$$\begin{pmatrix} s+1 & -4 \\ -3 & s+4 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$Y_1 = \frac{\begin{vmatrix} 3 & -4 \\ 4 & s+4 \end{vmatrix}}{\begin{vmatrix} s+1 & -4 \\ 3 & s+4 \end{vmatrix}} = \frac{3s+28}{s^2+5s+16} = \frac{3s+28}{\left(s+\frac{5}{2}\right)^2 + \frac{39}{4}}$$

$$Y_2 = \frac{\begin{vmatrix} s+1 & 3 \\ -3 & 4 \end{vmatrix}}{\begin{vmatrix} s+1 & -4 \\ 3 & s+4 \end{vmatrix}} = \frac{4s+13}{s^2+5s+16} = \frac{4s+13}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}$$

Their inverse Laplace transforms are

$$\begin{aligned}
y_1(t) &= \mathcal{L}^{-1}\left\{\frac{3s+28}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{3s}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} + \mathcal{L}^{-1}\left\{\frac{28}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} \\
&= 3\mathcal{L}^{-1}\left\{\frac{s}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} + 28\frac{2}{\sqrt{39}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{39}}{2}}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} \\
&= 3\cos\left(\frac{\sqrt{39}}{2}t\right) + \frac{56}{\sqrt{39}}\sin\left(\frac{\sqrt{39}}{2}t\right)
\end{aligned}$$

$$\begin{aligned}
y_2(t) &= \mathcal{L}^{-1}\left\{\frac{4s+13}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{4s}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} + \mathcal{L}^{-1}\left\{\frac{13}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} \\
&= 4\mathcal{L}^{-1}\left\{\frac{s}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} + 13\frac{2}{\sqrt{39}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{39}}{2}}{\left(s+\frac{5}{2}\right)^2+\frac{39}{4}}\right\} \\
&= 4\cos\left(\frac{\sqrt{39}}{2}t\right) + \sqrt{\frac{26}{3}}\sin\left(\frac{\sqrt{39}}{2}t\right)
\end{aligned}$$

7 Chapter 11

Kreyszig, 11.1.14

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Fourier series of the function x^2 (between $-\pi < x < \pi$) which is assumed to be periodic outside with period 2π

Solution: Since x^2 is an even function in the domain $-\pi < x < \pi$, we only need to compute the a_0 and a_n terms, since the b_n terms will all be 0.

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{\pi^2}{3} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)dx = \frac{1}{\pi}\left.\frac{(n^2x^2-2)\sin(nx)+2nx\cos(nx)}{n^3}\right|_{-\pi}^{\pi} \\
&= \frac{4}{n^2}\cos(\pi n) = (-1)^n\frac{4}{n^2}
\end{aligned}$$

Finally, the Fourier series of x^2 between $-\pi < x < \pi$ is

$$\boxed{x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos(nx)}$$

Kreyszig, 11.1.15

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Fourier series of the function x^2 (between $0 < x < 2\pi$) which is assumed to be periodic outside with period 2π

Solution: x^2 is not an even or odd function in the domain $0 < x < 2\pi$, so we need to compute all the terms of the Fourier series

$$\begin{aligned}
a_0 &= \tfrac{1}{2\pi}\int_0^{2\pi} x^2 dx = \tfrac{1}{2\pi}\left.\tfrac{x^3}{3}\right|_0^{2\pi} = \tfrac{8\pi^2}{3} \\
a_n &= \tfrac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)dx = \tfrac{1}{\pi}\left.\tfrac{(n^2x^2-2)\sin(nx)+2nx\cos(nx)}{n^3}\right|_0^{2\pi} = \tfrac{4}{n^2} \\
b_n &= \tfrac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)dx = \tfrac{1}{\pi}\left.\tfrac{(2-n^2x^2)\cos(nx)+2nx\sin(nx)}{n^3}\right|_0^{2\pi} = -\tfrac{4\pi}{n}
\end{aligned}$$

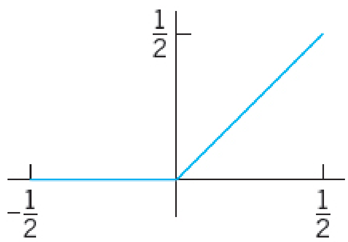
Finally, the Fourier series of x^2 between $0 < x < 2\pi$ is

$$\boxed{x^2 = \frac{8\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx) - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin(nx)}$$

Kreyszig, 11.2.13

Carlos Oscar Sorzano, Jan. 15th, 2015

Calculate the Fourier series of period $p = 1$ of the function below



Solution: We apply the definition of the different coefficients, where $L = \frac{1}{2}$

$$\begin{aligned}
a_0 &= \tfrac{1}{2L}\int_{-L}^{L} f(x) = \int_0^{\frac{1}{2}} x dx = \left.\tfrac{x^2}{2}\right|_0^{\frac{1}{2}} = \tfrac{1}{8} \\
a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{n\pi}{L}x\right)dx = 2\int_0^{\frac{1}{2}} x\cos(2n\pi x)\,dx \\
&= 2\left.\tfrac{2\pi nx\sin(2\pi nx)+\cos(2\pi nx)}{4\pi^2n^2}\right|_0^{\frac{1}{2}} = 2\left(\tfrac{\cos(\pi n)}{4\pi^2n^2} - \tfrac{1}{4\pi^2n^2}\right) \\
&= \tfrac{(-1)^n-1}{2\pi^2n^2} = \begin{cases} 0 & n = \dot{2} \\ -\frac{1}{\pi^2n^2} & n \neq \dot{2} \end{cases} \\
b_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{n\pi}{L}x\right)dx = 2\int_0^{\frac{1}{2}} x\sin(2n\pi x)\,dx \\
&= 2\left.\tfrac{\sin(2\pi nx)-2\pi nx\cos(2\pi nx)}{4\pi^2n^2}\right|_0^{\frac{1}{2}} = 2\left(\tfrac{-\pi n\cos(\pi n)}{4\pi^2n^2} - 0\right) \\
&= \tfrac{(-1)^{n+1}}{2\pi n}
\end{aligned}$$

The Fourier series is, then

$$
\boxed{y(x)} = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\tfrac{\pi n}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{\pi n}{L}x\right)
$$
$$
= \boxed{\frac{1}{8} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{2\pi^2 n^2} \cos(2\pi n x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2\pi n} \sin(2\pi n x)}
$$

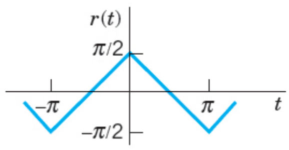
Kreyszig, 11.3.4

Carlos Oscar Sorzano, Aug. 31st, 2014

Let us assume we have mass-spring system responding to the ODE

$$my'' + cy' + ky = r(t)$$

Let $r(t)$ be the function



The solution can be expressed as

$$y_n = C_0 + \sum_{n=1}^{\infty} C_n \cos(nt + \delta_n)$$

What happens if we replace $r(t)$ with its derivative, the rectangular wave? What is the ratio of the new C_n to the old ones?
Solution: Let us consider the Fourier series of the input function $r(t)$

$$r(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(nt + \theta_n)$$

Its derivative, assuming the series is convergent, can be calculated as

$$r'(t) = \sum_{n=1}^{\infty} (-c_n n \sin(nt + \theta_n)) = \sum_{n=1}^{\infty} c_n n \cos(nt + \theta_n + \tfrac{\pi}{2})$$

Since the parameters m, c and k are constant, then the system is linear. For this reason, for the input $r(t)$, each harmonic of the input excites the corresponding harmonic of the output, that is

$$c_n \rightarrow C_n$$

If now the amplitude of the input is nc_n, then amplitude of the output is nC_n

$$\boxed{nc_n \rightarrow nC_n}$$

Kreyszig, 11.3.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Find a general solution of the ODE

$$y'' + \omega^2 y = \sin(\alpha t) + \sin(\beta t)$$

with $\omega^2 \neq \alpha^2, \beta^2$.
Solution: The general solution of the homogeneous equation is

$$y_h = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

For the particular solution, we look for a solution of the form

$$y_p = a \sin(\alpha t) + b \cos(\alpha t) + A \sin(\beta t) + B \cos(\beta t)$$

$$y_p' = a\alpha \cos(\alpha t) - b\alpha \sin(\alpha t) + A\beta \cos(\beta t) - B\beta \sin(\beta t)$$

$$y_p'' = -a\alpha^2 \sin(\alpha t) - b\alpha^2 \cos(\alpha t) - A\beta^2 \sin(\beta t) - B\beta^2 \cos(\beta t)$$

Substituting into the differential equation, we get

$$\begin{gathered}[-a\alpha^2 \sin(\alpha t) - b\alpha^2 \cos(\alpha t) - A\beta^2 \sin(\beta t) - B\beta^2 \cos(\beta t)]+ \\ \omega^2[a \sin(\alpha t) + b \cos(\alpha t) + A \sin(\beta t) + B \cos(\beta t)] = \\ \sin(\alpha t) + \sin(\beta t)\end{gathered}$$

or what is the same

$$\begin{gathered}a(\omega^2 - \alpha^2) \sin(\alpha t) + b(\omega^2 - \alpha^2) \cos(\alpha t)+ \\ +A(\omega^2 - \beta^2) \sin(\beta t) + B(\omega^2 - \beta^2) \cos(\beta t) = \\ \sin(\alpha t) + \sin(\beta t)\end{gathered}$$

Equating coefficients we find that

$$b = B = 0$$

$$a = \frac{1}{\omega^2 - \alpha^2}$$

$$A = \frac{1}{\omega^2 - \beta^2}$$

The general solution of the equation is

$$\boxed{y = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{1}{\omega^2 - \alpha^2} \sin(\alpha t) + \frac{1}{\omega^2 - \beta^2} \sin(\beta t)}$$

Kreyszig, 11.3.11

Carlos Oscar Sorzano, June 15th, 2015

Find the eigenvalue and eigenfunctions of

$$\left(\frac{y'}{x}\right)' + (\lambda + 1)\frac{y}{x^3} = 0 \quad y(1) = 0, y(e^\pi) = 0$$

The following strategy is suggested:

- Do the change of variable $x = e^t$.
- Find the general solution without considering boundary constraints.
- Apply the boundary conditions to find the eigenvalues and eigenfunctions of the ODE.

Solution: 1) Change of variable
Let us analyze the change of variable

$$x = e^t \Rightarrow t = \log(x)$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \dot{y}\frac{1}{x} = \dot{y}e^{-t}$$

Substituting in the differential equation:

$$\frac{d\left(\frac{\dot{y}e^{-t}}{e^t}\right)}{dt}e^{-t} + (\lambda+1)\frac{y}{e^{3t}} = 0$$

$$\frac{d\left(\dot{y}e^{-2t}\right)}{dt}e^{-t} + (\lambda+1)ye^{-3t} = 0$$

$$\left(\ddot{y}e^{-2t} - 2\dot{y}e^{-2t}\right)e^{-t} + (\lambda+1)ye^{-3t} = 0$$

$$\ddot{y} - 2\dot{y} + (\lambda+1)y = 0 \quad y(0) = 0 = y(\pi)$$

2) Find the general solution
The general solution of this equation (without considering the boundary constraints) is given by the roots of the polynomial

$$s^2 - 2s + (\lambda+1) = 0 \rightarrow s = 1 \pm \sqrt{-\lambda}$$

That is

$$y(t) = e^t\left(C_1e^{-\sqrt{-\lambda}t} + C_2e^{\sqrt{-\lambda}t}\right)$$

3) Find the eigenvalues and eigenfunctions
Case $\lambda < 0$: If $\lambda < 0$, then $\sqrt{-\lambda} > 0$ and the boundary conditions imply

$$\left.\begin{array}{rcl} y(0) &=& C_1 + C_2 = 0 \\ y(\pi) &=& e^{\pi}(C_1e^{-\sqrt{-\lambda}\pi} + C_2e^{\sqrt{-\lambda}\pi}) = 0 \end{array}\right\} \Rightarrow C_1 = C_2 = 0$$

Case $\lambda = 0$: If $\lambda = 0$, then the general solution is

$$y(t) = e^t(C_1 + C_2t)$$

The boundary conditions imply

$$\left.\begin{array}{rcl} y(0) &=& C_1 = 0 \\ y(\pi) &=& e^{\pi}(C_1 + C_2\pi) = 0 \end{array}\right\} \Rightarrow C_1 = C_2 = 0$$

Case $\lambda > 0$: Then the general solution becomes

$$y(t) = e^t(C_1\cos(\sqrt{\lambda}t) + C_2\sin(\sqrt{\lambda}t)$$

From the boundary conditions we get

$$\left.\begin{array}{rcl} y(0) &=& C_1 = 0 \\ y(\pi) &=& e^{\pi}(-C_1) = 0 \end{array}\right\} \Rightarrow C_1 = 0$$

Consequently, the eigenfunctions are the functions of the form

$$y(t) = e^t \sin(\sqrt{\lambda} t)$$

and the associated eigenvalue is λ. Undoing the change of variable we get the eigenfunctions

$$\boxed{y(x) = x \sin(\sqrt{\lambda} \log(x))}$$

Kreyszig, 11.5.6

Carlos Oscar Sorzano, Aug. 31st, 2014

Tranformation to Sturm–Liouville form. Show that

$$y'' + fy' + (g + \lambda h)y = 0$$

takes the form

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0$$

if you set $p = \exp(\int f dx)$, $q = pg$ and $r = hp$. Why would you do such a transformation?

Solution: Let us substitute the proposed functions into the Sturm-Liouville form

$$(py')' + (pg + \lambda hp)y = 0$$

$$p'y' + py'' + (pg + \lambda hp)y = 0$$

Note that

$$p' = f \exp\left(\int f dx\right) = fp$$

Then

$$fpy' + py'' + (pg + \lambda hp)y = 0$$

Note that p is never 0, then dividing by p

$$fy' + y'' + (g + \lambda h)y = 0$$

$$y'' + fy' + (g + \lambda h)y = 0$$

that is the original ODE.

Kreyszig, 11.5.9

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$y'' + \lambda y = 0 \quad y(0) = 0, y'(L) = 0$$

Solution: We can rewrite the ODE as

$$(y')' + \lambda y = 0$$

with the constraints

$$1y(0) + 0y'(0) = 0$$

$$0y(L) + 1y'(L) = 0$$

That is, this is a Sturm-Liouville problem.

If $\lambda < 0$, $\lambda = -\nu^2$, then the general solution is

$$y = c_1 e^{\nu x} + c_2 e^{-\nu x}$$

Imposing the two boundary conditions

$$\begin{array}{rcl} y(0) = 0 & = & c_1 + c_2 \\ y'(L) = 0 & = & c_1 \nu e^{\nu L} - c_2 \nu e^{-\nu L} \end{array}$$

Its unique solution is $c_1 = c_2 = 0$.

If $\lambda = 0$, then the general solution is

$$y = c_1 + c_2 x$$

Imposing the two boundary conditions

$$\begin{array}{rcl} y(0) = 0 & = & c_1 \\ y'(L) = 0 & = & c_2 \end{array}$$

If $\lambda > 0$, $\lambda = \nu^2$, then the general solution is

$$y = c_1 \cos(\nu x) + c_2 \sin(\nu x)$$

Imposing the two boundary conditions

$$\begin{array}{rcl} y(0) = 0 & = & c_1 \\ y'(L) = 0 & = & -c_1 \sin(\nu L) + c_2 \nu \cos(\nu L) = c_2 \nu \cos(\nu L) \Rightarrow \nu L = \frac{\pi}{2} + \pi k \Rightarrow \nu = \frac{\pi + 2\pi k}{2L} \end{array}$$

That is the functions

$$\boxed{y_\nu = \sin(\nu x) \quad \nu = \frac{\pi + 2\pi k}{2L}}$$

are the eigenfunctions of the Sturm-Liouville problem and their eigenvalues are $\lambda = \nu^2$.

Kreyszig, 11.5.11

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$\left(\frac{y'}{x}\right)' + (\lambda + 1)\frac{y}{x^3} = 0 \quad y(1) = 0, y(e^\pi) = 0$$

(Set $x = e^t$).

Solution: If we make the change of variable $x = e^t \Rightarrow t = \log(x)$, then

$$\begin{array}{rcl} y' & = & \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x} = \frac{dy}{dt}e^{-t} \\ y'' & = & \frac{dy'}{dx} = \frac{dy'}{dt}\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dt}e^{-t}\right)\frac{1}{x} = \left(\frac{d^2y}{dt^2}e^{-t} - \frac{dy}{dt}e^{-t}\right)e^{-t} = \frac{d^2y}{dt^2}e^{-2t} - \frac{dy}{dt}e^{-2t} \end{array}$$

We note that

$$\left(\frac{y'}{x}\right)' = \frac{y''x - y'}{x^2} = \frac{\left(\frac{d^2y}{dt^2}e^{-2t} - \frac{dy}{dt}e^{-2t}\right)e^t - \frac{dy}{dt}e^{-t}}{e^{2t}} = \frac{d^2y}{dt^2}e^{-3t} - 2\frac{dy}{dt}e^{-3t}$$

Then we can rewrite the problem as a function of $y(t)$

$$\left(\frac{d^2y}{dt^2}e^{-3t} - 2\frac{dy}{dt}e^{-3t}\right) + (\lambda+1)ye^{-3t} = 0 \quad y(0) = 0, y(\pi) = 0$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + (\lambda+1)y = 0 \quad y(0) = 0, y(\pi) = 0$$

We now check if the problem is a Sturm-Liouville problems using Kreyszig 11.5.6 with $f = -2$, $g = 1$, $h = 1$. We calculate

$$p = e^{\int(-2)dt} = e^{-2t}$$

$$q = pg = e^{-2t}$$

$$h = hp = e^{-2t}$$

So, in the Sturm-Liouville form the problem becomes

$$\frac{d}{dt}\left(e^{-2t}\frac{dy}{dt}\right) + (e^{-2t} + \lambda e^{-2t})y = 0$$

We go back to the problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + (\lambda+1)y = 0$$

and look for solutions of the form $y = e^{st}$

$$s^2 - 2s + (\lambda+1) = 0 \Rightarrow s = 1 \pm \sqrt{-\lambda}$$

If $\lambda < 0$, $\lambda = -\nu^2$, then the general solution is of the form

$$y = c_1e^{s_1t} + c_2e^{s_2t}$$

with $s_1 = 1+\nu$ and $s_2 = 1-\nu$. Imposing the two boundary conditions

$$\begin{aligned} y(0) = 0 &= c_1 + c_2 \\ y(\pi) = 0 &= c_1e^{s_1\pi} + c_2e^{s_2\pi} \end{aligned}$$

whose unique solution is $c_1 = c_2 = 0$.

If $\lambda = 0$, then the general solution is of the form

$$y = c_1e^t + c_2te^t$$

Imposing the two boundary conditions

$$\begin{aligned} y(0) = 0 &= c_1 \\ y(\pi) = 0 &= c_1e^{\pi} + c_2\pi e^{\pi} \end{aligned}$$

whose unique solution is $c_1 = c_2 = 0$.

If $\lambda > 0$, $\lambda = \nu^2$, then the general solution is of the form

$$y = c_1 e^t \cos(\nu t) + c_2 e^t \sin(\nu t)$$

Imposing the two boundary conditions

$$\begin{aligned} y(0) = 0 &= c_1 \\ y(\pi) = 0 &= c_2 e^{\pi} \sin(\nu \pi) \Rightarrow \nu = k \end{aligned}$$

So, all the functions of the form

$$y = e^t \sin(kt) = e^{\log x} \sin(k \log(x)) = \boxed{x \sin(k \log(x)) \quad k \in \mathbb{Z}}$$

are eigenfunctions of the Sturm-Liouville problem, and their associated eigenvalue is $\lambda = k^2$.

Kreyszig, 11.6.2

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the Fourier-Legendre series of the polynomial $(x+1)^2$

Solution: The Fourier-Legendre series of the function f is a series expansion of the form

$$f = \sum_{m=0}^{\infty} \frac{\langle f, P_m \rangle}{\|P_m\|^2} P_m(x)$$

where

$$\|P_m\|^2 = \frac{2}{2m+1}$$

and the Legendre polynomials are given by

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ (n+1)P_{n+1}(x) &= (2n+1)xP_n(x) - nP_{n-1}(x) \end{aligned}$$

In particular

$$\begin{aligned} P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\ P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \end{aligned}$$

To perform the Fourier-Legendre expansion, let us perform the following calculations

$$\begin{aligned} \left\langle (x+1)^2, P_0(x) \right\rangle &= \int_{-1}^{1} (x+1)^2 dx = \tfrac{8}{3} \\ \|P_0\|^2 &= \tfrac{2}{2\cdot 0+1} = 2 \\ \left\langle (x+1)^2, P_1(x) \right\rangle &= \int_{-1}^{1} (x+1)^2 x dx = \tfrac{4}{3} \\ \|P_1\|^2 &= \tfrac{2}{2\cdot 1+1} = \tfrac{2}{3} \\ \left\langle (x+1)^2, P_2(x) \right\rangle &= \int_{-1}^{1} (x+1)^2 \tfrac{1}{2}(3x^2 - 1) dx = \tfrac{4}{15} \\ \|P_2\|^2 &= \tfrac{2}{2\cdot 2+1} = \tfrac{2}{5} \\ \left\langle (x+1)^2, P_3(x) \right\rangle &= \int_{-1}^{1} (x+1)^2 \tfrac{1}{2}(5x^3 - 3x) dx = 0 \\ \|P_3\|^2 &= \tfrac{2}{2\cdot 3+1} = \tfrac{2}{7} \end{aligned}$$

Actually, since f is a polynomial of degree 2, and Legendre polynomials are a basis of polynomials in the domain $[-1, 1]$, we have that all coefficients for $m \geq 3$ are 0 ($\left\langle (x+1)^2, P_m(x) \right\rangle = 0$).

Finally, the Fourier-Legendre expansion is

$$\begin{aligned}
(x+1)^2 &= \frac{\left\langle (x+1)^2, P_0(x) \right\rangle}{\|P_0\|^2} P_0 + \frac{\left\langle (x+1)^2, P_1(x) \right\rangle}{\|P_1\|^2} P_1 + \frac{\left\langle (x+1)^2, P_2(x) \right\rangle}{\|P_2\|^2} P_2 \\
&= \frac{\frac{8}{3}}{2} + \frac{\frac{4}{3}}{\frac{2}{3}} x + \frac{\frac{4}{15}}{\frac{2}{5}} \frac{1}{2}(3x^2 - 1) \\
&= \boxed{\frac{4}{3} + 2x + \frac{2}{3}\frac{1}{2}(3x^2 - 1)}
\end{aligned}$$

Kreyszig, 11.9.6

Carlos Oscar Sorzano, Dec. 19th, 2014

Find the Fourier transform of $f(x) = e^{-|x|} \quad (-\infty < x < \infty)$ by integration.

Solution: The definition of the Fourier transform is

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Substituting in this formula the value of f, we have

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x} dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{-i\omega x} dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1+i\omega)x} dx \\
&= \frac{2}{\sqrt{2\pi}} \left. \frac{e^{-(1+i\omega)x}}{-(1+i\omega)} \right|_0^{\infty} \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+i\omega}
\end{aligned}$$

8 Chapter 12

Kreyszig, 12.1.2

Carlos Oscar Sorzano, Aug. 31st, 2014

Verify that the function

$$u = x^2 + t^2$$

is a solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

for a suitable c.

Solution: Let us calculate the different partial derivatives needed to substitute in the wave equation

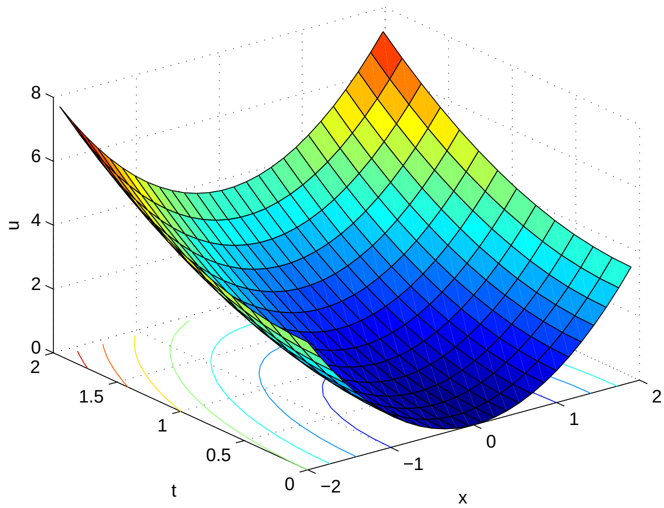
$$\begin{aligned} u_t &= 2t \\ u_{tt} &= 2 \\ u_x &= 2x \\ u_{xx} &= 2 \end{aligned}$$

The wave equation states

$$2 = c^2 2$$

which is true for $c = 1$. In Matlab:

```
[x,t]=meshgrid(-2:0.15:2,0:0.15:2);
u=x.^2+t.^2;
surfc(x,t,u)
xlabel('x'); ylabel('t'); zlabel('u')
```



Kreyszig, 12.1.5

Carlos Oscar Sorzano, Aug. 31st, 2014

Verify that the function

$$u = \sin(at)\sin(bx)$$

is a solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

for a suitable c.

Solution: Let us calculate the different partial derivatives needed to substitute in the wave equation

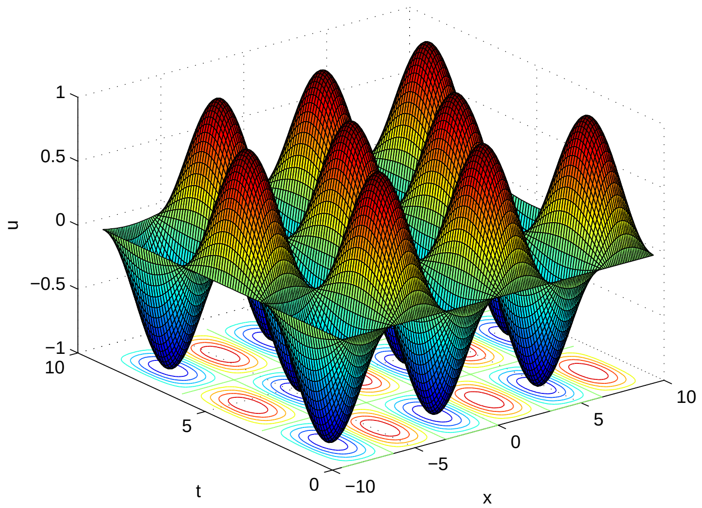
$$\begin{aligned} u_t &= a\cos(at)\sin(bx) \\ u_{tt} &= -a^2\sin(at)\sin(bx) \\ u_x &= b\sin(at)\cos(bx) \\ u_{xx} &= -b^2\sin(at)\sin(bx) \end{aligned}$$

The wave equation states

$$-a^2 \sin(at) \sin(bx) = c^2(-b^2 \sin(at) \sin(bx))$$

which is true for $c = \frac{a^2}{b^2}$. In Matlab:

```
[x,t]=meshgrid(-3*pi:0.1:3*pi,0:0.1:3*pi);
u=sin(t).*sin(x);
surfc(x,t,u)
xlabel('x'); ylabel('t'); zlabel('u')
```



Kreyszig, 12.1.19

Carlos Oscar Sorzano, Aug. 31st, 2014

Solve

$$u_y + y^2 u = 0$$

Solution: Since the PDE is only depending on y, we can treat x as if it were a parameter, then we can solve the PDE as if it were an ODE on y

$$u_y = -y^2 u$$

$$\frac{du}{u} = -y^2$$

$$\log |u| = -\frac{y^3}{3} + C(x)$$

$$\boxed{u = C(x)\exp\left(-\frac{y^3}{3}\right)}$$

Kreyszig, 12.4.11

Carlos Oscar Sorzano, Aug. 31st, 2014

Find the type, transform to normal form and solve

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

Solution: The prototypical equation for the method of characteristics is of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

which corresponds to the equation in the problem with

$$A = B = C = 1$$

Consequently,

$$AC - B^2 = (1)(1) - 1^2 = 0$$

that is, the PDE is a parabolic PDE. Its characteristic equation is

$$A(y')^2 - 2By' + C = 0$$

$$(y')^2 - 2y' + 1 = 0$$

$$(y' - 1)^2 = 0$$

whose solution is

$$y = c_1 + x \Rightarrow \Psi(x, y) = y - x = c_1$$

We now do the change of variables

$$v = x$$

$$w = y - x$$

The standard form of a parabolic PDE is

$$u_{ww} = 0$$

Integrating in w we have

$$u_w = \phi(w)$$

Integrating again in w

$$u = \int \phi(w)dw + \psi(w) = \zeta(w) + \psi(w) = \eta(w)$$

Undoing the change of variable

$$\boxed{u = \eta(y - x)}$$

being η any function.

Kreyszig, 12.4.19

Carlos Oscar Sorzano, Aug. 31st, 2014

Longitudinal Vibrations of an Elastic Bar or Rod. These vibrations in the direction of the x-axis are modeled by the wave equation

$$u_{tt} = c^2 u_{xx}$$

with $c^2 = \frac{E}{\rho}$ (see Tolstov [C9], p. 275). If the rod is fastened at one end, $x = 0$, and free at the other, $x = L$, we have $u(0,t) = 0$ and $u_x(L,t) = 0$. Show that the motion corresponding to initial displacement $u(x,0) = f(x)$ and initial velocity zero is

$$u = \sum_{n=0}^{\infty} A_n \sin(p_n x)\cos(p_n ct)$$

with

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(p_n x)dx$$

and

$$p_n = \frac{(2n+1)\pi}{2L}$$

Solution: Let us first check that the suggested solution satisfies the boundary conditions:

$\underline{u(0,t) = 0}$

$$u(0,t) = \sum_{n=0}^{\infty} A_n \sin(p_n 0)\cos(p_n ct) = 0$$

$\underline{u_x(L,t) = 0}$

$$\begin{aligned} u_x &= \sum_{n=0}^{\infty} A_n p_n \cos(p_n x)\cos(p_n ct) \\ u_x(L,t) &= \sum_{n=0}^{\infty} A_n p_n \cos(p_n L)\cos(p_n ct) = 0 \end{aligned}$$

But

$$p_n L = \frac{(2n+1)\pi}{2L}L = \frac{(2n+1)\pi}{2}$$

that is

$$p_n L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

and

$$\cos(p_n L) = 0 \Rightarrow u_x(L,t) = 0$$

Let us check now the initial conditions $\underline{u(x,0) = f(x)}$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \sin(p_n x)\cos(p_n c0) = \sum_{n=0}^{\infty} A_n \sin(p_n x)$$

That is $u(x,0)$ is a Fourier sine series, but A_n are precisely the corresponding Fourier coefficients, so the series add up to $f(x)$.

Let us check now that u is a solution of the PDE

$$\begin{aligned} u_t &= -c\sum_{n=0}^{\infty} A_n p_n \sin(p_n x)\sin(p_n ct) \\ u_{tt} &= -c^2\sum_{n=0}^{\infty} A_n p_n^2 \sin(p_n x)\cos(p_n ct) \\ u_x &= \sum_{n=0}^{\infty} A_n p_n \cos(p_n x)\cos(p_n ct) \\ u_{xx} &= -\sum_{n=0}^{\infty} A_n p_n^2 \sin(p_n x)\cos(p_n ct) \end{aligned}$$

The PDE states

$$u_{tt} = c^2 u_{xx}$$

$$-c^2 \sum_{n=0}^{\infty} A_n p_n^2 \sin(p_n x)\cos(p_n ct) = c^2 \left(-\sum_{n=0}^{\infty} A_n p_n^2 \sin(p_n x)\cos(p_n ct) \right)$$

The equation above is obviously true, so the proposed function is a solution of the PDE and it satifies the boundary conditions.

Kreyszig, 12.6.11

Carlos Oscar Sorzano, Aug. 31st, 2014

Show that for the completely insulated bar, $u_x(0,t) = 0$, $u_x(L,t) = 0$ and $u(x,0) = f(x)$ and separation of variables the solution of the heat equation

$$u_t = c^2 u_{xx}$$

gives the solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

with

$$A_0 = \frac{1}{L}\int_0^L f(x)dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Solution: Let us first check that the suggested solution satisfies the boundary conditions:

$\underline{u_x(0,t) = 0}$

$$\begin{aligned} u_x &= -\sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} \\ u_x(0,t) &= -\sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sin\left(\frac{n\pi 0}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} = 0 \end{aligned}$$

$\underline{u_x(L,t) = 0}$

$$u_x(L,t) = -\sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sin\left(\frac{n\pi L}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} = 0$$

because

$$\sin\left(\frac{n\pi L}{L}\right) = \sin(n\pi) = 0$$

Let us check now the initial conditions $\underline{u(x,0) = f(x)}$

$$\begin{aligned} u(x,0) &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 0} \\ &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \end{aligned}$$

That is $u(x,0)$ is a Fourier cosine series, but A_n are precisely the corresponding Fourier coefficients, so the series add up to $f(x)$.

Let us check now that u is a solution of the PDE

$$\begin{aligned} u_t &= -\sum_{n=1}^{\infty} A_n \cos\left(\tfrac{n\pi x}{L}\right)\left(\tfrac{cn\pi}{L}\right)^2 e^{-\left(\frac{cn\pi}{L}\right)^2 t} \\ u_x &= -\sum_{n=1}^{\infty} A_n \tfrac{n\pi}{L} \sin\left(\tfrac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} \\ u_{xx} &= -\sum_{n=1}^{\infty} A_n \left(\tfrac{n\pi}{L}\right)^2 \cos\left(\tfrac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t} \end{aligned}$$

The PDE states

$$u_t = c^2 u_{xx}$$

$$-\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\left(\frac{cn\pi}{L}\right)^2 e^{-\left(\frac{cn\pi}{L}\right)^2 t} = c^2\left(-\sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}\right)$$

The equation above is obviously true, so the proposed function is a solution of the PDE and it satifies the boundary conditions.

Kreyszig, 12.7.3

Carlos Oscar Sorzano, Aug. 31st, 2014

Using

$$u(x,t) = \int_0^{\infty} (A_p \cos(px) + B_p \sin(px)) e^{-c^2p^2t} dp$$

with

$$A_p = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(pv)dv \quad B_p = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(pv)dv$$

solve the 1D heat equation

$$u_t = c^2 u_{xx}$$

when

$$u(x,0) = f(x) = \frac{1}{1+x^2}$$

Solution: We simply need to substitute $f(x) = \frac{1}{1+x^2}$ in the formulas for A_p and B_p

$$\begin{aligned} A_p &= \tfrac{1}{\pi}\int_{-\infty}^{\infty} \tfrac{1}{1+v^2}\cos(pv)dv = \tfrac{1}{\pi}(\pi e^{-|p|}) = e^{-|p|} \\ B_p &= \tfrac{1}{\pi}\int_{-\infty}^{\infty} \tfrac{1}{1+v^2}\sin(pv)dv = 0 \end{aligned}$$

So the solution of the 1D heat problem is

$$\boxed{u(x,t) = \int_0^{\infty} e^{-|p|}\cos(px) e^{-c^2p^2t} dp}$$