Chapter 1. First-order ODEs

C.O.S. Sorzano

Biomedical Engineering

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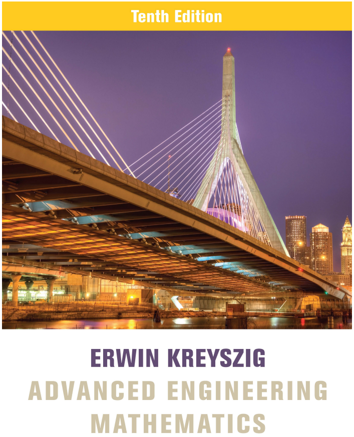
Outline

1. First-order ODEs
   - Basic concepts
   - Geometric meaning, direction fields
   - Separable ODEs
   - Exact ODEs. Integrating factors.
   - Linear ODEs. Bernouilli equation. Population dynamics.
   - Orthogonal trajectories
   - Existence and uniqueness of IVPs

References



E. Kreyszig. Advanced Engineering Mathematics. John Wiley & sons. Chapter 1.

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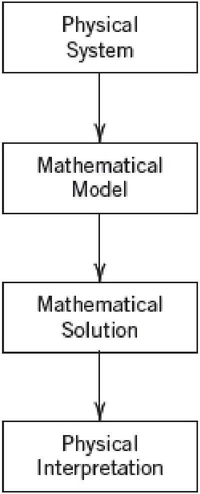
Modeling



Modeling workflow

A mathematical model is an equation that helps us to understand a physical process.
A first-order **Ordinary Differential Equation** is an equation of the form

$$F(y', y, x) = 0 \tag{1}$$

Examples

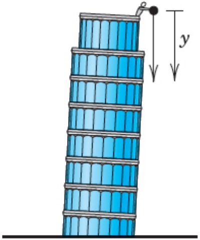
$$\begin{aligned} y' &= \cos(x)(\text{1st order}) \\ y'' + 9y &= e^{-2x}(\text{2nd order}) \\ y'y''' - \tfrac{3}{2}(y')^2 &= 0(\text{3rd order}) \end{aligned} \tag{2}$$

Drug concentration in plasma

$$C' = -K_e C \Rightarrow C(t) = C(0)e^{-K_e t}u(t) \tag{3}$$

Modeling

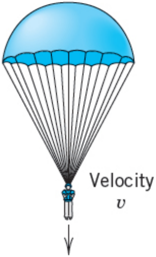
Examples



Falling stone

$y'' = g = const.$

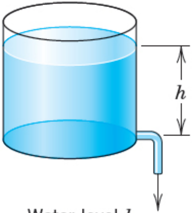
(Sec. 1.1)



Parachutist

$mv' = mg - bv^2$

(Sec. 1.2)



Water level h

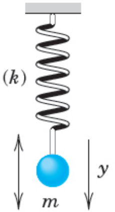
Outflowing water

$h' = -k\sqrt{h}$

(Sec. 1.3)

Modeling

Examples

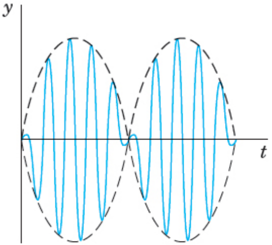


Displacement y

Vibrating mass on a spring

$my'' + ky = 0$

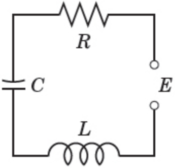
(Secs. 2.4, 2.8)



Beats of a vibrating system

$y'' + \omega_0^2 y = \cos\, \omega t, \quad \omega_0 \approx \omega$

(Sec. 2.8)



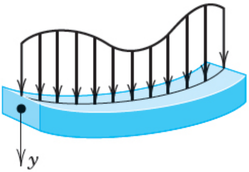
Current I in an RLC circuit

$LI'' + RI' + \frac{1}{C} I = E'$

(Sec. 2.9)

Modeling

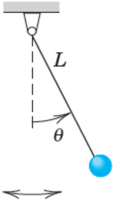
Examples



Deformation of a beam

$EIy^{iv} = f(x)$

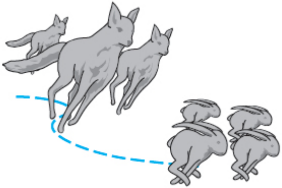
(Sec. 3.3)



Pendulum

$L\theta'' + g \sin\theta = 0$

(Sec. 4.5)



Lotka–Volterra predator–prey model

$y_1' = ay_1 - by_1y_2$

$y_2' = ky_1y_2 - ly_2$

(Sec. 4.5)

Basic concepts

Definitions

Ordinary Diff.Eq. $f(x), f', \frac{df}{dx}, \frac{d^2f}{dx^2}, \ldots$	Partial Diff. Eq. $f(x,y), f'_x, f'_y, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots$
Implicit F(x,y,y')=0	Explicit y'=f(x,y)
Order n $y^{(n)}$	

Solution

Solution

A function $y = h(x)$ is a solution of a given ODE $F(x, y, y') = 0$ on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced by h and h'.

Example

$y = \frac{c}{x}$ is a solution of $xy' = -y$.
<u>*Proof*</u>
$y' = -\frac{c}{x^2}$
$xy' = x\left(-\frac{c}{x^2}\right) = -\frac{c}{x} = -y$

Solution

Example

Solve $y' = \cos(x)$

Solution

$$\frac{dy}{dx} = \cos(x)$$
$$dy = \cos(x)dx$$
$$\int dy = \int \cos(x)dx$$
$$y = \sin(x) + C$$

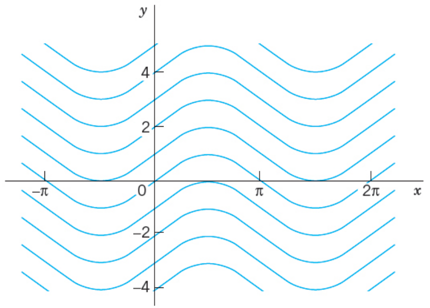


Fig. 3. Solutions $y = \sin x + c$ of the ODE $y' = \cos x$

Solution

Example

Solve $y' = ky$

Solution

$$\frac{dy}{dx} = ky$$
$$\frac{dy}{y} = kdx$$
$$\int \frac{dy}{y} = \int kdx$$
$$\log|y| = kx + C$$
$$y = e^{kx+C} = e^C e^{kx} = Ce^{kx}$$

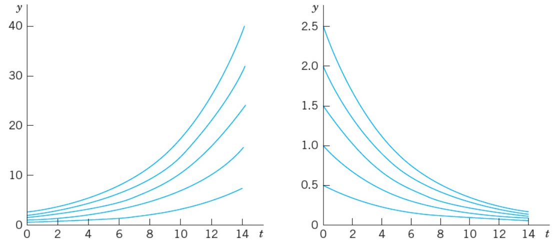


Fig. 4A. Solutions of $y' = 0.2y$ in Example 3 **(exponential growth)**

Fig. 4B. Solutions of $y' = -0.2y$ in Example 3 **(exponential decay)**

Basic concepts

Definitions

General solution $y = Ce^{kx}$	Particular solution $y = 0.2e^{kx}$
Initial value problem	$y' = f(x, y) \quad y(x_0) = y_0$

Initial value problem

$$y' = 3y \quad y(0) = 5.7$$
$$y = Ce^{3x} \quad y(0) = 5.7$$
$$Ce^{3\cdot 0} = 5.7 \Rightarrow C = 5.7$$

$$y = 5.7e^{3x}$$

Modeling

Radioactivity

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

Physical Information. Experiments show that at each instant a radioactive substance decomposes—and is thus decaying in time—proportional to the amount of substance present.

Modeling

Radioactivity

Step 1. Setting up a mathematical model of the physical process.

$$y' = -ky$$

The value of k is known from experiments for various radioactive substances (e.g., $k = 1.4 \cdot 10^{-11} s^{-1}$ approximately, for radium ${}^{226}_{88}Ra$).

Modeling

Radioactivity

Step 1. Setting up a mathematical model of the physical process.

$$y' = -ky$$

The value of k is known from experiments for various radioactive substances (e.g., $k = 1.4 \cdot 10^{-11} s^{-1}$ approximately, for radium ${}^{226}_{88}Ra$).

Radioactivity

Step 2. Mathematical solution.

$$y' = -ky \Rightarrow y = Ce^{-kt} \quad y(0) = 0.5 \Rightarrow C = 0.5$$
$$y = 0.5e^{-kt}$$

Modeling

Radioactivity

Step 2. Mathematical solution.

$$y = 0.5e^{-kt}$$

Radioactivity

Step 3. Interpretation of result.

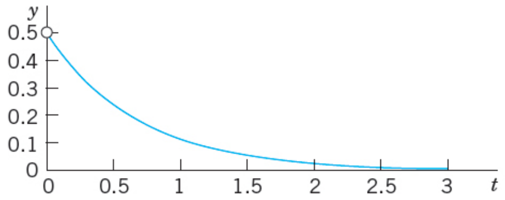


Fig. 5. Radioactivity (Exponential decay, $y = 0.5e^{-kt}$, with $k = 1.5$ as an example)

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 1, Section 1:

- 1.1.2
- 1.1.5
- 1.1.6
- 1.1.8
- 1.1.10
- 1.1.12
- 1.1.16
- 1.1.18
- 1.1.19

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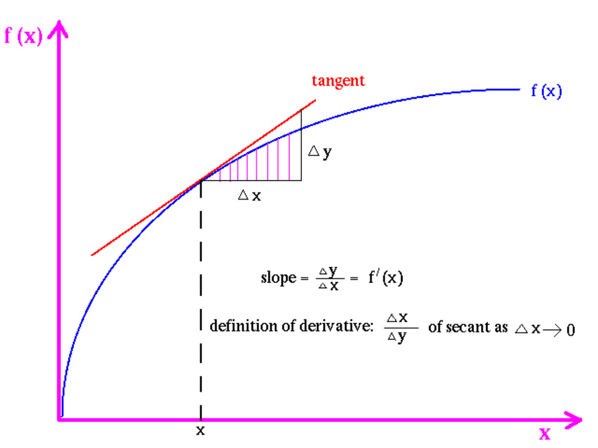
Geometric meaning

Derivative and tangent slope

$$y' = f(x, y) \Rightarrow y'(x_0) = f(x_0, y_0)$$



Geometric meaning

Direction field

$$y' = y + x$$

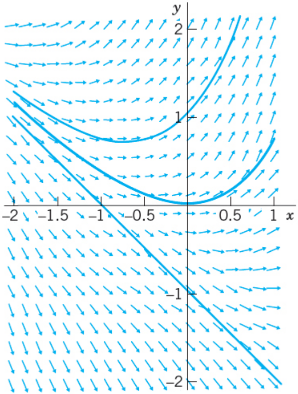


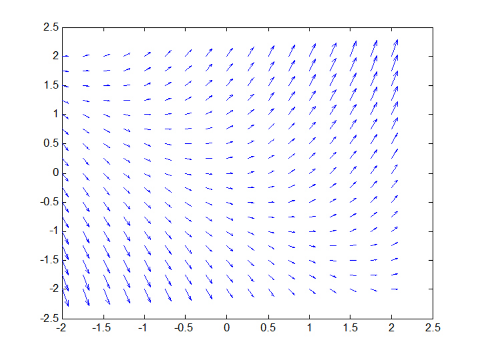
Fig. 7. Direction field of $y' = y + x$, with three approximate solution curves passing through (0, 1), (0, 0), (0, −1), respectively

Geometric meaning

Direction field

MATLAB:

```
[x,y]=meshgrid(-2:0.25:2,-2:0.25:2);
dy=y+x;
dx=ones(size(dy));
quiver(x,y,dx,dy);
```



Euler method (numerical)

Algorithm

$$y' = f(x, y)$$

$$\frac{\Delta y}{\Delta x} \approx f(x, y)$$

$$y_{k+1} = y_k + \Delta x f(x_k, y_k)$$

x_0	y_0
$x_1 = x_0 + h$	$y_1 = y_0 + \Delta x f(x_0, y_0)$
$x_2 = x_1 + h$	$y_2 = y_1 + \Delta x f(x_1, y_1)$
$x_3 = x_2 + h$	$y_3 = y_2 + \Delta x f(x_2, y_2)$
...	

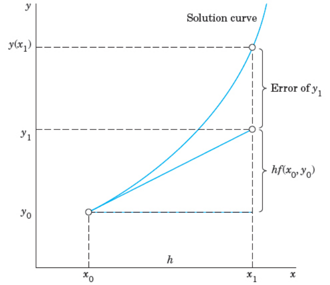
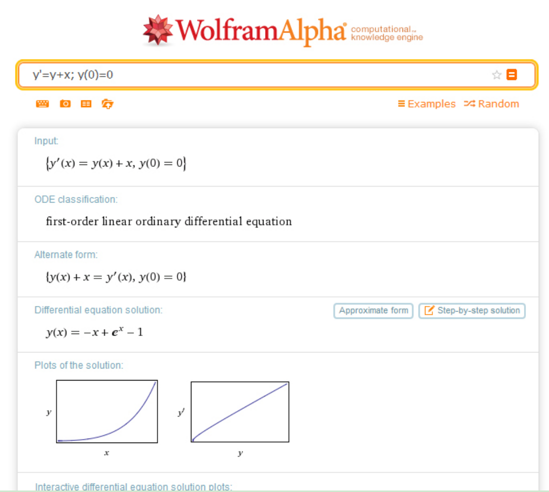


Fig. 8. First Euler step, showing a solution curve, its tangent at (x_0, y_0), step h and increment $hf(x_0, y_0)$ in the formula for y_1

Euler method (numerical)

Example (exact solution)

$$y' = y + x$$



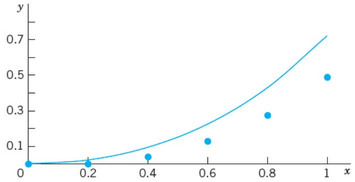
Euler method (numerical)

Example (approximate solution)

$$y' = y + x$$

Table 1.1. Euler method for $y' = y + x, y(0) = 0$ for $x = 0, \cdots, 1.0$ with step $h = 0.2$

n	x_n	y_n	$y(x_n)$	Error
0	0.0	0.000	0.000	0.000
1	0.2	0.000	0.021	0.021
2	0.4	0.04	0.092	0.052
3	0.6	0.128	0.222	0.094
4	0.8	0.274	0.426	0.152
5	1.0	0.488	0.718	0.230



ig. 9. Euler method: Approximate values in Table 1.1 and solution curve

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 1, Section 2:

- 1.2.4
- 1.2.5
- 1.2.11
- 1.2.15
- 1.2.20

Exercises

16. **CAS PROJECT. Direction Fields.** Discuss direction fields as follows.

(a) Graph portions of the direction field of the ODE (2) (see Fig. 7), for instance, $-5 \leqq x \leqq 2$, $-1 \leqq y \leqq 5$. Explain what you have gained by this enlargement of the portion of the field.

(b) Using implicit differentiation, find an ODE with the general solution $x^2 + 9y^2 = c$ $(y > 0)$. Graph its direction field. Does the field give the impression that the solution curves may be semi-ellipses? Can you do similar work for circles? Hyperbolas? Parabolas? Other curves?

(c) Make a conjecture about the solutions of $y' = -x/y$ from the direction field.

(d) Graph the direction field of $y' = -\frac{1}{2}y$ and some solutions of your choice. How do they behave? Why do they decrease for $y > 0$?

Outline

Separable ODEs

Method of separating variables

An ODE is separable if it can be written as

$$g(y)y' = f(x) \tag{4}$$

We integrate both sides with respect to x to get

$$\int g(y)y'dx = \int f(x)dx$$

But we know that

$$dy = y'dx$$

Consequently,

$$\boxed{\int g(y)dy = \int f(x)dx}$$

Separable ODEs

Example

$$y' = 1 + y^2$$

<u>Solution:</u>

$$\frac{dy}{1+y^2} = dx$$

$$\arctan(y) = x + C$$

$$y = \tan(x + C)$$

Separable ODEs

Example

$$y' = (x+1)e^{-x}y^2$$

Solution:

$$y^{-2}dy = (x+1)e^{-x}dx$$

$$\int y^{-2}dy = \int (x+1)e^{-x}dx$$

$$-y^{-1} = -(x+2)e^{-x} + C$$

$$y = \frac{1}{(x+2)e^{-x} + C}$$

Example

In September 1991 the famous Iceman (Oetzi), a mummy from the Neolithic period of the Stone Age found in the ice of the Oetztal Alps (hence the name "Oetzi") in Southern Tyrolia near the Austrian–Italian border, caused a scientific sensation. When did Oetzi approximately live and die if the ratio of carbon ${}^{14}_{6}C$ to carbon ${}^{12}_{6}C$ in this mummy is 52.5% of that of a living organism?

Physical Information. In the atmosphere and in living organisms, the ratio of radioactive carbon ${}^{14}_{6}C$ (made radioactive by cosmic rays) to ordinary carbon ${}^{12}_{6}C$ is constant. When an organism dies, its absorption of ${}^{14}_{6}C$ by breathing and eating terminates. Hence one can estimate the age of a fossil by comparing the radioactive carbon ratio in the fossil with that in the atmosphere. To do this, one needs to know the half-life of ${}^{14}_{6}C$, which is 5715 years (*CRC Handbook of Chemistry and Physics,* 83rd ed., Boca Raton: CRC Press, 2002, page 11–52, line 9).

Separable ODEs

Example (continued)

Solution:

Radioactive decay follows the model

$$y' = -ky$$

$$y^{-1}dy = -kdt$$

$$\log|y| = -kt + C$$

$$y = Ce^{-kt} = y(0)e^{-kt}$$

The half life is defined as the time, τ, at which

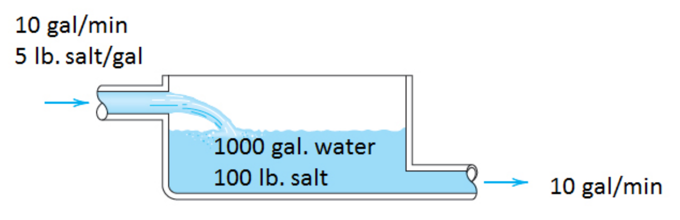
$$y(t) = \frac{1}{2}y(0) \Rightarrow \cancel{y(0)}e^{-k\tau} = \frac{1}{2}\cancel{y(0)}$$

$$-k\tau = -\log(2) \Rightarrow k = \frac{\log(2)}{\tau} = \frac{\log(2)}{5715} = 1.213 \cdot 10^{-4}[\text{years}^{-1}]$$

Separable ODEs

Example

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 11 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time *t*.



Separable ODEs

Example (continued)

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<u>Solution:</u>

$$y' = \text{Salt inflow rate} - \text{Salt outflow rate}$$

$$y' = 5\left[\frac{\text{lb}}{\text{gal}}\right] 10 \left[\frac{\text{gal}}{\text{min}}\right] - 10\left[\frac{\text{gal}}{\text{min}}\right]\frac{y}{1000}\left[\frac{\text{lb}}{\text{gal}}\right]$$

$$y' = 50 - 0.01y = -0.01(y - 5000)$$

$$\frac{dy}{y - 5000} = -0.01dt$$

$$\log|y - 5000| = -0.01t + C$$

$$y - 5000 = Ce^{-0.01t} \Rightarrow y = 5000 + Ce^{-0.01t}$$

Separable ODEs

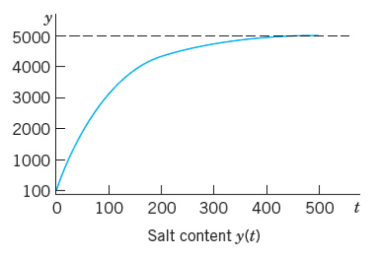
Example (continued)

Solution:

$$y = 5000 + Ce^{-0.01t} \quad y(0) = 100 \text{ [lb]}$$

$$100 = 5000 + C \Rightarrow C = -4900$$

$$y = 5000 - 4900e^{-0.01t}$$



Separable ODEs

Example

Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M. was found to be 65°F. The outside temperature was 50°F at 10 P.M. and had dropped to 40°F by 6 A.M. What was the temperature inside the building when the heat was turned on at 6 A.M.?

Physical information. Experiments show that the time rate of change of the temperature T of a body B (which conducts heat well, for example, as a copper ball does) is proportional to the difference between T and the temperature of the surrounding medium (**Newton's law of cooling).**

Solution:

$$T' = k(T - T_{out})$$

being T_{out} the temperature outside. Since there is no information about the temperature outside at any time, we take an average

$$T_{out} = \frac{50 + 40}{2} = 45[^\circ F]$$

Separable ODEs

Example (continued)

Solution: General solution

$$T' = k(T - 45)$$

$$\frac{dT}{T - 45} = kdt$$

$$\log|T - 45| = kt + C \Rightarrow T = 45 + Ce^{kt}$$

Solution: Particular solution

We choose $t = 0[h]$ at 10PM. Then, $T(0) = 70[^\circ F]$. We also know that at 2AM ($t = 4[h]$), $T(4) = 65[^\circ F]$.

$$T = 45 + Ce^{kt} \quad T(0) = 70, T(4) = 65$$

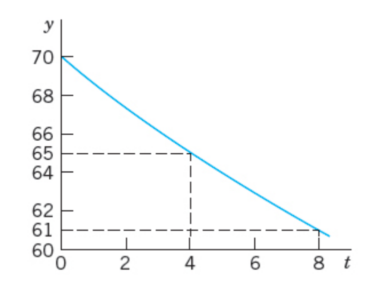
$$\left.\begin{array}{r} 70 = 45 + C \\ 65 = 45 + Ce^{k4} \end{array}\right\} \Rightarrow C = 25, k = -0.056$$

Separable ODEs

Example (continued)

Solution: Particular solution

$$T = 45 + 25e^{-0.056t}$$

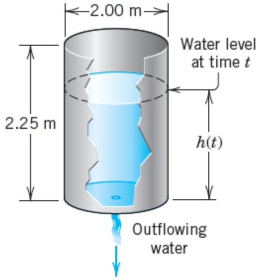


At 6AM, $t = 8[h]$, the temperature is

$$T(8) = 45 + 25e^{-0.056 \cdot 8} = 61[^{\circ}F]$$

Separable ODEs

Example



Physical information. Under the influence of gravity the outflowing water has velocity

(7) $$v(t) = 0.600\sqrt{2gh(t)}$$ **(Torricelli's law[4]),**

where $h(t)$ is the height of the water above the hole at time t, and $g = 980\ \text{cm/sec}^2 = 32.17\ \text{ft/sec}^2$ is the acceleration of gravity at the surface of the earth.

Separable ODEs

Example (continued)

Solution:

The amount of volume, ΔV outflowing by a hole of surface A in a short time Δt is

$$\Delta V = Av\Delta t$$

This volume must be equal to the change in height in the tank (of base surface B)

$$\Delta V = -B\Delta h$$

$$-B\Delta h = Av\Delta t$$

$$\frac{\Delta h}{\Delta t} = -\frac{A}{B}v$$

$$h' = -\frac{A}{B}\left(0.6\sqrt{2gh}\right)$$

Separable ODEs

Example (continued)

Solution: General solution

$$h' = -\frac{A}{B}\left(0.6\sqrt{2gh}\right)$$

$$h^{-\frac{1}{2}}dh = -\frac{A}{B}0.6\sqrt{2g}dt = -26.56\frac{A}{B}$$

$$2h^{\frac{1}{2}} = -26.56\frac{A}{B}t + C$$

$$h = \left(-13.28\frac{A}{B}t + C\right)^2$$

We have that $B = \pi R^2 = \pi(100)^2$, and $A = \pi r^2 = \pi(0.5)^2$. Substituting we have

$$h = \left(-13.28\frac{0.5^2}{100^2}t + C\right)^2 = (C - 0.000332t)^2\,[cm]$$

Separable ODEs

Example (continued)

Solution: Particular solution

$$h = (C - 0.000332t)^2$$

At $t = 0$, we have $h = 2.25[m]$

$$225 = (C - 0.000332 \cdot 0)^2 \Rightarrow C = \sqrt{225} = 15$$

$$h = (15 - 0.000332t)^2 \, [cm]$$

Separable ODEs

Reduction to separable form

An ODE that can be written as

$$y' = f\left(\frac{y}{x}\right) \tag{5}$$

We make the change of variables

$$u = \frac{y}{x}$$

$$y = ux \Rightarrow y' = u'x + u$$

Then, the ODE can be written as

$$u'x + u = f(u)$$

$$u'x = f(u) - u$$

$$\boxed{\frac{du}{f(u) - u} = \frac{dx}{x}}$$

that can now be integrated.

Separable ODEs

Example

$$2xyy' = y^2 - x^2$$

Solution:

$$y' = \frac{y^2 - x^2}{2xy} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$$

We do the change of variable $u = \frac{y}{x}$, then

$$u'x + u = \frac{1}{2}\left(u - \frac{1}{u}\right)$$

$$u'x = -u + \frac{1}{2}\left(u - \frac{1}{u}\right)$$

$$u'x = -\frac{1}{2}\left(u + \frac{1}{u}\right) = -\frac{u^2 + 1}{2u}$$

Separable ODEs

Example (continued)

Solution:

$$u'x = -\frac{1}{2}\left(u + \frac{1}{u}\right) = -\frac{u^2+1}{2u}$$

$$\frac{2u}{1+u^2}du = -\frac{dx}{x}$$

$$\log(1+u^2) = -\log|x| + C$$

$$1+u^2 = \frac{C}{x}$$

$$1+\left(\frac{y}{x}\right)^2 = \frac{C}{x}$$

$$x^2+y^2 = Cx$$

Example (continued)

Solution:

$$x^2 + y^2 = Cx$$

$$\left(x - \frac{C}{2}\right)^2 + y^2 = \frac{C^2}{4}$$

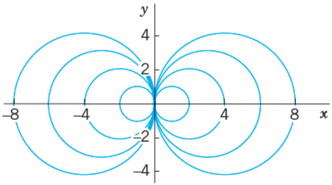


Fig. 14. General solution (family of circles) in Example 8

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 1, Section 3:

- 1.3.2
- 1.3.8
- 1.3.19
- 1.3.20
- 1.3.23
- 1.3.26

Outline

1. First-order ODEs
   - Basic concepts
   - Geometric meaning, direction fields
   - Separable ODEs
   - Exact ODEs. Integrating factors.
   - Linear ODEs. Bernouilli equation. Population dynamics.
   - Orthogonal trajectories
   - Existence and uniqueness of IVPs

Exact ODEs

Method of exact ODEs

If a function $u(x, y)$ has continuous partial derivatives, then

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \tag{6}$$

If $u(x, y) = C$, then $du = 0$.

Example

$$u = x + x^2 y^3 = c$$

$$du = (1 + 2xy^3)dx + (3x^2 y^2)dy = 0$$

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2 y^2}$$

Exact ODEs

Method of exact ODEs

A first-order ODE

$$M(x, y) + N(x, y)y' = 0$$

can be rewritten as

$$M(x, y)dx + N(x, y)dy = 0$$

This ODE is an **exact differential equation** if there is a C^1 function $u(x, y)$ such that

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

and

$$\frac{\partial u}{\partial x} = M(x, y)$$

$$\frac{\partial u}{\partial y} = N(x, y)$$

Its **implicit** solution is $u(x, y) = 0$.

Exact ODEs

Method of exact ODEs

To check whether there exists such a u function we should compute

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

Consequently, if the ODE is exact, then

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

and conversely, if the previous condition is met, then the ODE is exact.

Exact ODEs

Method of exact ODEs

We can find u by inspection or by integrating with respect to x

$$\frac{\partial u}{\partial x} = M(x,y) \Rightarrow \boxed{u(x,y) = \int M(x,y)dx + C(y)}$$

To determine $C(y)$ we differentiate with respect to y and equate it to $N(x,y)$

$$\frac{\partial}{\partial y}\left(\int M(x,y)dx + C(y)\right) = N(x,y)$$

$$\boxed{\frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + C'(y) = N(x,y)}$$

Exact ODEs

Method of exact ODEs

Alternatively, we can perform a similar approach integrating with respect to y

$$\frac{\partial u}{\partial y} = N(x,y) \Rightarrow \boxed{u(x,y) = \int N(x,y)dy + C(x)}$$

To determine $C(x)$ we differentiate with respect to y and equate it to $M(x,y)$

$$\frac{\partial}{\partial x}\left(\int N(x,y)dy + C(x)\right) = M(x,y)$$

$$\boxed{\frac{\partial}{\partial x}\left(\int N(x,y)dy\right) + C'(x) = M(x,y)}$$

Exact ODEs

Example

$$\cos(x+y)dx + (3y^2 + 2y + \cos(x+y))dy = 0$$

Solution:

Test for exactness

$$\frac{\partial \cos(x+y)}{\partial y} = -\sin(x+y)$$

$$\frac{\partial(3y^2 + 2y + \cos(x+y))}{\partial x} = -\sin(x+y)$$

Let's find a general solution

$$u = \int \cos(x+y)dx = \sin(x+y) + C(y)$$

Now we differentiate u with respect to y

$$\frac{\partial u}{\partial y} = \cos(x+y) + C'(y) = 3y^2 + 2y + \cos(x+y)$$

Exact ODEs

Example (continued)

$$\frac{\partial u}{\partial y} = \cos(x+y) + C'(y) = 3y^2 + 2y + \cos(x+y)$$

$$C'(y) = 3y^2 + 2y$$

$$C(y) = y^3 + y^2$$

Finally

$$u = \sin(x+y) + y^3 + y^2$$

and the solution

$$\sin(x+y) + y^3 + y^2 = C$$

It's an implicit solution because there is not a closed form of y as a function of x.

Exact ODEs

Example

$$(\cos(y)\sinh(x)+1)dx-\sin(y)\cosh(x)dy=0 \quad y(1)=2$$

Solution:
Test for exactness

$$\frac{\partial(\cos(y)\sinh(x)+1)}{\partial y}=-\sin(y)\sinh(x)$$

$$\frac{\partial(-\sin(y)\cosh(x))}{\partial x}=-\sin(y)\sinh(x)$$

Let's find a general solution

$$u=-\int \sin(y)\cosh(x)dy=\cos(y)\cosh(x)+C(x)$$

Now we differentiate u with respect to x

$$\frac{\partial u}{\partial x}=\cos(y)\sinh(x)+C'(x)=\cos(y)\sinh(x)+1$$

Exact ODEs

Example (continued)

$$\cos(y)\sinh(x) + C'(x) = \cos(y)\sinh(x) + 1$$

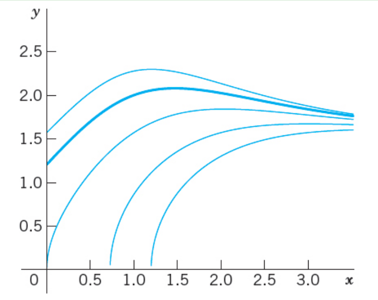
$$C'(x) = 1$$

$$C(x) = x$$

The general solution is

$$\cos(y)\cosh(x) + x = C$$

The particular solution comes from the initial condition $y(1) = 2$



$\cos(2)\cosh(1) + 1 = C \Rightarrow C = 0.358$

MATLAB: `ezplot('cos(y).*cosh(x)+x-0.358',[0 3 0 3])`

Integrating factors

Example

The equation

$$-ydx + xdy = 0$$

is not exact, but it becomes exact if we multiply by $\frac{1}{x^2}$

$$\frac{1}{x^2}(-ydx + xdy) = -\frac{y}{x^2}dx + \frac{1}{x}dy = d\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{y}{x} = C$$

Method of integrating factors

An **integrating factor** is a function $F(x, y)$ such that the equation

$$P(x, y)dx + Q(x, y)dy = 0$$

becomes an exact ODE after multiplication

$$FPdx + FQdy = 0$$

Integrating factors

Example (continued)

$$-ydx + xdy = 0$$

In fact, the integrating factor is not unique. We can find other integrating factors for the same equation

$$\begin{array}{c|l}
\frac{1}{x^2} & \frac{1}{x^2}(-ydx + xdy) = d\left(\frac{y}{x}\right) = 0 \\
\frac{1}{y^2} & \frac{1}{y^2}(-ydx + xdy) = d\left(\frac{x}{y}\right) = 0 \\
\frac{1}{xy} & \frac{1}{xy}(-ydx + xdy) = d\left(\log\frac{x}{y}\right) = 0 \\
\frac{1}{x^2+y^2} & \frac{1}{x^2+y^2}(-ydx + xdy) = d\left(\arctan\frac{y}{x}\right) = 0
\end{array}$$

Integrating factors

How to find integrating factors

The condition for the ODE being exact is

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

$$F_y P + F P_y = F_x Q + F Q_x$$

If we are looking for integrating factors depending on a single variable, say x, then $F_y = 0$, that is

$$F P_y = F' Q + F Q_x$$

Dividing by FQ

$$\frac{P_y}{Q} = \frac{F'}{F} + \frac{Q_x}{Q}$$

$$\frac{F'}{F} = \frac{P_y - Q_x}{Q}$$

Integrating factors

How to find integrating factors

$$\frac{F'}{F} = \frac{P_y - Q_x}{Q}$$

If the right-hand side only depends on x, then by integration we find the integrating factor

$$\log|F| = \int \frac{P_y - Q_x}{Q} dx \Rightarrow \boxed{F = \exp\left(\int \frac{P_y - Q_x}{Q} dx\right)}$$

Similarly, if $\frac{Q_x - P_y}{P}$ depends only on y, then there exists an integrating factor

$$\log|F| = \int \frac{Q_x - P_y}{P} dy \Rightarrow \boxed{F = \exp\left(\int \frac{Q_x - P_y}{P} dy\right)}$$

Integrating factors

Example

$$(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$$

Solution:
Let's check if it is exact:

$$P_y = \frac{\partial}{\partial y}\left(e^{x+y} + ye^y\right) = e^{x+y} + e^y + ye^y$$

$$Q_x = \frac{\partial}{\partial x}\left(xe^y - 1\right) = e^y$$

So it is not exact. Let's check if it has an integrating factor depending on y

$$\frac{Q_x - P_y}{P} = -\frac{e^{x+y} + ye^y}{e^{x+y} + ye^y} = -1$$

It does.

Integrating factors

Example (continued)

$$\frac{Q_x - P_y}{P} = -\frac{e^{x+y} + ye^y}{e^{x+y} + ye^y} = -1$$

$$F = \exp\left(\int (-1)dy\right) = e^{-y}$$

This integrating factor transforms the ODE into

$$(e^x + y)dx + (x - e^{-y})dy = 0$$

That is exact

$$M_y = \frac{\partial}{\partial y}(e^x + y) = 1$$

$$N_x = \frac{\partial}{\partial x}\left(x - e^{-y}\right) = 1$$

Example (continued)

Its general solution is

$$u(x, y) = e^x + xy + e^{-y} = C$$

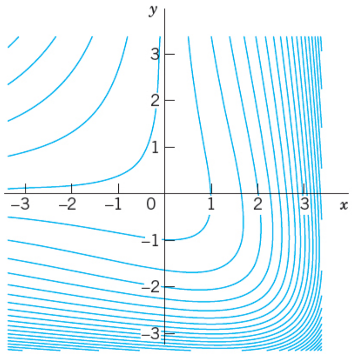


Fig. 18. Particular solutions in Example 5

Integrating factors

How to find integrating factors

Condition	Integrating factor
$\frac{P_y - Q_x}{Q} = f(x)$	$F = \exp \int f(x) dx$
$\frac{Q_x - P_y}{P} = f(y)$	$F = \exp \int f(y) dy$
$\frac{P_y - Q_x}{yQ - xP} = f(xy)$	$F(z) = \exp \int f(z) dz \quad z = xy$
$\frac{1}{2} \frac{P_y - Q_x}{xQ - yP} = f(x^2 + y^2)$	$F(r) = \exp \int f(r) dr \quad r = x^2 + y^2$
$\frac{y^2(P_y - Q_x)}{xP + yQ} = f\left(\frac{x}{y}\right)$	$F(z) = \exp \int f(z) dz \quad z = \frac{x}{y}$
$yf_1(xy)dx + xf_2(xy)dy$	$F = \frac{1}{xy(f_1(xy) - f_2(xy))}$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 1, Section 4:

- 1.4.8
- 1.4.11

Outline

1. First-order ODEs
   - Basic concepts
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Linear ODEs

Linear ODE

A first-order ODE is said to be **linear** if it can be written in the form

$$y' + p(x)y = r(x) \tag{7}$$

The equation abopve is linear in y and y'. In an engineering setup, $r(x)$ is called the **input** to the system, while $y(x)$ is the system's output.

Linear ODEs

Homogeneous Linear ODE

A linear, first-order ODE is said to be **homogeneous** if $r(x) = 0$

$$y' + p(x)y = 0 \tag{8}$$

Then we can solve it by separation of variables

$$\frac{dy}{y} = -p$$

$$\log|y| = -\int p dx + C$$

$$\boxed{y = Ce^{-\int p dx}}$$

We have also the trivial solution $\boxed{y = 0}$.

Linear ODEs

Non-homogeneous Linear ODE

If $r(x)$ is not zero everywhere in the open interval being studied, then the linear ODE is non-homogeneous.

$$y' + py = r$$

$$(py - r)dx + dy = 0$$

Let's look for an integrating factor

$$\frac{P_y - Q_x}{Q} = \frac{p - 0}{1} = p$$

This function only depends on x so there exists an integrating factor in x given by

$$F = e^{\int p dx}$$

Linear ODEs

Non-homogeneous Linear ODE

Let's call h to $\int p dx$, and multiply the linear equation by the integrating factor $F = e^h$

$$e^h y' + p e^h y = r e^h$$

Note that $h' = p$, then

$$e^h y' + h' e^h y = r e^h$$

$$(e^h y)' = r e^h$$

$$e^h y = \int r e^h dx + C$$

$$\boxed{y = e^{-h} \left(\int r e^h dx + C \right) \quad h = \int p dx}$$

If $r = 0$, we are back to the homogeneous solution

$$y = Ce^{-h}$$

Linear ODEs

Non-homogeneous Linear ODE

$$y = e^{-h}\left(\int re^{h}dx + C\right)$$

$$y = e^{-h}\int re^{h}dx + Ce^{-h}$$

We distinguish two terms, the first one, $e^{-h}\int re^{h}dx$, is the response of the system to the input r, while the second one, Ce^{-h} is the response of the system to the initial conditions.

Linear ODEs

Example

$$y' + y\tan(x) = \sin(2x) \quad y(0) = 1$$

Solution:

$$h = \int \tan(x)dx = \log\left|\frac{1}{\cos(x)}\right|$$

$$e^h = \frac{1}{\cos(x)}$$

$$e^{-h} = \cos(x)$$

The general solution is

$$y = \cos(x)\left(\int \frac{\sin(2x)}{\cos(x)}dx + C\right)$$

$$y = \cos(x)\left(-2\cos(x) + C\right)$$

Linear ODEs

Example (continued)

$$y = \cos(x)\,(-2\cos(x) + C) \quad y(0) = 1$$

The particular solution is

$$1 = \cos(0)\,(-2\cos(0) + C) \Rightarrow C = 3$$

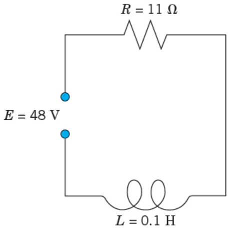
$$y = \cos(x)\,(3 - 2\cos(x))$$

$$y = 3\cos(x) - 2\cos^2(x)$$

The term $3\cos(x)$ is the response to the initial conditions, while the term $-2\cos^2(x)$ is the response to the input.

Example

Find the circulating current in the RL circuit



Circuit

Physical Laws. A current I in the circuit causes a **voltage drop** RI across the resistor **(Ohm's law)** and a voltage drop $LI' = L\,dI/dt$ across the conductor, and the sum of these two voltage drops equals the EMF **(Kirchhoff's Voltage Law, KVL)**.

Solution:

$$LI' + RI = E$$

Linear ODEs

Example (continued)

It is a linear equation of the form

$$I' + \frac{R}{L} I = \frac{E}{L}$$

$$h = \int \frac{R}{L} dt = \frac{R}{L} t$$

$$\left[y = e^{-h} \left(\int e^{h} r dt + C \right) \right]$$

$$I = e^{-\frac{R}{L}t} \left(\int e^{\frac{R}{L}t} \frac{E}{L} dt + C \right)$$

$$I = e^{-\frac{R}{L}t} \left(\frac{L}{R} e^{\frac{R}{L}t} \frac{E}{L} + C \right)$$

$$I = \frac{E}{R} + C e^{-\frac{R}{L}t}$$

Linear ODEs

Example (continued)

The general solution is

$$I = \frac{E}{R} + Ce^{-\frac{R}{L}t}$$

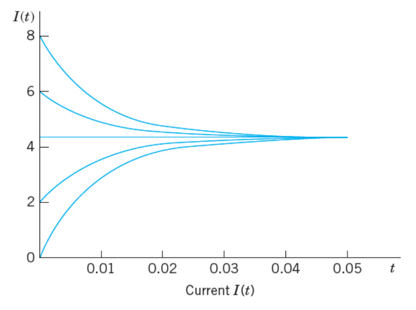
The initial condition is $I(0) = 0$, and the particular solution

$$0 = \frac{E}{R} + Ce^{-\frac{R}{L}0} = \frac{E}{R} + C \Rightarrow C = -\frac{E}{R}$$

$$I = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$I = \frac{48}{11}\left(1 - e^{-\frac{11}{0.1}t}\right)$$

$$I = \frac{48}{11}\left(1 - e^{-110t}\right)$$



Linear ODEs

Example

Assume that the level of a certain hormone in the blood of a patient varies with time. Suppose that the time rate of change is the difference between a sinusoidal input of a 24-hour period from the thyroid gland and a continuous removal rate proportional to the level present. Set up a model for the hormone level in the blood and find its general solution. Find the particular solution satisfying a suitable initial condition.

Linear ODEs

Example (continued)

Assume that the level of a certain hormone in the blood of a patient varies with time. Suppose that the time rate of change is the difference between a sinusoidal input of a 24-hour period from the thyroid gland and a continuous removal rate proportional to the level present. Set up a model for the hormone level in the blood and find its general solution. Find the particular solution satisfying a suitable initial condition.

Solution:

$$y' = \text{In} - \text{Out}$$

$$y' = (A + B\cos(\omega t)) - Ky$$

$$y' + Ky = (A + B\cos(\omega t))$$

$$h = \int K dt = Kt$$

$$y = e^{-Kt}\left(\int (A + B\cos(\omega t))e^{Kt} dt + C\right)$$

$$y = e^{-Kt}\left(e^{Kt}\left(\frac{A}{K} + \frac{B}{K^2+\omega^2}\left(K\cos(\omega t) + \omega\sin(\omega t)\right)\right) + C\right)$$

Linear ODEs

Example (continued)

$$y = e^{-Kt}\left(e^{Kt}\left(\frac{A}{K} + \frac{B}{K^2+\omega^2}\left(K\cos(\omega t) + \omega\sin(\omega t)\right)\right) + C\right)$$

$$y = \left(\frac{A}{K} + \frac{B}{K^2+\omega^2}\left(K\cos(\omega t) + \omega\sin(\omega t)\right)\right) + Ce^{-Kt}$$

Since the variation is every 24h, the frequency $\omega = \frac{2\pi}{24} = \frac{\pi}{12}$. Then, the general solution becomes

$$y = \left(\frac{A}{K} + \frac{B}{K^2+\frac{\pi^2}{12^2}}\left(K\cos(\frac{\pi}{12}t) + \frac{\pi}{12}\sin(\frac{\pi}{12}t)\right)\right) + Ce^{-Kt}$$

Example (continued)

$$y = \left(\frac{A}{K} + \frac{B}{K^2 + \frac{\pi^2}{12^2}} \left(K \cos(\frac{\pi}{12} t) + \frac{\pi}{12} \sin(\frac{\pi}{12} t) \right) \right) + Ce^{-Kt}$$

If we assume $y(0) = 0$, then the particular solution is

$$0 = \left(\frac{A}{K} + \frac{B}{K^2 + \frac{\pi^2}{12^2}} \left(K \cdot 1 + \frac{\pi}{12} \cdot 0 \right) \right) + C$$

$$C = -\left(\frac{A}{K} + \frac{BK}{K^2 + \frac{\pi^2}{12^2}} \right)$$

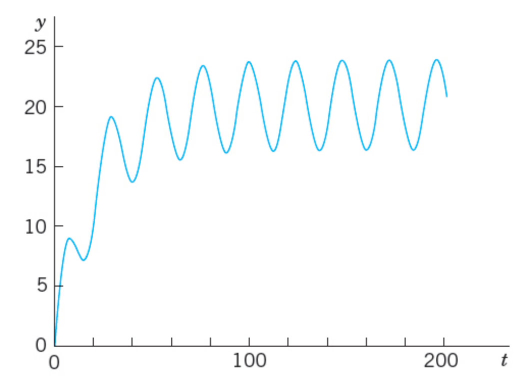
$$y = \frac{A}{K}(1 - e^{-Kt}) + \frac{B}{K^2 + \frac{\pi^2}{12^2}} \left(K \cos(\frac{\pi}{12} t) + \frac{\pi}{12} \sin(\frac{\pi}{12} t) - Ke^{-Kt} \right)$$

Linear ODEs

Example (continued)

$$y = \frac{A}{K}(1 - e^{-Kt}) + \frac{B}{K^2 + \frac{\pi^2}{12^2}}\left(K\cos(\frac{\pi}{12}t) + \frac{\pi}{12}\sin(\frac{\pi}{12}t) - Ke^{-Kt}\right)$$

For $A = B = 1$, $K = 0.05$ we have



Reduction to Linear ODEs

Bernouilli equation

$$y' + p(x)y = g(x)y^a$$

This equation is non-linear except for $a = 0$ or $a = 1$. Let's make the change of variable

$$u = y^{1-a}$$

$$u' = (1-a)y^{-a}y'$$

$$u' = (1-a)y^{-a}(-py + gy^a)$$

$$u' = (1-a)(-py^{1-a} + g)$$

$$u' = (1-a)(-pu + g)$$

$$\boxed{u' + (1-a)pu = (1-a)g}$$

This is a linear equation.

Reduction to Linear ODEs

Example: Logistic equation

$$y' = Ay - By^2$$

<u>Solution:</u>

$y = 0$ is a solution. Otherwise, this is Bernouilli equation with $a = 2$.

$$u = y^{1-2} = y^{-1}$$

$$u' = (-1)y^{-2}y'$$

$$u' = -y^{-2}(Ay - By^2)$$

$$u' = -(Ay^{-1} - B)$$

$$u' = -(Au - B)$$

$$u' + Au = B$$

Example: Logistic equation (continued)

$$u' + Au = B$$

$$h = \int A dx = Ax$$

$$u = e^{-Ax} \left(\int B e^{Ax} dx + C \right)$$

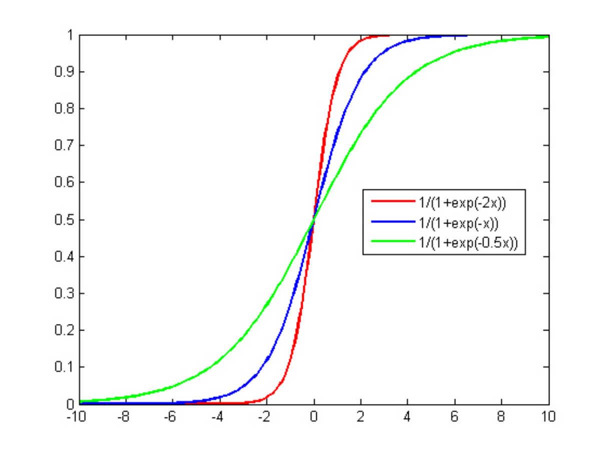
$$u = e^{-Ax} \left(\frac{B}{A} e^{Ax} + C \right)$$

$$u = \frac{B}{A} + Ce^{-Ax}$$

$$y^{-1} = \frac{B}{A} + Ce^{-Ax}$$

$$y = \frac{1}{\frac{B}{A} + Ce^{-Ax}}$$

Example: Logistic equation (continued)



Example: Population dynamics

For a small population, its growth can be described by Malthus law

$$y' = Ay$$

This is a particular case of the logistic equation whose solution is

$$y = \frac{1}{Ce^{-At}} = \frac{1}{C}e^{At}$$

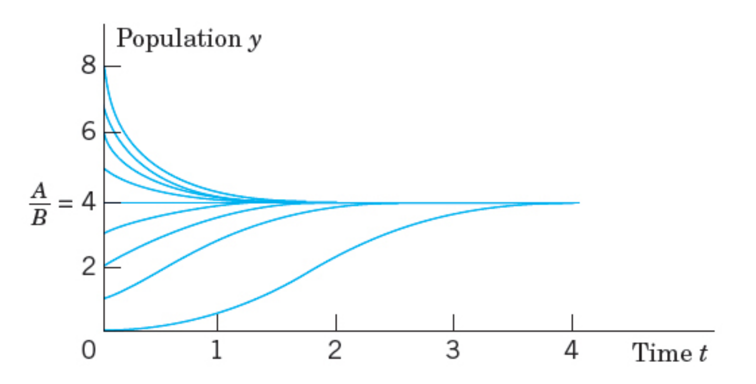
The term $-By^2$ acts as a "braking" term that prevents the population of growing infinitely. If we rewrite the logistic equation as

$$y' = Ay\left(1 - \frac{B}{A}y\right)$$

If $y < \frac{A}{B}$, then $y' > 0$ and the population grows.
If $y > \frac{A}{B}$, then $y' < 0$ and the population decreases.

Reduction to Linear ODEs

Example: Population dynamics (continued)



Critical points

Autonomous ODE and critical points

An equation

$$y' = f(x, y)$$

in which the independent variable does not appear explicitly

$$y' = f(y)$$

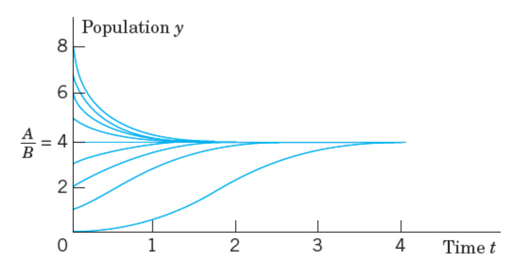
is called **autonomous**. Autonomous ODEs have **critical or equilibrium points** at those values at which $f(y) = 0$ because there is no change ($y' = 0$). A critical point may be **stable** (if solutions close to it for some t remain close to it for all further t) or **unstable** (if solutions initially close to it do not remain close as t increases).

Reduction to Linear ODEs

Example: Population dynamics (continued)

$$y' = Ay\left(1 - \frac{B}{A}y\right)$$

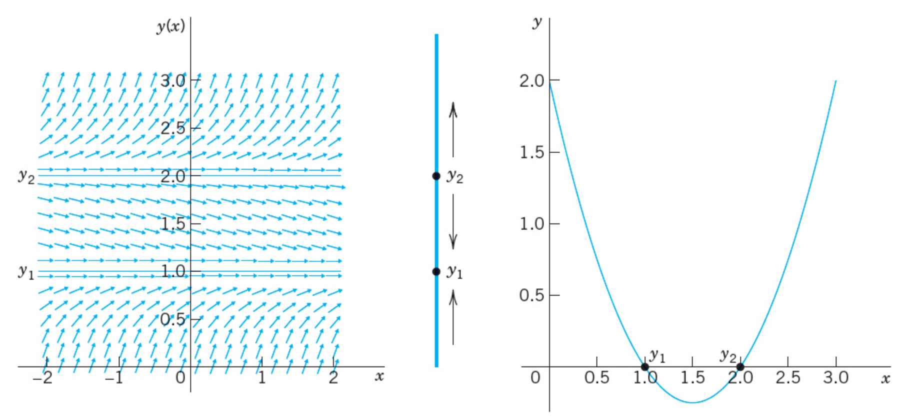
Equilibrium points are $y = 0$ (unstable) and $y = \frac{A}{B}$ (stable).



Example

$$y' = (y-1)(y-2)$$

Equilibrium points are $y = 1$ (stable) and $y = 2$ (unstable).



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 1, Section 5:

- 1.5.7
- 1.5.13
- 1.5.15
- 1.5.16
- 1.5.17
- 1.5.18
- 1.5.21
- 1.5.24
- 1.5.28
- 1.5.33
- 1.5.34

Exercises

30. TEAM PROJECT. Riccati Equation. Clairaut Equation. Singular Solution.

A **Riccati equation** is of the form

(14) $$y' + p(x)y = g(x)y^2 + h(x).$$

A **Clairaut equation** is of the form

(15) $$y = xy' + g(y').$$

(a) Apply the transformation $y = Y + 1/u$ to the Riccati equation (14), where Y is a solution of (14), and obtain for u the linear ODE $u' + (2Yg - p)u = -g$. Explain the effect of the transformation by writing it as $y = Y + v$, $v = 1/u$.

(b) Show that $y = Y = x$ is a solution of the ODE $y' - (2x^3 + 1)\, y = -x^2y^2 - x^4 - x + 1$ and solve this Riccati equation, showing the details.

(c) Solve the Clairaut equation $y'^2 - xy' + y = 0$ as follows. Differentiate it with respect to x, obtaining $y''(2y' - x) = 0$. Then solve (A) $y'' = 0$ and (B) $2y' - x = 0$ separately and substitute the two solutions (a) and (b) of (A) and (B) into the given ODE. Thus obtain (a) a general solution (straight lines) and (b) a parabola for which those lines (a) are tangents (Fig. 6 in Prob. Set 1.1); so (b) is the envelope of (a). Such a solution (b) that cannot be obtained from a general solution is called a **singular solution**.

(d) Show that the Clairaut equation (15) has as solutions a family of straight lines $y = cx + g(c)$ and a singular solution determined by $g'(s) = -x$, where $s = y'$, that forms the envelope of that family.

Outline

1. First-order ODEs
   - Basic concepts
   - Geometric meaning, direction fields
   - Separable ODEs
   - Exact ODEs. Integrating factors.
   - Linear ODEs. Bernouilli equation. Population dynamics.
   - Orthogonal trajectories
   - Existence and uniqueness of IVPs

Orthogonal trajectories

Orthogonal trajectories

Let's consider the family of curves that are the solution of a given ODE

$$G(x, y, c) = 0$$

For each c we have a different curve. The question now is which is the family of curves that is orthogonal to the first family? For instance,

$$\frac{1}{2}x^2 + y^2 = C$$

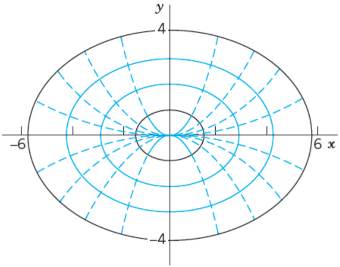


Fig. 24. Electrostatic field between two ellipses (elliptic cylinders in space): Elliptic equipotential curves (equipotential surfaces) and orthogonal trajectories (parabolas)

Orthogonal trajectories

Example

Step 1: Find the ODE for the family of curves (differentiate the family).

$$d\left(\frac{1}{2}x^2 + y^2 = C\right)$$

$$x + 2yy' = 0$$

$$y' = -\frac{x}{2y}$$

$$\boxed{y' = f(x, y)}$$

Orthogonal trajectories

Example

Step 2: Find the ODE of the orthogonal family. Remind that two lines in the plane are orthogonal if

$$m_1 m_2 = -1$$

At the point $(x, \tilde{y})$ they are orthogonal if

$$f(x,\tilde{y})\tilde{y}' = -1 \Rightarrow \boxed{\tilde{y}' = -\frac{1}{f(x,\tilde{y})}}$$

$$\tilde{y}' = -\frac{1}{-\frac{x}{2\tilde{y}}} = \frac{2\tilde{y}}{x}$$

Orthogonal trajectories

Example

Step 3: Solve the differential equation

$$\tilde{y}' = \frac{2\tilde{y}}{x}$$

$$\frac{\tilde{y}'}{\tilde{y}} = \frac{2}{x}$$

$$\log|\tilde{y}| = 2\log|x| + C$$

$$\tilde{y} = Cx^2$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 1, Section 6:

- 1.6.12
- 1.6.13

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Existence and uniqueness

Example: Lack of solution

$$|y'| + |y| = 0 \quad y(0) = 1$$

The only solution of the ODE is

$$y = 0$$

and it does not meet $y(0) = 1$. There is **no solution** to the Initial Value Problem.

Existence and uniqueness of IVPs

Example: Unique solution

$$y' = 2x \quad y(0) = 1$$

The general solution of the ODE is

$$y = x^2 + C$$

To fulfill the Initial Value we need

$$1 = 0^2 + C \Rightarrow C = 1$$

Therefore, there is a **unique solution** to the Initial Value Problem

$$y = x^2 + 1$$

Example: Infinite solutions

$$xy' = y - 1 \quad y(0) = 1$$

The function

$$y = 1 + Cx$$

is a solution of the ODE and it fulfills the Initial Value Problem for any value of C

Existence and uniqueness of IVPs

Existence theorem

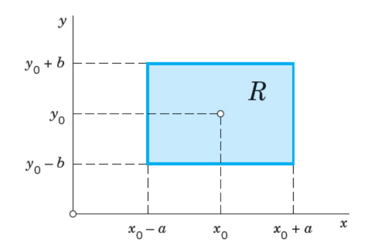
Given the IVP

$$y' = f(x, y) \quad y(x_0) = y_0$$

If $f(x, y)$ is continuous in a rectangle R

$$R = \{(x, y) \in \mathbb{R}^2 | |x - x_0| < a, |y - y_0| < b\}$$

and bounded in R, that is, there exists $K \in \mathbb{R}$ such that



$$|f(x, y)| \leq K$$

Then, the IVP has at least one solution $y(x)$. This solution exists at least for all x in $|x - x_0| < \alpha$ where $\alpha = \min\{a, \frac{b}{K}\}$.

Existence and uniqueness of IVPs

Existence theorem

The fact that f is bounded by K means that any solution y cannot "grow" as much as it likes and that it must be confined within a certain region. The slop of any solution is at least $-K$ and at most K.

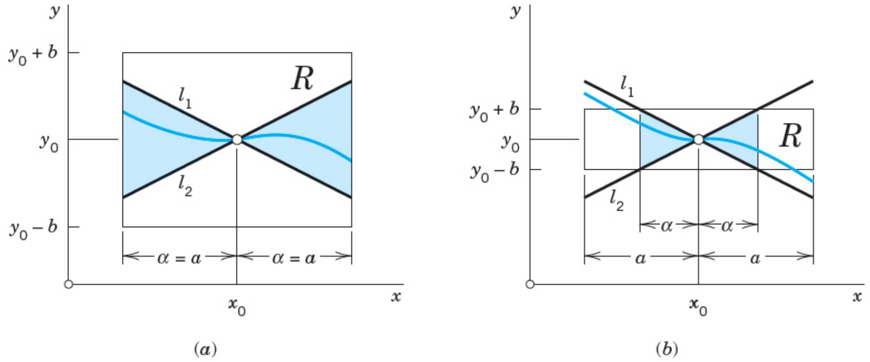


Fig. 27. The condition (2) of the existence theorem. (a) First case. (b) Second case

Existence and uniqueness of IVPs

Example: Lack of solution (continued)

$$|y'| + |y| = 0 \quad y(0) = 1$$

$$y' = \begin{cases} 1 - |y| & y' \geq 0 \\ -(1 - |y|) & y' < 0 \end{cases}$$

This IVP does not have a solution because f is not continuous.

Existence and uniqueness of IVPs

Uniqueness theorem

Let the IVP meet the conditions for existence. If $f_y = \frac{\partial f}{\partial y}$ is continuous in R and it is bounded in R, that is, there exists $M \in \mathbb{R}$ such that

$$|f_y(x, y)| \leq M$$

Then, the IVP has a unique solution $y(x)$. This solution exists at least for all x in $|x - x_0| < \alpha$ where $\alpha = \min\{a, \frac{b}{K}\}$.

Existence and uniqueness of IVPs

Example: Infinite solutions (continued)

$$xy' = y - 1 \quad y(0) = 1$$

$$y' = \frac{y-1}{x} \quad y(0) = 1$$

The IVP has not a unique solution because

$$f_y = \frac{1}{x}$$

is not continuous around $x = 0$.

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