Chapter 2. Second-order linear ODEs

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Biomedical Engineering

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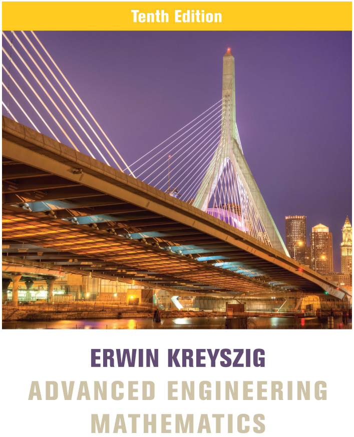
Outline

1. Second-order linear ODEs
   - Homogeneous linear ODEs
   - Homogeneous linear ODEs with constant coefficients
   - Differential operators
   - Modeling of free oscillations of a mass-spring system
   - Euler-Cauchy equations
   - Existence and uniqueness of solutions. Wronskian
   - Nonhomogeneous ODEs
   - Forced oscillations. Resonance.
   - Electric circuits
   - Solution by variation of parameters

References



E. Kreyszig. Advanced Engineering Mathematics. John Wiley & sons. Chapter 2.

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Homogeneous linear ODEs of second-order

Definition

A second-order ODE is **linear** if it can be written as

$$y'' + p(x)y' + q(x)y = r(x)$$

Otherwise, it is **nonlinear**. It is **homogeneous** if $r(x) = 0$.

Examples

$y'' + 25y = e^{-x}\cos(x)$	Linear, non-homogeneous
$xy'' + y' + xy = 0$	
$y'' + \frac{1}{x}y' + y = 0$	Linear, homogeneous
$y''y + (y')^2 = 0$	Nonlinear

Principle of superposition

Theorem: Principle of superposition

The linear combination of any two solutions of a homogeneous, linear ODE is also a solution.

Proof:

if y_1 and y_2 are solutions, then

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$

$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

Let's study the linear combination

$$y = c_1 y_1 + c_2 y_2$$

$$y'' + p(x)y' + q(x)y = 0$$

$$(c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2) = 0$$

Principle of superposition

Theorem: Principle of superposition (continued)

$$(c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2) = 0$$

$$(c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2) = 0$$

$$(c_1 y_1'' + c_1 p(x) y_1' + c_1 q(x) y_1) + (c_2 y_2'' + c_2 p(x) y_2' + c_2 q(x) y_2) = 0$$

$$c_1(y_1'' + p(x) y_1' + q(x) y_1) + c_2(y_2'' + p(x) y_2' + q(x) y_2) = 0$$

$$c_1 \cdot 0 + c_2 \cdot 0 = 0$$

$$0 = 0$$

Principle of superposition

Example

$$y'' + y = 0$$

Two solutions of the ODE are

$$y_1 = \cos(x)$$

$$y_2 = \sin(x)$$

The linear combination

$$y = c_1 y_1 + c_2 y_2 = c_1 \cos(x) + c_2 \sin(x)$$

is also a solution.

Principle of superposition

Example

It does not work with nonhomogeneous, linear ODEs. For instance,

$$y_1 = 1 + \cos(x)$$

$$y_2 = 1 + \sin(x)$$

are solutions of

$$y'' + y = 1$$

but

$$y = y_1 + y_2$$

is not.

Principle of superposition

Example

It does not work with nonlinear ODEs. For instance,

$$y_1 = x^2$$

$$y_2 = 1$$

are solutions of

$$y''y - xy' = 0$$

but

$$y = y_1 + y_2$$

is not.

Initial Value Problem

Initial Value Problem

An Initial Value Problem consists of two initial conditions

$$y(x_0) = y_0$$

$$y'(x_0) = m_0$$

These two conditions are used to determine the constants of a the general solution

$$y = c_1 y_1 + c_2 y_2$$

y_1 and y_2 must be linearly independent, that is,

$$c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1 = c_2 = 0$$

and they form a **basis** or **fundamental system** of solutions.

Basis of solutions

Example

$$y'' + y = 0$$

We know that

$$y_1 = \cos(x)$$

$$y_2 = \sin(x)$$

are solutions. Let's check if they are linearly independent

$$c_1 \cos(x) + c_2 \sin(x) = 0$$

$$\frac{\cos(x)}{\sin(x)} = -\frac{c_2}{c_1}$$

That is, if they were linearly dependent, their ratio would be constant. But this is not the case

$$\frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$$

The ratio is a function of x and not a constant.

Initial Value Problem

Example

$$y'' + y = 0 \quad y(0) = 3, y'(0) = -0.5$$

Solution:
The general solution is

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Now we impose the two initial conditions

$$\left.\begin{array}{r} y(0) = c_1 \cos(0) + c_2 \sin(0) = 3 \\ y'(0) = c_1(-\sin(0)) + c_2 \cos(0) = -0.5 \end{array}\right\} \Rightarrow c_1 = 3, c_2 = -0.5$$

The particular solution is

$$y = 3\cos(x) - 0.5\sin(x)$$

Basis of solutions: Reduction of order

Example

$$(x^2 - x)y'' - xy' + y = 0$$

We can easily see that $y_1 = x$ is a solution of the equation

$$(x^2 - x)(x)'' - x(x)' + x = 0$$

$$(x^2 - x)(0) - x(1) + x = 0$$

$$-x + x = 0$$

How can we find the second element of the basis? Reduction of order. Let's find a solution of the form

$$y_2 = uy_1$$

In this case

$$y_2 = ux$$

Basis of solutions: Reduction of order

Example (continued)

$$y_2 = ux$$

$$y_2' = u'x + u$$

$$y_2'' = u''x + u' + u' = u''x + 2u'$$

We substitute y_2 in the equation to get

$$(x^2 - x)y_2'' - xy_2' + y_2 = 0$$

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0$$

$$x^3u'' + 2x^2u' - x^2u'' - 2xu' - x^2u' - ux + ux = 0$$

$$(x^3 - x^2)u'' + (x^2 - 2x)u' = 0$$

$$x(x^2 - x)u'' + x(x - 2)u' = 0$$

$$(x^2 - x)u'' + (x - 2)u' = 0$$

Basis of solutions: Reduction of order

Example (continued)

$$(x^2 - x)u'' + (x-2)u' = 0$$

We now make a change of variable

$$U = u' \Rightarrow U' = u''$$

$$(x^2 - x)U' + (x-2)U = 0$$

That is a linear equation

$$\frac{U'}{U} = -\frac{x-2}{x^2 - x} = -\frac{x-2}{x(x-1)}$$

$$\frac{dU}{U} = \left(\frac{1}{x-1} - \frac{2}{x}\right) dx$$

$$\log|U| = \log|x-1| - 2\log|x| \Rightarrow U = \frac{x-1}{x^2}$$

Basis of solutions: Reduction of order

Example (continued)

We now solve for u in the change of variable

$$u' = U = \frac{x-1}{x^2}$$

$$u = \int \frac{x-1}{x^2}\,dx = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$u = \log|x| + \frac{1}{x}$$

Finally,

$$y_2 = ux = \left(\log|x| + \frac{1}{x}\right) x = x\log|x| + 1$$

Since y_1 and y_2 are not proportional, they form a basis of solutions of the ODE.

Basis of solutions: Reduction of order

Reduction of order

Consider the ODE

$$y'' + p(x)y' + q(x)y = 0$$

and a solution of it y_1. The second solution will be designed as

$$y_2 = uy_1$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

We now subtitute it in the ODE

$$(u''y_1 + 2u'y_1' + uy_1'') + p(x)(u'y_1 + uy_1') + q(x)(uy_1) = 0$$

Basis of solutions: Reduction of order

Reduction of order (continued)

Consider the ODE

$$y'' + p(x)y' + q(x)y = 0$$

and a solution of it y_1. The second solution will be designed as

$$y_2 = uy_1$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

We now subtitute it in the ODE

$$(u''y_1 + 2u'y_1' + uy_1'') + p(u'y_1 + uy_1') + q(uy_1) = 0$$

$$u''y_1 + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u = 0$$

Basis of solutions: Reduction of order

Reduction of order (continued)

$$u''y_1 + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u = 0$$

But the coefficient of u is 0 because y_1 is a solution of the ODE.

$$u''y_1 + (2y_1' + py_1)u' = 0$$

We now make the change of variable

$$U = u'$$

$$U'y_1 + (2y_1' + py_1)U = 0$$

$$U' + \frac{2y_1' + py_1}{y_1}U = 0$$

$$U' + \left(2\frac{y_1'}{y_1} + p\right)U = 0$$

Basis of solutions: Reduction of order

Reduction of order (continued)

$$U' + \left(2\frac{y_1'}{y_1} + p\right)U = 0$$

$$\frac{dU}{U} = -\left(2\frac{y_1'}{y_1} + p\right)dx$$

$$\log|U| = -2\log|y_1| - \int p dx$$

$$U = \frac{1}{y_1^2}e^{-\int p dx}$$

$$U = u' \Rightarrow u = \int U dx$$

Finally,

$$\boxed{y_2} = uy_1 = y_1\int U dx = \boxed{y_1\int \frac{1}{y_1^2}e^{-\int p dx}dx} \tag{1}$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 2, Section 1:

- 2.1.1
- 2.1.2
- 2.1.5
- 2.1.6
- 2.1.12
- 2.1.13
- 2.1.17

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Homogeneous linear ODEs with constant coefficients

Characteristic equation

$$y' + ky = 0$$

Try $y = e^{\lambda x}$

$$\lambda e^{\lambda x} + ke^{\lambda x} = 0$$

$$e^{\lambda x}(\lambda + k) = 0$$

$$\lambda + k = 0 \Rightarrow \lambda_1 = -k$$

$$y_1 = e^{\lambda_1 x}$$

General solution

$$y = c_1 y_1$$

$$y'' + ay' + by = 0$$

Try $y = e^{\lambda x}$

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

$$e^{\lambda x}(\lambda^2 + a\lambda + b) = 0$$

$$\lambda^2 + a\lambda + b = 0 \Rightarrow \lambda_1, \lambda_2$$

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

$$y_1 = e^{\lambda_1 x}$$

$$y_2 = e^{\lambda_2 x}$$

General solution

$$y = c_1 y_1 + c_2 y_2$$

Homogeneous linear ODEs with constant coefficients

Characteristic equation: Two distinct real roots, $a^2 - 4b > 0$

General solution:

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right)$$

$$\lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

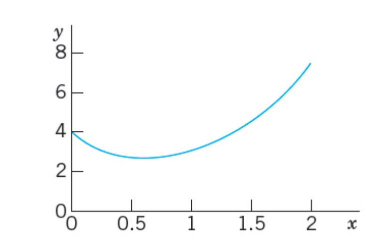
Homogeneous linear ODEs with constant coefficients

Example

$$y'' + y' - 2y = 0 \quad y(0) = 4, y'(0) = -5$$

Solution:

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -2$$



General solution:

$$y = c_1 e^x + c_2 e^{-2x}$$

Particular solution:

$$\left.\begin{array}{r} 4 = c_1 e^0 + c_2 e^{-2\cdot 0} = c_1 + c_2 \\ -5 = c_1 e^0 + c_2(-2)e^{-2\cdot 0} = c_1 - 2c_2 \end{array}\right\} \Rightarrow c_1 = 1, c_2 = 3$$

$$y = e^x + 3e^{-2x}$$

Homogeneous linear ODEs with constant coefficients

Characteristic equation: Double root $a^2 - 4b = 0$

$$\lambda_1 = -\frac{a}{2}$$

One of the solutions is

$$y_1 = e^{\lambda_1 x}$$

Let's look for another using the reduction of order

$$y_2 = uy_1 = ue^{\lambda_1 x}$$

$$y_2' = u'e^{\lambda_1 x} + \lambda_1 ue^{\lambda_1 x}$$

$$y_2'' = u''e^{\lambda_1 x} + 2\lambda_1 u'e^{\lambda_1 x} + \lambda_1^2 ue^{\lambda_1 x}$$

$$y_2'' + ay_2' + by_2 = 0$$

$$(u''e^{\lambda_1 x} + 2\lambda_1 u'e^{\lambda_1 x} + \lambda_1^2 ue^{\lambda_1 x}) + a(u'e^{\lambda_1 x} + \lambda_1 ue^{\lambda_1 x}) + bue^{\lambda_1 x} = 0$$

$$e^{\lambda_1 x}\left[u'' + (2\lambda_1 + a)u' + (\lambda_1^2 + a\lambda_1 + b)u\right] = 0$$

Homogeneous linear ODEs with constant coefficients

Characteristic equation: Double root $a^2 - 4b = 0$ (continued)

$$e^{\lambda_1 x}\left[u'' + (2\lambda_1 + a)u' + (\lambda_1^2 + a\lambda_1 + b)u\right] = 0$$

The coefficient of u $(\lambda_1^2 + a\lambda_1 + b)$ is 0 because λ_1 is a root of the characteristic equation:

$$e^{\lambda_1 x}\left[u'' + (2\lambda_1 + a)u'\right] = 0$$

$$u'' + (2\lambda_1 + a)u' = 0$$

Note that

$$2\lambda_1 + a = 2\left(-\frac{a}{2}\right) + a = 0$$

then,

$$u'' = 0$$

whose general solution is

$$u = c_1 x + c_2$$

and a particular solution

$$u = x$$

Homogeneous linear ODEs with constant coefficients

Characteristic equation: Double root $a^2 - 4b = 0$ (continued)

The second element of the basis of solutions is

$$y_2 = uy_1 = xe^{\lambda_1 x}$$

The general solution of

$$y'' + ay' + by = 0$$

is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

$$y = (c_1 + c_2 x) e^{\lambda_1 x}$$

$$\boxed{y = (c_1 + c_2 x) e^{-\frac{a}{2} x}}$$

Homogeneous linear ODEs with constant coefficients

Example

$$y'' + 6y' + 9y = 0$$

Solution:

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0$$

The general solution is

$$y = (c_1 + c_2 x)e^{-3x}$$

Homogeneous linear ODEs with constant coefficients

Characteristic equation: Complex roots $a^2 - 4b < 0$

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) = -\frac{a}{2} + i\omega$$

$$\lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right) = -\frac{a}{2} - i\omega$$

Two independent solutions are

$$y_1 = e^{(-\frac{a}{2}+i\omega)x} = e^{-\frac{a}{2}x}e^{i\omega} = e^{-\frac{a}{2}x}(\cos(\omega x) + i\sin(\omega x))$$

$$y_2 = e^{(-\frac{a}{2}-i\omega)x} = e^{-\frac{a}{2}x}e^{-i\omega} = e^{-\frac{a}{2}x}(\cos(\omega x) - i\sin(\omega x))$$

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{(-\frac{a}{2}+i\omega)x} + c_2 e^{(-\frac{a}{2}-i\omega)x}$$

Characteristic equation: Complex roots $a^2 - 4b < 0$ (continued)

Let's calculate two other independent solutions

$$\begin{aligned}
y_1^* &= \frac{y_1+y_2}{2} \\
&= e^{-\frac{a}{2}x}\frac{(\cos(\omega x)+i\sin(\omega x))+(\cos(\omega x)-i\sin(\omega x))}{2} \\
&= e^{-\frac{a}{2}x}\frac{2\cos(\omega x)}{2} \\
&= e^{-\frac{a}{2}x}\cos(\omega x) \\
y_2^* &= \frac{y_1-y_2}{2i} \\
&= e^{-\frac{a}{2}x}\frac{(\cos(\omega x)+i\sin(\omega x))-(\cos(\omega x)-i\sin(\omega x))}{2i} \\
&= e^{-\frac{a}{2}x}\frac{2i\sin(\omega x)}{2i} \\
&= e^{-\frac{a}{2}x}\sin(\omega x)
\end{aligned}$$

Since they are independent, they are another basis, so the general solution can also be written as

$$y = c_1 y_1^* + c_2 y_2^*$$

$$y = e^{-\frac{a}{2}x}(c_1\cos(\omega x) + c_2\sin(\omega x))$$

Homogeneous linear ODEs with constant coefficients

Characteristic equation: Complex roots $a^2 - 4b < 0$ (continued)

$$y = e^{-\frac{a}{2}x}(c_1 \cos(\omega x) + c_2 \sin(\omega x))$$

This can also be written as

$$\boxed{y = e^{\mathrm{Re}\{\lambda_1\}x}(c_1 \cos(\mathrm{Im}\{\lambda_1\}x) + c_2 \sin(\mathrm{Im}\{\lambda_1\}x))}$$

Example

$$y'' + 0.4y' + 9.04y = 0 \quad y(0) = 0, y'(0) = 3$$

Solution:

$$\lambda^2 + 0.4\lambda + 9.04 = 0 \Rightarrow \lambda_1, \lambda_2 = -0.2 \pm 3i$$

The general solution is

$$y = e^{-0.2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Homogeneous linear ODEs with constant coefficients

Example (continued)

The initial conditions are $y(0) = 0, y'(0) = 3$

$$y(0) = 0 = e^{-0.2\cdot 0}(c_1 \cos(3 \cdot 0) + c_2 \sin(3 \cdot 0)) = c_1 \Rightarrow c_1 = 0$$

The particular solution is of the form

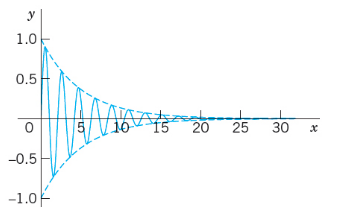
$$y = c_2 e^{-0.2x} \sin(3x)$$

$$y' = c_2 e^{-0.2x}(-0.2 \sin(3x) + 3 \cos(3x))$$

$$y'(0) = 3 = c_2 e^{-0.2\cdot 0}(-0.2 \sin(3\cdot 0)+3 \cos(3\cdot 0))$$

$$\Rightarrow c_2 = 1$$

$$y = 3e^{-0.2x} \sin(3x)$$



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 2, Section 2:

- 2.2.16
- 2.2.17
- 2.2.31
- 2.2.35

Exercises

38. TEAM PROJECT. General Properties of Solutions

(a) Coefficient formulas. Show how a and b in (1) can be expressed in terms of λ_1 and λ_2. Explain how these formulas can be used in constructing equations for given bases.

(b) Root zero. Solve $y'' + 4y' = 0$ (i) by the present method, and (ii) by reduction to first order. Can you explain why the result must be the same in both cases? Can you do the same for a general ODE $y'' + ay' = 0$?

(c) Double root. Verify directly that $xe^{\lambda x}$ with $\lambda = -a/2$ is a solution of (1) in the case of a double root. Verify and explain why $y = e^{-2x}$ is a solution of $y'' - y' - 6y = 0$ but xe^{-2x} is not.

(d) Limits. Double roots should be limiting cases of distinct roots λ_1, λ_2 as, say, $\lambda_2 \to \lambda_1$. Experiment with this idea. (Remember l'Hôpital's rule from calculus.) Can you arrive at $xe^{\lambda_1 x}$? Give it a try.

Outline

Differential operators

Differential operators

$$\boxed{D = \frac{d}{dx}}$$

$$Dy = \frac{dy}{dx} = y'$$

$$D\sin(x) = \cos(x)$$

$$D(y_1 + y_2) = D(y_1) + D(y_2) = y_1' + y_2'$$

$$D(ay) = aD(y) = ay'$$

$$D(Dy) = D^2y = y''$$

$$D^2\sin(x) = -\sin(x)$$

Differential operators

$$y'' + ay' + by = 0$$

$$D^2y + aDy + by = 0$$

$$(D^2 + aD + bI)y = 0$$

We may define the operator

$$L = D^2 + aD + bI$$

Then,

$$(D^2 + aD + bI)y = 0 \leftrightarrow Ly = 0$$

If we apply L to $e^{\lambda x}$, we get

$$Le^{\lambda x} = \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + e^{\lambda x} = (\lambda^2 + a\lambda + b)e^{\lambda x}$$

Differential operators

Example

$$y'' - 3y' - 40y = 0$$

Solution:

$$(D^2 - 3D - 40I)y = 0$$

Now we factorize the differential operator

$$(D - 8I)(D + 5I)y = 0$$

We can check that it is equivalent to the differential equation

$$(D - 8I)(y' + 5y) = 0$$

$$D(y' + 5y) - 8I(y' + 5y) = 0$$

$$y'' + 5y' - 8y' - 40y = 0$$

$$y'' - 3y' - 40y = 0$$

Differential operators

Example (continued)

$$(D - 8I)(D + 5I)y = 0$$

To construct a basis of solutions we realize that

$$(D - 8I)y = 0 \Rightarrow y_1 = e^{8x}$$

$$(D + 5I)y = 0 \Rightarrow y_2 = e^{-5x}$$

Exercises

From Kreyszig (10th ed.), Chapter 2, Section 3:

- 2.3.14

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Free oscillations

Free oscillations of a mass-spring system

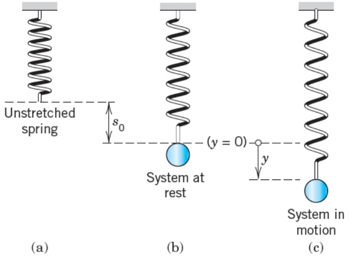


Fig. 33. Mechanical mass–spring system

If we pull the ball down, there is force

$$F = -ky \quad \text{Hooke's law}$$

k is the **spring constant**. Stiff springs have large k.

Free oscillations

Free oscillations of a mass-spring system (continued)

Newton's second law states

$$\sum F = ma$$

$$-ky = my''$$

We can easily solve it

$$\boxed{y'' + \frac{k}{m}y = 0}$$

$$\lambda^2 + \frac{k}{m} = 0 \Rightarrow \lambda_1, \lambda_2 = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

The general solution is

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

This is called an **harmonic oscillation** and its associated natural frequency is $f_0 = \frac{\omega_0}{2\pi}[Hz]$, the oscillation period is $T_0 = \frac{1}{f_0}[s]$.

Free oscillations of a mass-spring system (continued)

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

$$y = C \cos(\omega_0 t - \delta)$$

where $C = \sqrt{(}c_1^2 + c_2^2)$ and $\delta = \arctan \frac{c_2}{c_1}$.

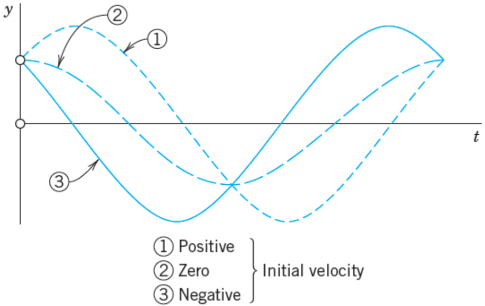
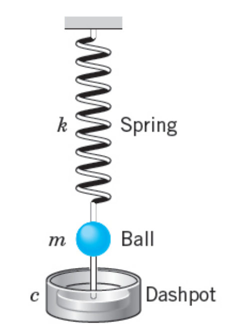


Fig. 34. Typical harmonic oscillations (4) and (4*) with the same $y(0) = A$ and different initial velocities $y'(0) = \omega_0 B$, positive ①, zero ②, negative ③

Damped oscillations

Damped oscillations of a mass-spring system



The dashpot introduces a braking force that at low speed can be modelled as $-cy'$. The overall model is

$$-ky - cy' = my''$$

$$\boxed{y'' + \frac{k}{m}y' + \frac{c}{m}y = 0}$$

$$\lambda^2 + \frac{k}{m}\lambda + \frac{c}{m} = 0$$

$$\lambda_1, \lambda_2 = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$

$$\lambda_1, \lambda_2 = -\alpha \pm \beta$$

Overdamping: $c^2 - 4mk > 0$

The general solution is

$$y = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$

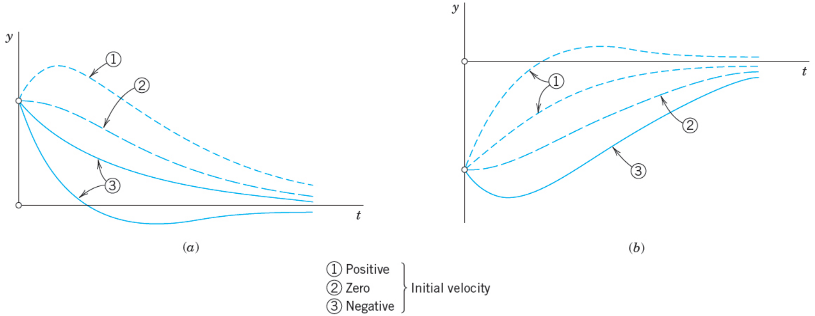


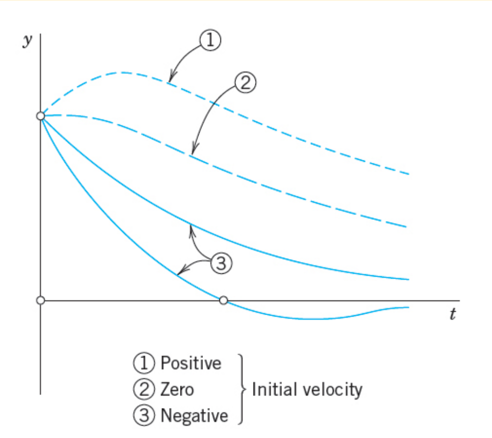
Fig. 37. Typical motions (7) in the overdamped case
(*a*) Positive initial displacement
(*b*) Negative initial displacement

Critical damping

Critical damping: $c^2 - 4mk = 0$

The general solution is

$$y = (c_1 + c_2 t)e^{-\alpha t}$$



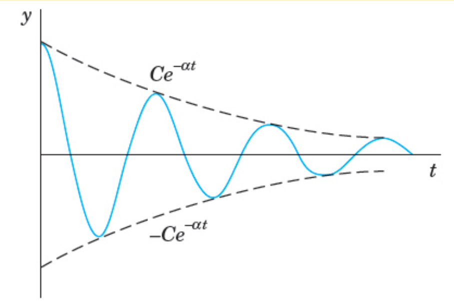
Underdamping

Underdamping: $c^2 - 4mk < 0$

$$\beta = i\frac{1}{2m}\sqrt{4mk - c^2} = i\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = i\omega^*$$

Note that if $c \to 0$, then $\omega^* \to \omega_0 = \sqrt{\frac{k}{m}}$ (harmonic oscillation). The general solution is

$$y = Ce^{-\alpha t}\cos(\omega^* t - \delta)$$



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 2, Section 4:

- 2.4.5
- 2.4.6
- 2.4.7
- 2.4.14
- 2.4.18

Outline

Euler-Cauchy equations

Euler-Cauchy equations

They are equations of the form

$$x^2y'' + axy' + by = 0$$

We subsitute

$$y = x^m$$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

to get

$$x^2(m(m-1)x^{m-2}) + ax(mx^{m-1}) + bx^m = 0$$

$$m(m-1)x^m + amx^m + bx^m = 0$$

$$x^m(m(m-1) + am + b) = 0$$

$$x^m(m^2 + (a-1)m + b) = 0$$

Euler-Cauchy equations

Euler-Cauchy equations (continued)

$$x^m(m^2 + (a-1)m + b) = 0$$

Hence, x^m is a solution of the ODE iff m is a solution of

$$m^2 + (a-1)m + b = 0$$

$$m_1, m_2 = \frac{1-a}{2} \pm \sqrt{\frac{1}{4}(1-a)^2 - b}$$

Euler-Cauchy equations

Euler-Cauchy equations: Two distinct real roots

The general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Example

$$y = x^2 y'' + 1.5xy' - 0.5y = 0$$

Solution:

$$m^2 + 0.5m - 0.5 = 0 \Rightarrow m_1 = 0.5, m_2 = -1$$

$$y = c_1 \sqrt{x} + c_2 \frac{1}{x}$$

Note that because of the square root, it must be $x > 0$ for this solution to exist.

Euler-Cauchy equations: A real double root

This happens if

$$\frac{1}{4}(1-a)^2 - b = 0 \Rightarrow b = \frac{(1-a)^2}{4}$$

Consequently the ODE can be rewritten as

$$x^2 y'' + axy' + \frac{(1-a)^2}{4} y = 0$$

$$y'' + \frac{a}{x} y' + \frac{(1-a)^2}{4x^2} y = 0$$

The real double root is

$$m_1 = \frac{1-a}{2}$$

and one of the solutions is

$$y_1 = x^{\frac{1-a}{2}}$$

Euler-Cauchy equations: A real double root (continued)

The other solution is obtained by reduction of order (Eq. (1))

$$U = \frac{1}{y_1^2} e^{-\int p dx}$$

That is

$$U = \frac{1}{\left(x^{\frac{1-a}{2}}\right)^2} e^{-\int \frac{a}{x} dx} = \frac{1}{x^{1-a}} e^{-a \log|x|} = \frac{x^{-a}}{x^{1-a}} = \frac{1}{x}$$

$$u = \int U dx = \int \frac{1}{x} dx = \log|x|$$

$$y_2 = u y_1 = x^{m_1} \log|x|$$

The general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 x^{m_1} + c_2 x^{m_1} \log|x| = (c_1 + c_2 \log|x|) x^{m_1}$$

Euler-Cauchy equations

Euler-Cauchy equations: Complex roots

$$m_1, m_2 = \alpha \pm i\omega$$

Two independent solutions are

$$y_1 = x^{\alpha+i\omega} = x^{\alpha}(e^{\log(x)})^{i\omega} = x^{\alpha}(e^{i\omega \log(x)}) = x^{\alpha}(\cos(\omega \log(x)) + i \sin(\omega \log(x)))$$

$$y_2 = x^{\alpha-i\omega} = x^{\alpha}(e^{\log(x)})^{-i\omega} = x^{\alpha}(e^{-i\omega \log(x)}) = x^{\alpha}(\cos(\omega \log(x)) - i \sin(\omega \log(x)))$$

We may obtain two other independent solutions as

$$y_1^* = \frac{y_1 + y_2}{2} = \mathrm{Re}\{y_1\} = x^{\alpha} \cos(\omega \log(x))$$

$$y_2^* = \frac{y_1 - y_2}{2i} = \mathrm{Im}\{y_1\} = x^{\alpha} \sin(\omega \log(x))$$

The general solution is

$$\boxed{y = c_1 y_1^* + c_2 y_2^* = x^{\alpha}(c_1 \cos(\omega \log(x)) + c_2 \sin(\omega \log(x)))}$$

Euler-Cauchy equations

Examples

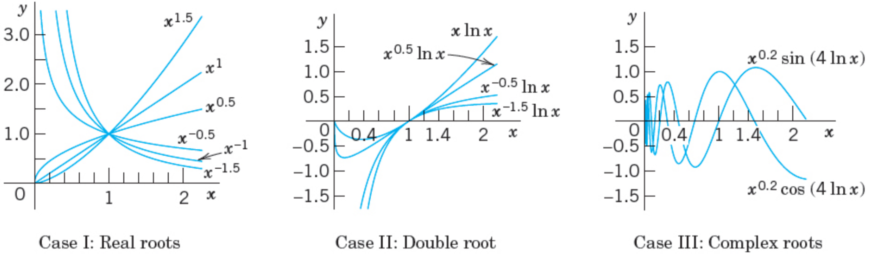


Fig. 48. Euler–Cauchy equations

Euler-Cauchy equations

Example

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

Physical Information. $v(r)$ is a solution of the Euler–Cauchy equation $rv'' + 2v' = 0$, where $v' = dv/dr$.

Solution: The constitutive equation

$$rv'' + 2v' = 0$$

is not Euler-Cauchy, but multiplying by r, it is

$$r^2 + 2rv' = 0$$

$$m^2 + m = 0 \Rightarrow m_1 = 0, m_2 = -1$$

The general solution is

$$v = c_1 x^0 + c_2 x^{-1} = c_1 + \frac{c_2}{x}$$

Euler-Cauchy equations

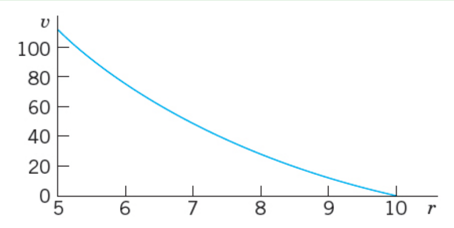
Example (continued)

$$v = c_1 x^0 + c_2 x^{-1} = c_1 + \frac{c_2}{x}$$

The particular solution comes from the boundary constraints

$$\left.\begin{array}{c} v(5) = 110 = c_1 + \frac{c_2}{5} \\ v(10) = 0 = c_1 + \frac{c_2}{10} \end{array}\right\} \Rightarrow c_1 = -110, c_2 = 1100$$

$$v = -110 + \frac{1100}{x}$$



Outline

1. Second-order linear ODEs
   - Homogeneous linear ODEs
   - Homogeneous linear ODEs with constant coefficients
   - Differential operators
   - Modeling of free oscillations of a mass-spring system
   - Euler-Cauchy equations
   - Existence and uniqueness of solutions. Wronskian
   - Nonhomogeneous ODEs
   - Forced oscillations. Resonance.
   - Electric circuits
   - Solution by variation of parameters

Existence and uniqueness of solutions

Existence and uniqueness Theorem

Let us analyze the existence and solutions of the Initial Value Problem

$$y'' + p(x)y' + q(x) = 0 \quad y(x_0) = K_0, y'(x_0) = K_1$$

If $p(x)$ and $q(x)$ are continuous on some open interval I and $x_0 \in I$, then the IVP has a unique solution in I.

Existence of a general solution

If p and q are continuous functions on an open interval I, then there exists a general solution on I and any solution is of the form

$$y = c_1 y_1 + c_2 y_2$$

where y_1 and y_2 are a basis of solutions on I. Hence, the IVP has no **singular solution** (that is, solutions that cannot be obtained from the general solution).

Linear independence of solutions: Wronskian

Linear independence of solutions: Wronskian

Considering the previous problem with continuous p and q functions on an open interval I. Two solutions, y_1 and y_2, on I are linearly independent if their Wroskian is different from 0 at some point $x \in I$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

If y_1 and y_2 are linearly dependent, then $W(x) = 0$ for all points $x \in I$.

Linear independence of solutions: Wronskian

Example

$y_1 = \cos(\omega x)$ and $y_2 = \sin(\omega x)$ are solutions of $y'' + \omega^2 y = 0$. Check if they are linearly independent.

Solution:

$$W(x) = \begin{vmatrix} \cos(\omega x) & \sin(\omega x) \\ -\omega \sin(\omega x) & \omega \cos(\omega x) \end{vmatrix} = \omega \cos^2(\omega x) + \omega \sin^2(\omega x) = \omega$$

The Wronskian is 0 only if $\omega = 0$. So, in general, the two functions are linearly independent (also their ratio, $\frac{\sin(x)}{\cos(x)} = \tan(x)$, is not a constant; this would be another way of checking).

However, if $\omega = 0$, then $y_1 = 1$, $y_2 = 0$. These two functions are linearly dependent and they are not a basis of solutions. In fact, in this case the basis is given by $y_1 = 1$, $y_2 = x$.

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 2, Section 6:

- 2.6.5
- 2.6.12

Outline

Nonhomogeneous ODEs

Nonhomogeneous ODEs

$$y'' + p(x)y' + q(x) = r(x) \quad (NH)$$

A general solution of the nonhomogeneous ODE is of the form

$$y = y_h + y_p$$

where y_h is the general solution of the homogeneous problem

$$y'' + p(x)y' + q(x) = r(x) \quad (H)$$

and y_p is a particular solution of NH. A particular solution of NH is obtained by determining the constants of the general solution.

If p, q, and r are continuous functions in an open interval I, then there is no singular solution in I (that is, all solutions can be obtained from the general solution).

Nonhomogeneous ODEs

Theorem: Relationship between H and NH

1. $y_H + y_{NH,1} = y_{NH,2}$. The sum of a solution of H and a solution of NH is a solution of NH.
2. $y_{NH,1} - y_{NH,2} = y_H$. The subtraction of two solutions of NH is a solution of H.

Proof: Let us denote the H and NH problems as

$$Ly = 0 \quad H$$

$$Ly = r \quad NH$$

1. $L(y_H + y_{NH,1}) = Ly_H + Ly_{NH,1} = 0 + r = r$
2. $L(y_{NH,1} - y_{NH,2}) = Ly_{NH,1} - Ly_{NH,2} = r - r = 0$

Nonhomogeneous ODEs

Transient and steady-state solutions

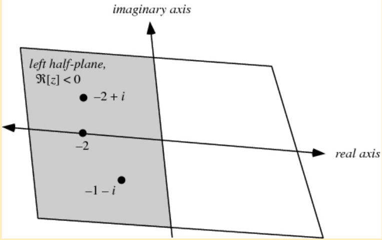
Since the general solution of the NH problem is

$$y = y_h + y_p$$

if $\mathrm{Re}\{\lambda_i\} < 0$ for all i, then the term coming from the homogeneous solution vanishes with increasing x and the solution tends to be that given by the input signal

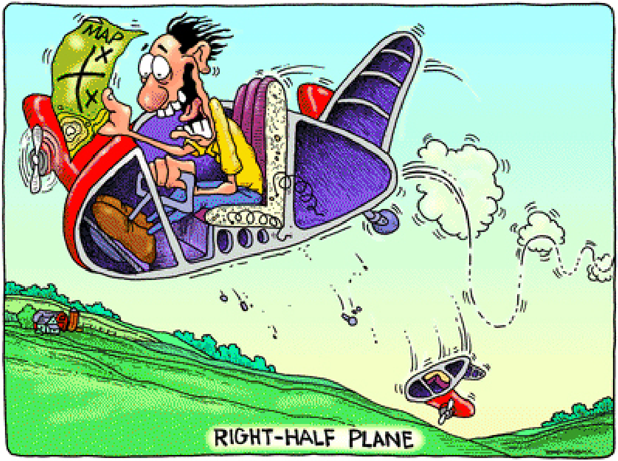
$$\lim_{x \to \infty} y_h + y_p = y_p$$

This condition is important in system theory to define stable systems.



Nonhomogeneous ODEs

Transient and steady-state solutions



Method of undetermined coefficients

Method of undetermined coefficients

$$y'' + ay' + by = r(x)$$

Rules:

- Basic: Depending on $r(x)$ choose y_p as

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

- Modification: If the term in r is also a solution of H, multiply it by x or x^2 depending if it is a single or double root of the characteristic polynomial.
- Sum: If r is a sum of functions, choose a sum of y_p's.

Method of undetermined coefficients

Example

$$y'' + y = 0.001x^2 \quad y(0) = 0, y'(0) = 1.5$$

Solution:

The general solution of the H problem is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

For the particular solution of the NH problem we choose

$$y_p = K_2x^2 + K_1x + K_0$$

$$y_p' = 2K_2x + K_1$$

$$y_p'' = 2K_2$$

Method of undetermined coefficients

Example (continued)

And substitute it in the NH ODE

$$(2K_2) + (K_2x^2 + K_1x + K_0) = 0.001x^2$$

$$K_2x^2 + K_1x + (K_0 + 2K_2) = 0.001x^2 \Rightarrow K_2 = 0.001, K_1 = 0, K_0 = -0.002$$

So

$$y_p = 0.001x^2 - 0.002$$

The general solution of the NH problem is

$$y = c_1\cos(x) + c_2\sin(x) + 0.001x^2 - 0.002$$

Method of undetermined coefficients

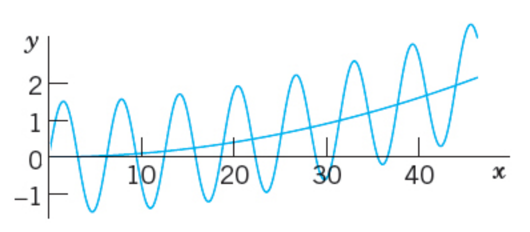
Example (continued)

For the particular solution we impose the initial conditions

$$y(0) = 0 = c_1 \cos(0) + c_2 \sin(0) + 0.001(0)^2 - 0.002 \Rightarrow c_1 = 0.002$$

$$y'(0) = 1.5 = c_1(-\sin(0)) + c_2 \cos(0) + 2 \cdot 0.001(0) \Rightarrow c_2 = 1.5$$

$$y = 0.002 \cos(x) + 1.5 \sin(x) + 0.001x^2 - 0.002$$



Method of undetermined coefficients

Example

$$y'' + 3y' + 2.25y = -10e^{-1.5x} \quad y(0) = 1, y'(0) = 0$$

Solution:
The characteristic equation of the H problem is

$$\lambda^2 + 3\lambda + 2.25 = 0$$

$$(\lambda + 1.5)^2 = 0$$

So the general solution of the H problem is

$$y_h = (c_1 + c_2 x)e^{-1.5x}$$

Since the excitation signal, r, corresponds to one of the solutions of the H problem (a double root) we choose

$$y_p = Cx^2 e^{-1.5x}$$

Method of undetermined coefficients

Example (continued)

$$y_p = Cx^2 e^{-1.5x}$$
$$y_p' = C(2x - 1.5x^2)e^{-1.5x}$$
$$y_p'' = C(2 - 6x - 2.25x^2)e^{-1.5x}$$

And substitute it in the NH problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}$$

$$C(2 - 6x - 2.25x^2)e^{-1.5x} + 3C(2x - 1.5x^2)e^{-1.5x} + 2.25Cx^2 e^{-1.5x} = -10e^{-1.5x}$$
$$C(2 - 6x - 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10$$
$$0x^2 + 0x + 2C = -10 \Rightarrow C = -5$$

So the general solution of the NH problem is

$$y = (c_1 + c_2 x)e^{-1.5x} - 5x^2 e^{-1.5x} = (c_1 + c_2 x - 5x^2)e^{-1.5x}$$

Method of undetermined coefficients

Example (continued)

$$y = (c_1 + c_2 x - 5x^2)e^{-1.5x}$$

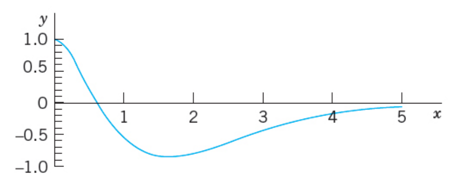
To determine c_1 and c_2 we impose the initial conditions

$$y(0) = 1 = (c_1 + c_2 0 - 5(0)^2)e^{-1.5\cdot 0} \Rightarrow c_1 = 1$$

$$y'(0) = 0 = (c_2 - 10(0) - 1.5(c_1 + c_2 0 - 5(0)^2))e^{-1.5\cdot 0} = c_2 - 1.5c_1 \Rightarrow c_2 = 1.5$$

Finally, the particular solution is

$$y = (1 + 1.5x - 5x^2)e^{-1.5x}$$



Method of undetermined coefficients

Example

$$y'' + 2y' + 0.75y = 2\cos(x) - 0.25\sin(x) + 0.99x \quad y(0) = 2.78, y'(0) = -0.43$$

Solution:
The characteristic equation of the H problem is

$$\lambda^2 + 2\lambda + 0.75 = 0$$

$$(\lambda + 0.5)(\lambda + 1.5) = 0$$

So the general solution of the H problem is

$$y_h = c_1 e^{-0.5x} + c_2 e^{-1.5x}$$

Since the excitation signal, r, is a sum of functions we choose

$$y_p = K\cos(x) + M\sin(x) + K_1 x + K_0$$

Method of undetermined coefficients

Example (continued)

$$y_p = K\cos(x) + M\sin(x) + K_1 x + K_0$$

$$y_p' = -K\sin(x) + M\cos(x) + K_1$$

$$y_p'' = -K\cos(x) - M\sin(x)$$

And substitute it in the NH problem

$$y'' + 2y' + 0.75y = 2\cos(x) - 0.25\sin(x) + 0.99x$$

$$\begin{gathered}(-K\cos(x) - M\sin(x)) + 2(-K\sin(x) + M\cos(x) + K_1) + \\ +0.75(K\cos(x) + M\sin(x) + K_1 x + K_0) = \\ 2\cos(x) - 0.25\sin(x) + 0.99x\end{gathered}$$

$$\begin{gathered}(2M - 0.25K)\cos(x) - (1.25M + 2K)\sin(x) + (0.75K_1)x + (2K_1 + 0.75K_0) = \\ 2\cos(x) - 0.25\sin(x) + 0.99x\end{gathered}$$

$$\Rightarrow K = 0, M = 1, K_1 = 0.12, K_0 = -0.32$$

$$y_p = \sin(x) + 0.12x - 0.32$$

Method of undetermined coefficients

Example (continued)

So the general solution of the NH problem is

$$y = c_1 e^{-0.5x} + c_2 e^{-1.5x} + \sin(x) + 0.12x - 0.32$$

To find a particular solution we impose the initial conditions

$$y(0) = 2.78 = c_1 e^{-0.5\cdot 0} + c_2 e^{-1.5\cdot 0} + \sin(0) + 0.12 \cdot 0 - 0.32 = c_1 + c_2 - 0.32$$

$$\begin{aligned} y'(0) &= -0.43 = -0.5c_1 e^{-0.5\cdot 0} - 1.5c_2 e^{-1.5\cdot 0} + \cos(0) + 0.12 \\ &= -0.5c_1 - 1.5c_2 + 1 + 0.12 \end{aligned}$$

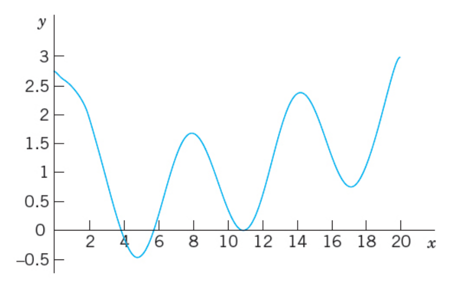
$$\Rightarrow c_1 = 3.1, c_2 = 0$$

Method of undetermined coefficients

Example (continued)

So the particular solution is

$$y = 3.1e^{-0.5x} + \sin(x) + 0.12x - 0.32$$



Exercises

Exercises

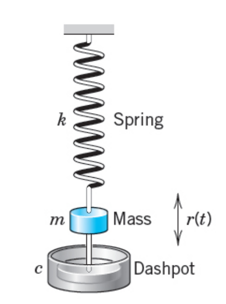
From Kreyszig (10th ed.), Chapter 2, Section 7:

- 2.7.6

Outline

Forced oscillations

Forced oscillations



If we now apply an external force to the mass, then the ODE model becomes

$$my'' = -cy' - ky + r(t)$$

Of special interest are external forces of the form

$$r(t) = F_0 \cos(\omega t)$$

Let us concentrate on the nonhomogeneous problem

$$my'' + cy' + ky = F_0 \cos(\omega t)$$

Forced oscillations

Forced oscillations (continued)

We remind that the solution of the homogeneous system is given by the roots

$$\lambda_1, \lambda_2 = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$

and that depending on the value of $c^2 - 4mk$ we have overdamping, critical damping or underdamping (see Section. 2.4). The particular solution is of the form

$$y_p = a\cos(\omega t) + b\sin(\omega t)$$

$$y_p' = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$$

$$y_p'' = -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t)$$

Substituting in the NH problem we get

Forced oscillations (continued)

$$my'' + cy' + ky = F_0 \cos(\omega t)$$

$$m(-a\omega^2 \cos(\omega t) - b\omega^2 \sin(\omega t)) + c(-a\omega \sin(\omega t) + b\omega \cos(\omega t)) + \\ + k(a \cos(\omega t) + b \sin(\omega t)) = F_0 \cos(\omega t)$$

$$(-ma\omega^2 + bc\omega + ka) \cos(\omega t) + (-mb\omega^2 - ca\omega + kb) \sin(\omega t) = F_0 \cos(\omega t)$$

$$\Rightarrow \begin{cases} (k - m\omega^2)a + c\omega b = F_0 \\ -c\omega a + (k - m\omega^2)b = 0 \end{cases}$$

$$\begin{pmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}$$

$$b = F_0 \frac{c\omega}{(k - m\omega^2)^2 + \omega^2 c^2}$$

Forced oscillations (continued)

Now we exploit that

$$\omega_0^2 = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_0^2$$

And we rewrite a and b

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

$$b = F_0 \frac{c\omega}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

The particular solution is

$$y_p = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \cos(\omega t) + F_0 \frac{c\omega}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \sin(\omega t)$$

And the general solution

$$y = y_h + y_p$$

Forced oscillations

Case: Undamped forced oscillations ($c = 0$)

The particular solution becomes

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

The absence of damping causes the homogeneous solution

$$y_h = C \cos(\omega_0 t - \delta)$$

The general solution is

$$y = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

This is valid as long as $\omega \neq \omega_0$.

Forced oscillations

Case: Undamped forced oscillations ($c = 0$)

For $C = \frac{F_0}{m(\omega_0^2 - \omega^2)}$ and $\delta = 0$ we get the particular solution:

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos(\omega_0 t) + \cos(\omega t)) = \frac{F_0}{m(\omega_0^2 - \omega^2)} 2\cos\left(\frac{\omega_0 + \omega}{2} t\right)\cos\left(\frac{\omega_0 - \omega}{2} t\right)$$

If $\omega_0 \approx \omega$ then, we get a solution like

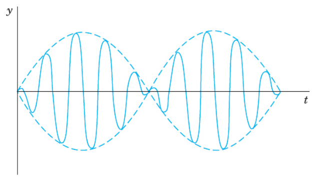


Fig. 56. Forced undamped oscillation when the difference of the input and natural frequencies is small (**"beats"**)

They are called beats. This is what musicians listen to when they tune their instruments.

Forced oscillations

Case: Undamped forced oscillations ($c = 0$), resonance

If $\omega = \omega_0$, then the situation is called resonance. In this case, the particular solution is no longer valid. Let's find it again. The ODE is

$$my'' + ky = F_0 \cos(\omega_0 t)$$

$$y'' + \frac{k}{m} y = \frac{F_0}{m} \cos(\omega_0 t)$$

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos(\omega_0 t)$$

The driving function, r, is one of those associated to a root of the characteristic equation. So we try

$$y_p = t(a \cos(\omega_0 t) + b \sin(\omega_0 t))$$

Forced oscillations

Case: Undamped forced oscillations ($c = 0$), resonance

$$y_p = t(a\cos(\omega_0 t) + b\sin(\omega_0 t))$$

$$y_p' = (a + bt\omega_0)\cos(\omega_0 t) + (b - at\omega_0)\sin(\omega_0 t)$$

$$y_p'' = (2b\omega_0 - at\omega_0^2)\cos(\omega_0 t) - (bt\omega_0^2 + 2a\omega_0)\sin(\omega_0 t))$$

Now we substitute this solution in the ODE

$$my'' + ky = F_0\cos(\omega_0 t)$$

$$y'' + \omega_0^2 y = F_0\cos(\omega_0 t)$$

$$\begin{gathered}(2b\omega_0 - at\omega_0^2)\cos(\omega_0 t) - (bt\omega_0^2 + 2a\omega_0)\sin(\omega_0 t) + \\ +\omega_0^2 t(a\cos(\omega_0 t) + b\sin(\omega_0 t)) = F_0\cos(\omega_0 t)\end{gathered}$$

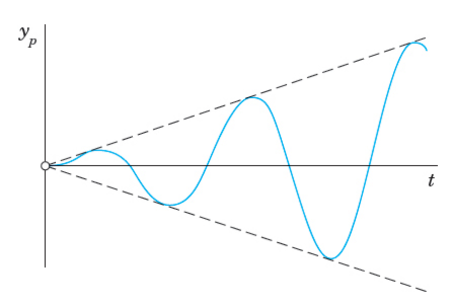
$$2b\omega_0\cos(\omega_0 t) - 2\omega_0 a\sin(\omega_0 t) = F_0\cos(\omega_0 t) \quad \Rightarrow a = 0, b = \frac{F_0}{2\omega_0}$$

$$y_p = \frac{F_0}{2\omega_0} t\sin(\omega_0 t)$$

Forced oscillations

Case: Undamped forced oscillations ($c = 0$), resonance

$$y_p = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$



Tacoma bridge resonance

Forced oscillations

Case: Damped forced oscillations, practical resonance

In practice, there is always some damping and the amplitude does not grow infinitely. Let's analyze the maximum amplitude. The particular solution was

$$y_p = a\cos(\omega t) + b\sin(\omega t)$$

with $a = F_0 \frac{m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}$ and $b = F_0 \frac{c\omega}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}$ We may rewrite the particular solution as

$$y_p = C^*\cos(\omega t - \eta)$$

with

$$C^* = \sqrt{a^2+b^2} = F_0 \frac{1}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}}$$

$$\eta = \arctan\left(\frac{b}{a}\right)$$

Forced oscillations

Case: Damped forced oscillations, practical resonance

$$C^* = \sqrt{a^2 + b^2} = F_0 \frac{1}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}}$$

Let's find the maximum amplitude

$$0 = \frac{dC^*}{d\omega} = F_0 \left(-\frac{1}{2} (m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2)^{-\frac{3}{2}} \right) \left[2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2 \right]$$

$$0 = 2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2$$
$$c^2 = 2m^2(\omega_0^2 - \omega^2)$$
$$\omega_{\max}^2 = \omega_0^2 - \frac{c^2}{2m^2}$$

That is, practical resonance occurs a little bit earlier than the natural frequency.

Forced oscillations

Case: Damped forced oscillations, practical resonance

It can be verified that the maximum amplitude at ω_{max} is

$$C^*_{max} = F_0 \frac{2m}{c\sqrt{4m^2\omega_0^2 - c^2}}$$

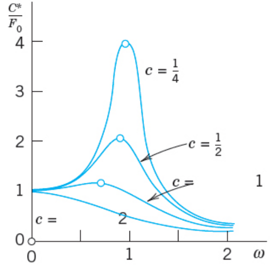


Fig. 57. Amplification C^*/F_0 as a function of ω for $m = 1, k = 1$, and various values of the damping constant c

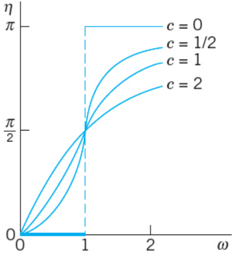


Fig. 58. Phase lag η as a function of ω for $m = 1, k = 1$, thus $\omega_0 = 1$, and various values of the damping constant c

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 2, Section 8:

- 2.8.13

Outline

1. Second-order linear ODEs
   - Homogeneous linear ODEs
   - Homogeneous linear ODEs with constant coefficients
   - Differential operators
   - Modeling of free oscillations of a mass-spring system
   - Euler-Cauchy equations
   - Existence and uniqueness of solutions. Wronskian
   - Nonhomogeneous ODEs
   - Forced oscillations. Resonance.
   - Electric circuits
   - Solution by variation of parameters

Electric circuits

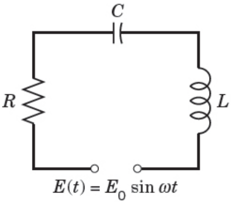


Fig. 61. *RLC*-circuit

Name	Symbol	Notation		Unit	Voltage Drop
Ohm's Resistor	—/\/\/\/—	R	Ohm's Resistance	ohms (Ω)	RI
Inductor	—0000—	L	Inductance	henrys (H)	$L\,\dfrac{dI}{dt}$
Capacitor	—)|—	C	Capacitance	farads (F)	Q/C

Electric circuits

Electric circuits (continued)

The relationship in the capacitor between charge and current is

$$I = \frac{dQ}{dt} = Q = \int I dt$$

The ODE modeling the RLC circuit is

$$LI' + RI + \frac{1}{C}\int I dt = E_0 \sin(\omega t)$$

$$LI'' + RI' + \frac{1}{C}I = E_0 \omega \cos(\omega t)$$

To solve the homogeneous equation, we solve the characteristic polynomial

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0 \Rightarrow \lambda = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

Electric circuits

Electric circuits (continued)

For a particular of the non-homogeneous problem we try with a function of the form

$$I_p = a\cos(\omega t) + b\sin(\omega t)$$

$$I_p' = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$$

$$I_p'' = -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t)$$

And subsitute it in the ODE

$$LI'' + RI' + \frac{1}{C}I = E_0\omega\cos(\omega t)$$

$$L(-a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t)) + R(-a\omega\sin(\omega t) + b\omega\cos(\omega t)) + \\ + \tfrac{1}{C}(a\cos(\omega t) + b\sin(\omega t)) = E_0\omega\cos(\omega t)$$

$$\left(\left(-L\omega^2 + \tfrac{1}{C}\right)a + R\omega b\right)\cos(\omega t) + \left(-R\omega a + \left(-L\omega^2 + \tfrac{1}{C}\right)b\right)\sin(\omega t) = \\ = E_0\omega\cos(\omega t)$$

Electric circuits

Electric circuits (continued)

$$\left(\left(-L\omega^2 + \tfrac{1}{C}\right) a + R\omega b\right) \cos(\omega t) + \left(-R\omega a + \left(-L\omega^2 + \tfrac{1}{C}\right) b\right) \sin(\omega t) = $$
$$= E_0 \omega \cos(\omega t)$$

$$\begin{pmatrix} -L\omega^2 + \frac{1}{C} & R\omega \\ -R\omega & -L\omega^2 + \frac{1}{C} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} E_0\omega \\ 0 \end{pmatrix}$$

$$\omega \begin{pmatrix} -L\omega + \frac{1}{C\omega} & R \\ -R & -L\omega + \frac{1}{C\omega} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \omega \begin{pmatrix} E_0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -S & R \\ -R & -S \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} E_0 \\ 0 \end{pmatrix} \Rightarrow a = \frac{-E_0 S}{R^2 + S^2}, b = \frac{E_0 R}{R^2 + S^2}$$

where S is the impedance

$$S = L\omega - \frac{1}{C\omega}$$

Electric circuits

Electric circuits (continued)

$$a = \frac{-E_0 S}{R^2 + S^2}, b = \frac{E_0 R}{R^2 + S^2}$$

The particular solution to the NH problem is

$$I_p = a\cos(\omega t) + b\sin(\omega t)$$

$$I_p = \sqrt{a^2 + b^2}\sin\left(\omega t - \arctan\frac{a}{b}\right)$$

$$\boxed{I_p = \frac{E_0}{\sqrt{R^2 + S^2}}\sin\left(\omega t - \arctan\frac{S}{R}\right)}$$

Electric circuits

RLC circuit

Find the current $I(t)$ in an RLC-circuit with $R = 11\ \Omega$ (ohms), $L = 0.1$ H (henry), $C = 10^{-2}$ F (farad), which is connected to a source of EMF $E(t) = 110 \sin(60 \cdot 2\pi t) = 110 \sin 377\, t$ (hence 60 Hz = 60 cycles/sec, the usual in the U.S. and Canada; in Europe it would be 220 V and 50 Hz). Assume that current and capacitor charge are 0 when $t = 0$.

Solution:

$$LI'' + RI' + \frac{1}{C}I = E_0\omega\cos(\omega t)$$

The homogeneous solution is given by

$$0.1\lambda^2 + 11\lambda + \frac{1}{0.01} = 0 \Rightarrow \lambda_1 = -10, \lambda_2 = -100$$

$$I_h = c_1 e^{-10t} + c_2 e^{-100t}$$

Electric circuits

RLC circuit (continued)

The particular solution

$$I_p = \frac{E_0}{\sqrt{R^2 + S^2}} \sin\left(\omega t - \arctan\frac{S}{R}\right)$$

with $E_0 = 110$ and

$$\omega = 60 \cdot 2\pi = 377$$

$$S = L\omega - \frac{1}{C\omega} = 0.1 \cdot 377 - \frac{1}{0.01 \cdot 377} = 37.7 - 0.3 = 37.4$$

$$I_p = \frac{110}{\sqrt{11^2 + 37.4^2}} \sin\left(60 \cdot 2\pi t + \arctan\frac{37.4}{11}\right)$$

$$I_p = 2.82 \sin\left(60 \cdot 2\pi t + 73.6^\circ\right)$$

The general solution is

$$I = c_1 e^{-10t} + c_2 e^{-100t} + 2.82 \sin\left(60 \cdot 2\pi t + 73.6^\circ\right)$$

RLC circuit (continued)

$$I = c_1 e^{-10t} + c_2 e^{-100t} + 2.82 \sin\left(60 \cdot 2\pi t - 73.6^\circ\right)$$

To find the constants c_1 and c_2 we apply the initial conditions $I(0) = 0$, $Q(0) = 0$. To use $Q(0) = 0$, we note that the ODE was originally written as

$$LI' + RI + \frac{1}{C} \int I dt = E_0 \sin(\omega t)$$

$$LI'(t) + RI(t) + \frac{1}{C} Q(t) = E_0 \sin(\omega t)$$

At $t = 0$ we have

$$LI'(0) + RI(0) + \frac{1}{C} Q(0) = E_0 \sin(\omega 0)$$

$$LI'(0) = 0 \Rightarrow I'(0) = 0$$

RLC circuit (continued)

So the initial conditions become $I(0) = 0$, $I'(0) = 0$

$$I(0) = 0 = c_1 e^{-10\cdot 0} + c_2 e^{-100\cdot 0} + 2.82 \sin\left(60 \cdot 2\pi 0 - 73.6^\circ\right) = c_1 + c_2 - 2.71$$

$$\begin{aligned} I'(0) &= 0 = -10 c_1 e^{-10\cdot 0} - 100 c_2 e^{-100\cdot 0} + 2.82(60 \cdot 2\pi) \cos\left(60 \cdot 2\pi 0 - 73.6^\circ\right) \\ &= -10 c_1 - 100 c_2 - 300.1 \end{aligned}$$

The solution is $c_1 = -0.323$, $c_2 = 3.033$. Finally,

$$I = -0.323 e^{-10t} + 3.033 e^{-100t} + 2.82 \sin\left(60 \cdot 2\pi t + 73.6^\circ\right)$$

Electric circuits

RLC circuit (continued)

$$I = -0.323e^{-10t} + 3.033e^{-100t} + 2.82\sin\left(60 \cdot 2\pi t + 73.6^\circ\right)$$

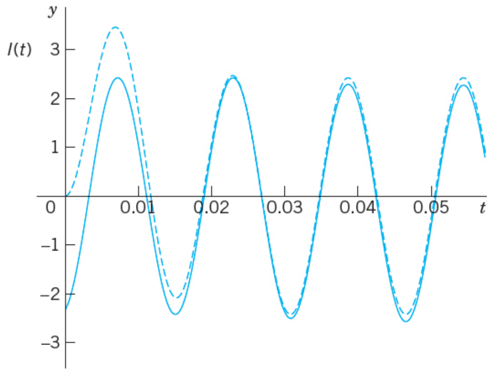


Fig. 63. Transient (upper curve) and steady-state currents in Example 1

Analogy Electric circuits-Mechanical systems

Analogy

$$LI'' + RI' + \frac{1}{C}I = r(t)$$

$$my'' + cy' + ky = r(t)$$

Table 2.2 Analogy of Electrical and Mechanical Quantities

Electrical System	Mechanical System
Inductance L	Mass m
Resistance R	Damping constant c
Reciprocal $1/C$ of capacitance	Spring modulus k
Derivative $E_0\omega \cos \omega t$ of electromotive force	Driving force $F_0 \cos \omega t$
Current $I(t)$	Displacement $y(t)$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 2, Section 9:

- 2.9.1

Outline

1. Second-order linear ODEs
   - Homogeneous linear ODEs
   - Homogeneous linear ODEs with constant coefficients
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   - Solution by variation of parameters

Variation of parameters

Variation of parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

The difference with undertermined coefficients is that now p and q do not need to be constant, although they must be continuous in an open interval I. Let's assume that y_1 and y_2 are two independent solutions of the H problem. Let us assume that there is a particular solution of the NH problem of the form

$$y_p = u(x)y_1 + v(x)y_2$$

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2' = (u'y_1 + v'y_2) + (uy_1' + vy_2')$$

Since we have one equation (the ODE) and two unknowns (u and v) we may impose an extra constraint

$$u'y_1 + v'y_2 = 0$$

Thus

$$y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$$

Variation of parameters

Variation of parameters (continued)

Now we substitute into the ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

$$(u'y_1' + uy_1'' + v'y_2' + vy_2'') + p(uy_1' + vy_2') + q(uy_1 + vy_2) = r$$

$$u'y_1' + v'y_2' + (y_1'' + py_1' + qy_1)u + (y_2'' + py_2' + qy_2) = r$$

$$u'y_1' + v'y_2' = r$$

Now we have two equations with two unknows

$$\left.\begin{array}{r} u'y_1 + v'y_2 = 0 \\ u'y_1' + v'y_2' = r \end{array}\right\} \Rightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

$$u' = -\frac{ry_2}{W}, v' = \frac{ry_1}{W}$$

Variation of parameters

Variation of parameters (continued)

$$u' = -\frac{ry_2}{W}, v' = \frac{ry_1}{W}$$

$$u = -\int \frac{ry_2}{W} dx, v = \int \frac{ry_1}{W}$$

Finally,

$$\boxed{y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx}$$

Variation of parameters

Example

$$y'' + y = \frac{1}{\cos(x)}$$

Solution:

$$y_1 = \cos(x)$$

$$y_2 = \sin(x)$$

$$\begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1$$

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

$$y_p = -\cos(x) \int \frac{\sin(x)}{\cos(x)} dx + \sin(x) \int \frac{\cos(x)}{\cos(x)} dx$$

$$y_p = -\cos(x) \log|\cos(x)| + x \sin(x)$$

Variation of parameters

Example

$$y_p = -\cos(x)\log|\cos(x)| + x\sin(x)$$

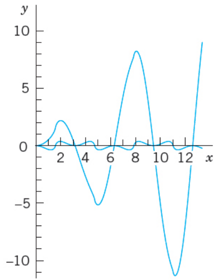

Fig. 70. Particular solution y_p and its first term in Example 1

The general solution is

$$y = c_1\cos(x) + c_2\sin(x) - \cos(x)\log|\cos(x)| + x\sin(x)$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 2, Section 10:

- 2.10.6

Outline

1. Second-order linear ODEs
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