Chapter 3. Higher-order linear ODEs

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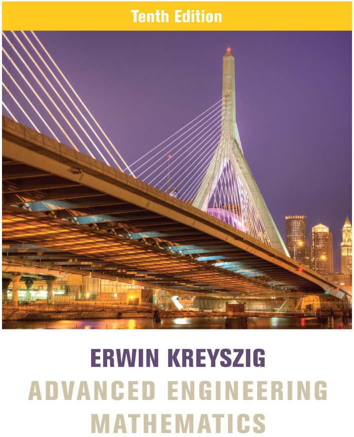
Outline

References



E. Kreyszig. Advanced Engineering Mathematics. John Wiley & sons. Chapter 3.

Outline

Homogeneous linear ODEs of higher-order

Definition

A nth-order ODE is **linear** if it can be written as

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = r(x)$$

Otherwise, it is **nonlinear**. It is **homogeneous** if $r(x) = 0$.

Theorem: Principle of superposition

The linear combination of any number solutions of a homogeneous, linear ODE is also a solution.

General solution

The general solution of the ODE on an open interval I is of the form

$$y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$$

where $\{y_1, y_2, ..., y_n\}$ is a set of linearly independent (in I) solutions.

Linear independence

Linear independence

A set $\{y_1, y_2, ..., y_n\}$ is linearly independent in I if

$$k_1 y_1 + k_2 y_2 + ... + k_n y_n = 0 \Rightarrow k_1 = k_2 = ... = k_n = 0$$

Example

$\{x^2, 5x, 2x\}$ is linearly dependent on any open interval because

$$0y_1 + y_2 - \frac{5}{2}y_3 = 0$$

$\{1, x, x^2\}$ is linearly independent on any open interval. We will learn a good

method of testing independence with the Wronskian.

Basis of a general solution

Example

$$y^{iv} - 5y'' + 4y = 0$$

Solution:
Let's try a solution of the form $e^{\lambda x}$

$$\lambda^4 e^{\lambda x} - 5\lambda^2 e^{\lambda x} + 4e^{\lambda x} = 0$$

$$e^{\lambda x}(\lambda^4 - 5\lambda^2 + 4) = 0$$

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

$$(\lambda^2)^2 - 5(\lambda^2) + 4 = 0 \Rightarrow \lambda^2 = 4, 1 \Rightarrow \lambda = \pm 2, \pm 1$$

The general solution is

$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}$$

Initial Value Problem

Initial Value Problem

An Initial Value Problem consists of n initial conditions

$$y(x_0) = K_0$$

$$y'(x_0) = K_1$$

$$...$$

$$y^{(n-1)}(x_0) = K_{n-1}$$

These n conditions are used to determine the constants of the general solution

$$y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$$

Initial Value Problem

Example

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0 \quad y(1) = 2, y'(1) = 1, y''(1) = -4$$

Solution:
This is a third-order Euler-Cauchy problem. Let's try a solution of the form $y = x^m$

$$x^3(m(m-1)(m-2)x^{m-3}) - 3x^2(m(m-1)x^{m-2}) + 6x(mx^{m-1}) - 6x^m = 0$$

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0$$

$$x^m(m(m-1)(m-2) - 3m(m-1) + 6m - 6) = 0$$

$$x^m(m^3 - 6m^2 + 11m - 6) = 0$$

$$x^m(m-1)(m-2)(m-3) = 0$$

So the general solution is

$$y = c_1x + c_2x^2 + c_3x^3$$

Initial Value Problem

Example (continued)

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

The particular solution is determined by imposing the initial conditions $y(1) = 2, y'(1) = 1, y''(1) = -4$

$$y(1) = 2 = c_1 + c_2 + c_3$$

$$y'(1) = 1 = c_1 + 2c_2 + 3c_3$$

$$y''(1) = -4 = 2c_2 + 6c_3$$

The solution is $c_1 = 2$, $c_2 = 1$, $c_3 = -1$ and the particular solution becomes

$$y_p = 2x + x^2 - x^3$$

Linear independence of solutions. Wronskian

Linear independence of solutions: Wronskian

Considering the previous problem with continuous p_{n-1}, p_{n-2}, ... and p_0 functions on an open interval I. A set of solutions $\{y_1, y_2, ..., y_n\}$ on I is linearly independent iff their Wroskian is different from 0 all $x \in I$

$$W(x) = \begin{vmatrix} y_1 & y_2 & ... & y_n \\ y_1' & y_2' & ... & y_n' \\ ... & ... & ... & ... \\ y_1^{(n-1)} & y_2^{(n-1)} & ... & y_n^{(n-1)} \end{vmatrix} \neq 0$$

If $W(x) = 0$ for a point $x \in I$, then it is also 0 for all $x \in I$ (and the two solutions are linearly dependent).

Linear independence of solutions. Wronskian

Example

$\{1, x, x^2\}$ are solutions of the ODE

$$y''' = 0$$

The p coefficients are continouous everywhere ($\mathbb{R}$). Then, by the previous theorem they are independent if their Wronskian is different from 0 at some point

$$W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$$

Since the Wronskian is not 0, the 3 functions are linearly independent.

Existence and uniqueness of solutions

Existence and uniqueness Theorem

Let us analyze the existence and solutions of the Initial Value Problem. If $p_{n-1}(x)$, $p_{n-2}(x)$, ..., $p_0(x)$ are continuous on some open interval I and $x_0 \in I$, then the IVP has a unique solution in I.

Existence of a general solution

If $p_{n-1}(x)$, $p_{n-2}(x)$, ..., $p_0(x)$ are continuous functions on an open interval I, then there exists a general solution on I and any solution is of the form

$$y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$$

where $\{y_1, y_2, ..., y_n\}$ is a basis of solutions on I. Hence, the IVP has no **singular solution** (that is, solutions that cannot be obtained from the general solution).

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 3, Section 1:

- 3.1.1
- 3.1.5

Outline

1. Higher-order linear ODEs
   - Homogeneous linear ODEs
   - Homogeneous linear ODEs with constant coefficients
   - Nonhomogeneous linear ODEs

Homogeneous linear ODEs with constant coefficients

Definition

$$y^{(n)} + p_{n-1}y^{(n-1)} + ... + p_1 y' + p_0 y = 0$$

If we try a solution of the form $y = e^{\lambda x}$, we get the characteristic equation

$$\lambda^n + p_{n-1}\lambda^{n-1} + ... + p_1\lambda + p_0 = 0$$

whose roots may be real or complex, and have any degree of multiplicity.

Distinct real roots

The general solution is

$$y = c_1 e^{\lambda_1 x} + ... + c_n e^{\lambda_n x}$$

Homogeneous linear ODEs with constant coefficients

Simple complex roots

The general solution associated to a couple of conjugated complex roots is

$$\lambda_1, \lambda_2 = \alpha \pm i\omega$$

$$y = e^{\alpha x}(c_1 \cos(\omega x) + c_2 \sin(\omega x)) = e^{\alpha x} \cos(\omega x - \delta)$$

Example

$$y''' - y'' + 100y' - 100y = 0$$

Solution:

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$$

$$(\lambda - 1)(\lambda^2 + 100) = 0 \Rightarrow \lambda = 1, \pm 10i$$

The general solution is

$$y = c_1 e^x + c_2 \cos(10x) + c_3 \sin(10x)$$

Homogeneous linear ODEs with constant coefficients

Multiple real roots

The general solution associated to a real root of order m is

$$y = (c_0 + c_1 x + ... + c_{m-1} x^{m-1}) e^{\lambda_1 x}$$

Example

$$y^{v} - 3y^{iv} + 3y''' - y'' = 0$$

Solution:

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$$

$$(\lambda - 1)^3 \lambda^2 = 0 \Rightarrow \lambda = 1(3), 0(2)$$

The general solution is

$$y = (c_1 + c_2 x + c_3 x^2) e^{x} + (c_4 + c_5 x)$$

Homogeneous linear ODEs with constant coefficients

Multiple real roots: Proof

Consider the ODE

$$y^{(n)} + p_{n-1}y^{(n-1)} + ... + p_1 y' + p_0 y = 0$$

And define the differential operator

$$L = D^n + p_{n-1}D^{(n-1)} + ... + p_1 D + p_0 I$$

The ODE can be written as

$$Ly = 0$$

For $y = e^{\lambda x}$ we have

$$L(e^{\lambda x}) = e^{\lambda x} P(\lambda) = 0$$

where $P(\lambda)$ is the characteristic polynomial.

Homogeneous linear ODEs with constant coefficients

Multiple real roots: Proof (continued)

$$L(e^{\lambda x}) = e^{\lambda x} P(\lambda) = 0$$

If λ_1 is a root of order m, then the previous equation can be written as

$$L(e^{\lambda x}) = e^{\lambda x} (\lambda - \lambda_1)^m P_1(\lambda)$$

Now we differentiate both sides with respect to λ

$$\frac{\partial L(e^{\lambda x})}{\partial \lambda} = \frac{\partial (e^{\lambda x} (\lambda - \lambda_1)^m P_1(\lambda))}{\partial \lambda}$$

$$\frac{\partial L(e^{\lambda x})}{\partial \lambda} = m(\lambda - \lambda_1)^{m-1} \left[e^{\lambda x} P_1(\lambda) \right] + (\lambda - \lambda_1)^m \frac{\partial \left[e^{\lambda x} P_1(\lambda) \right]}{\partial \lambda}$$

Homogeneous linear ODEs with constant coefficients

Multiple real roots: Proof (continued)

Since all the derivatives involved are continuous, we may interchange the order of derivation

$$\frac{\partial L(e^{\lambda x})}{\partial \lambda} = L\left(\frac{\partial e^{\lambda x}}{\partial \lambda}\right) = L(xe^{\lambda x})$$

In particular at $\lambda = \lambda_1$ we have

$$\left.\frac{\partial L(e^{\lambda x})}{\partial \lambda}\right|_{\lambda=\lambda_1} = L(xe^{\lambda_1 x})$$

$$m(\lambda_1 - \lambda_1)^{m-1}\left[e^{\lambda_1 x} P_1(\lambda_1)\right] + (\lambda_1 - \lambda_1)^m \left.\frac{\partial\left[e^{\lambda x} P_1(\lambda)\right]}{\partial \lambda}\right|_{\lambda=\lambda_1} = L(xe^{\lambda_1 x})$$

Homogeneous linear ODEs with constant coefficients

Multiple real roots: Proof (continued)

$$m(\lambda_1 - \lambda_1)^{m-1}\left[e^{\lambda_1 x} P_1(\lambda_1)\right] + (\lambda_1 - \lambda_1)^m \left.\frac{\partial\left[e^{\lambda x} P_1(\lambda)\right]}{\partial \lambda}\right|_{\lambda=\lambda_1} = L(xe^{\lambda_1 x})$$

$$m(0)^{m-1}\left[e^{\lambda_1 x} P_1(\lambda_1)\right] + (0)^m \left.\frac{\partial\left[e^{\lambda x} P_1(\lambda)\right]}{\partial \lambda}\right|_{\lambda=\lambda_1} = L(xe^{\lambda_1 x})$$

$$0 = L(xe^{\lambda_1 x})$$

But the last equation means that $xe^{\lambda_1 x}$ is a solution of the ODE. A similar derivation applies to $x^2 e^{\lambda_1 x}$, ..., $x^{m-1} e^{\lambda_1 x}$.

Homogeneous linear ODEs with constant coefficients

Multiple complex conjugate roots

The general solution associated to a couple of conjugated complex roots of order m is

$$\lambda_1, \lambda_2 = \alpha \pm i\omega (m)$$

$$y = e^{\alpha x}((a_0 + a_1 x + \ldots + a_{m-1} x^{m-1}) \cos(\omega x) + (b_0 + b_1 x + \ldots + b_{m-1} x^{m-1}) \sin(\omega x))$$

$$y = e^{\alpha x}(A_0 \cos(\omega x - \delta_0) + A_1 x \cos(\omega x - \delta_1) + \ldots + A_{m-1} x^{m-1} \cos(\omega x - \delta_{m-1}))$$

Exercises

From Kreyszig (10th ed.), Chapter 3, Section 2:

14. PROJECT. Reduction of Order. This is of practical interest since a single solution of an ODE can often be guessed. For second order, see Example 7 in Sec. 2.1.

(a) How could you reduce the order of a linear constant-coefficient ODE if a solution is known?

(b) Extend the method to a variable-coefficient ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0.$$

Assuming a solution y_1 to be known, show that another solution is $y_2(x) = u(x)y_1(x)$ with $u(x) = \int z(x)\,dx$ and z obtained by solving

$$y_1 z'' + (3y_1' + p_2 y_1)z' + (3y_1'' + 2p_2 y_1' + p_1 y_1)z = 0.$$

(c) Reduce

$$x^3 y''' - 3x^2 y'' + (6 - x^2)xy' - (6 - x^2)y = 0,$$

using $y_1 = x$ (perhaps obtainable by inspection).

Outline

1. Higher-order linear ODEs
   - Homogeneous linear ODEs
   - Homogeneous linear ODEs with constant coefficients
   - Nonhomogeneous linear ODEs

Nonhomogeneous linear ODEs

Nonhomogeneous linear ODEs

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = r(x)$$

The general solution is the sum of the general solution of the H problem and a particular solution of the NH problem

$$y = y_h + y_p$$

If the p_i $(i = 0, 1, ..., n-1)$ and r functions are continuous, then the general solution contains all solutions (there are no singular solutions).

Method of undetermined coefficients

Nonhomogeneous linear ODEs

Only valid for linear ODEs with constant coefficients.

$$y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_1 y' + p_0 y = r(x)$$

Same methodology as for second-order ODEs.

Example

$$y''' + 3y'' + 3y' + y = 30e^{-x} \quad y(0) = 3, y'(0) = -3, y''(0) = -47$$

Solution:

The general solution of the H problem comes from

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

$$(\lambda + 1)^3 = 0$$

$$y_h = (c_1 + c_2 x + c_3 x^2)e^{-x}$$

Method of undetermined coefficients

Example (continued)

For the NH problem, we see that $r = 30e^{-x}$ is similar to one of the functions in the basis of the homogeneous solutions. So we try

$$y_p = Cx^3e^{-x}$$

$$y_p' = C(3x^2 - x^3)e^{-x}$$

$$y_p'' = C(6x - 6x^2 + x^3)e^{-x}$$

$$y_p''' = C(6 - 18x + 9x^2 - x^3)e^{-x}$$

Substituting in the ODE and dropping the term e^{-x}

$$C(6 - 18x + 9x^2 - x^3) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30$$

$$6C = 30 \Rightarrow C = 5$$

$$y_p = 5x^3e^{-x}$$

Method of undetermined coefficients

Example (continued)

The general solution is

$$y = y_h + y_p = (c_1 + c_2 x + c_3 x^2 + 5x^3)e^{-x}$$

Imposing the initial conditions, $y(0) = 3, y'(0) = -3, y''(0) = -47$, we get

$$y(0) = 3 = (c_1 + c_2 0 + c_3(0)^2 + 5(0)^3)e^{-0} = c_1 \Rightarrow c_1 = 3$$

$$y' = (-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3)e^{-x}$$

$$y'(0) = -3 = -3 + c_2 \Rightarrow c_2 = 0$$

$$y'' = (3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3)e^{-x}$$

$$y''(0) = -47 = 3 + 2c_3 \Rightarrow c_3 = -25$$

The particular solution is

$$y = (3 - 25x^2 + 5x^3)e^{-x}$$

Method of undetermined coefficients

Example (continued)

$$y = (3 - 25x^2 + 5x^3)e^{-x}$$

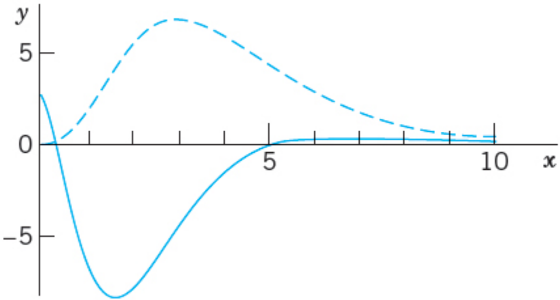


Fig. 74. y and y_p (dashed) in Example 1

Method of variation of parameters

Method of variation of parameters

It can be applied to arbitrary p_i functions. Given n solutions of the H problem, y_i ($i = 1, 2, ..., n$), the particular solution of the NH problem is of the form

$$y_p = \sum_{k=1}^{n} y_k \int \frac{W_k}{W} r dx$$

where W is the Wronskian of the n homogeneous solutions and W_k is obtained from W by subtituting the k-th column by $[00...01]^T$.

Method of variation of parameters

Example

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \log x \quad (x > 0)$$

This is an Euler-Cauchy equation. We try $y = x^m$ for the H problem

$$x^3m(m-1)(m-2)x^{m-3} - 3x^2m(m-1)x^{m-2} + 6xmx^{m-1} - 6x^m = 0$$

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0$$

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0 \Rightarrow m = 1, 2, 3$$

$$y_h = c_1x + c_2x^2 + c_3x^3$$

Method of variation of parameters

Example (continued)

For the particular solution, we need the following determinants

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

Method of variation of parameters

Example (continued)

We need the ODE in the standard form

$$y''' - 3\frac{1}{x}y'' + 6\frac{1}{x^2}y' - 6\frac{1}{x^3}y = x\log x$$

The particular solution of the NH problem is

$$y_p = y_1\int\frac{W_1}{W}rdx + y_2\int\frac{W_2}{W}rdx + y_3\int\frac{W_3}{W}rdx$$

$$y_p = x\int\frac{x^4}{2x^3}x\log x dx + x^2\int\frac{2x^3}{2x^3}x\log x dx + x^3\int\frac{x^2}{2x^3}x\log x dx$$

$$y_p = x\int\frac{x^2}{2}\log x dx + x^2\int x\log x dx + x^3\int\frac{1}{2}\log x dx$$

$$y_p = \frac{x}{2}\left(\frac{x^3}{3}\log x - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + \frac{x^3}{2}(x\log x - x)$$

Method of variation of parameters

Example (continued)

$$y_p = \frac{x}{2}\left(\frac{x^3}{3}\log x - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + \frac{x^3}{2}(x\log x - x)$$

$$y_p = \frac{1}{6}x^4\left(\log x - \frac{11}{6}\right)$$

$$y = y_h + y_p = c_1x + c_2x^2 + c_3x^3 + \frac{1}{6}x^4\left(\log x - \frac{11}{6}\right)$$

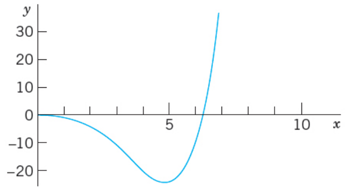


Fig. 75. Particular solution y_p of the nonhomogeneous Euler–Cauchy equation in Example 2

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 3, Section 3:

- 3.3.6
- 3.3.10

Outline