Chapter 4. Systems of ODEs. Phase plane. Qualitative methods

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Biomedical Engineering

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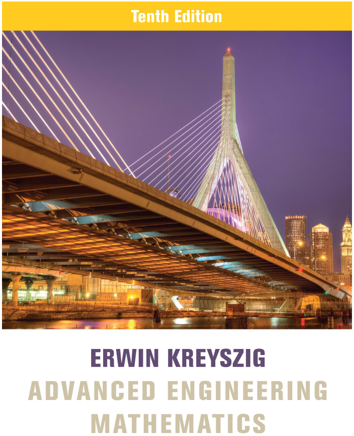
Outline

References



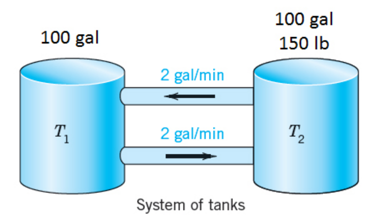
E. Kreyszig. Advanced Engineering Mathematics. John Wiley & sons. Chapter 4.

Outline

1. Systems of ODEs. Phase plane. Qualitative methods
   - Systems of ODEs as models
   - Basic theory of systems of ODEs. Wronskian
   - Constant-coefficient systems. Phase plane method
   - Criteria for critical points. Stability
   - Qualitative methods for nonlinear systems
   - Nonhomogeneous linear systems of ODEs

Systems of ODEs as models

Mixing tanks



Solution:

$$y_1' = \text{inflow-outflow} = \frac{y_2}{100}\left[\frac{lb}{gal}\right] 2 \left[\frac{gal}{min}\right] - \frac{y_1}{100}\left[\frac{lb}{gal}\right] 2 \left[\frac{gal}{min}\right]$$

$$y_2' = \text{inflow-outflow} = \frac{y_1}{100}\left[\frac{lb}{gal}\right] 2 \left[\frac{gal}{min}\right] - \frac{y_2}{100}\left[\frac{lb}{gal}\right] 2 \left[\frac{gal}{min}\right]$$

$$\left.\begin{array}{r} y_1' = -0.02y_1 + 0.02y_2 \\ y_2' = 0.02y_1 - 0.02y_2 \end{array}\right\} \Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \mathbf{y}' = A\mathbf{y}$$

Systems of ODEs as models

Mixing tanks (continued)

We try a solution of the form

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$\mathbf{y}' = \lambda \mathbf{x}e^{\lambda t}$$

Now we substitute into the ODE

$$\mathbf{y}' = A\mathbf{y}$$

$$\lambda \mathbf{x}e^{\lambda t} = A\mathbf{x}e^{\lambda t}$$

$$\lambda \mathbf{x} = A\mathbf{x}$$

That is $\mathbf{y} = \mathbf{x}e^{\lambda t}$ can be a solution of the ODE if $\mathbf{x}$ is an eigenvector of the matrix A and λ its associated eigenvalue.

$$A \Rightarrow \begin{cases} \lambda_1 = 0, \mathbf{x}_1 = (1,1)^T \\ \lambda_2 = -0.04, \mathbf{x}_2 = (1,-1)^T \end{cases}$$

Mixing tanks (continued)

The general solution is

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}$$

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-0.04t}$$

To determine the coefficients c_1 and c_2, we impose the initial conditions $y_1(0) = 0$, $y_2(150)$

$$c_1 \begin{pmatrix} 0 \\ 150 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow c_1 = 75, c_2 = -75$$

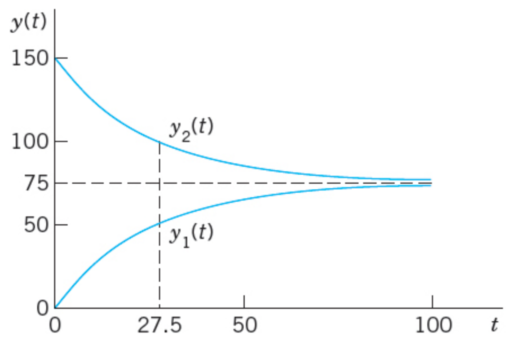
The solution to the problem is

$$\mathbf{y} = 75 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 75 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-0.04t} \Rightarrow \begin{cases} y_1 = 75 - 75e^{-0.04t} \\ y_2 = 75 + 75e^{-0.04t} \end{cases}$$

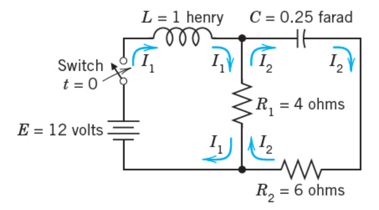
Systems of ODEs as models

Mixing tanks (continued)

$$\mathbf{y} = 75\begin{pmatrix}1\\1\end{pmatrix} - 75\begin{pmatrix}1\\-1\end{pmatrix}e^{-0.04t} \Rightarrow \begin{cases} y_1 = 75 - 75e^{-0.04t} \\ y_2 = 75 + 75e^{-0.04t} \end{cases}$$



Electrical network



Solution:

$$LI_1' + R_1(I_1 - I2) = E \Rightarrow I_1' = -R_1 I_1 + R_1 I_2 + E$$

$$R_1(I_2 - I_1) + \frac{1}{C}\int I_2 dt + R_2 I_2 = 0 \Rightarrow -R_1 I_1' + (R_1 + R_2)I_2' = -\frac{1}{C} I_2$$

Systems of ODEs as models

Electrical network (continued)

$$I_1' = -R_1 I_1 + R_1 I_2 + E$$

$$-R_1 I_1' + (R_1 + R_2) I_2' = -\frac{1}{C} I_2$$

$$\left.\begin{array}{r} I_1' = -4I_1 + 4I_2 + 12 \\ -4I_1' + 10I_2' = -4I_2 \end{array}\right\} \Rightarrow \begin{pmatrix} 1 & 0 \\ -4 & 10 \end{pmatrix} \begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$$

The eigenvalues and eigenvectors of the matrix A are

$$A \Rightarrow \left\{ \begin{array}{c} \lambda_1 = -2, \mathbf{x}_1 = (2, 1)^T \\ \lambda_2 = -0.8, \mathbf{x}_2 = (1, 0.8)^T \end{array} \right.$$

Systems of ODEs as models

Electrical network (continued)

$$A \Rightarrow \begin{cases} \lambda_1 = -2, \mathbf{x}_1 = (2,1)^T \\ \lambda_2 = -0.8, \mathbf{x}_2 = (1,0.8)^T \end{cases}$$

The general solution of the H problem is

$$\mathbf{I} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t}$$

For a particular solution of the NH problem we try a constant vector $\mathbf{I} = \mathbf{a}$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix} \Rightarrow a_1 = 3, a_2 = 0$$

So the general solution of the NH problem is

$$\mathbf{I} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Electrical network (continued)

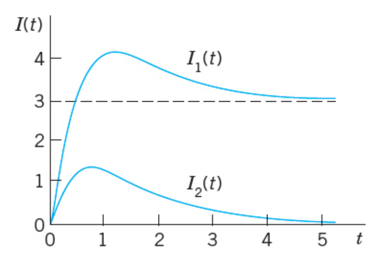
$$\mathbf{I} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

To determine the unknown coefficients we impose the initial condition $\mathbf{I}(0) = \mathbf{0}$

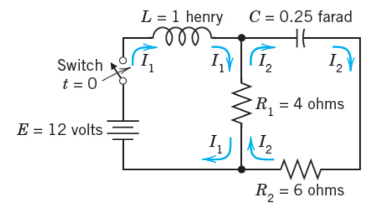
$$\mathbf{0} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \Rightarrow c_1 = -4, c_2 = 5$$

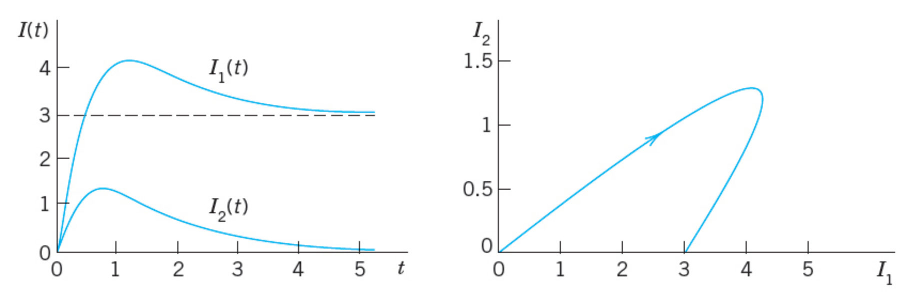
The solution to the problem is

$$\mathbf{I} = -4 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + 5 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$



Electrical network (continued)





Conversion of an n-th order ODE to a system

Conversion of an ODE

An n-th order ODE

$$y^{(n)} = F(t, y, y', ..., y^{(n-1)})$$

can be converted to a system of n first-order ODEs by setting $y_1 = y$ and

$$\begin{gathered} y_1' = y_2 \\ y_2' = y_3 \\ \ldots \\ y_{n-1}' = y_n \\ y_n' = F(t, y_1, y_2, ..., y_n) \end{gathered}$$

Conversion of an n-th order ODE to a system

Example

$$my'' + cy' + ky = 0$$

Solution:

$$y'' = -\frac{c}{m}y' - \frac{k}{m}y$$

Now we write it as

$$y_1' = y_2$$

$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Its characteristic polynomial is

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Conversion of an n-th order ODE to a system

Example (continued)

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

Let us now give values, $m = 1$, $c = 2$, $k = 0.75$

$$\lambda^2 + 2\lambda + 0.75 = 0 \Rightarrow \lambda_1 = -0.5, \lambda_2 = -1.5$$

With eigenvectors

$$\mathbf{v}_1 = (2, -1)^T, \mathbf{v}_2 = (1, -1.5)^T$$

So, the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} e^{-1.5t}$$

The first component is

$$y_1 = y = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

The second component is its derivative.

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 4, Section 1:

- 4.1.1
- 4.1.12

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Basic theory

In general, an ODE system is of the form

$$\begin{matrix} y_1' = f_1(t, y_1, ..., y_n) \\ y_2' = f_2(t, y_1, ..., y_n) \\ ... \\ y_n' = f_n(t, y_1, ..., y_n) \end{matrix} \Rightarrow \mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

An Initial Value Problem needs n initial conditions

$$y_1(t_0) = K_1, y_2(t_0) = K_2, ..., y_n(t_0) = K_n \Rightarrow \mathbf{y}(t_0) = \mathbf{K}$$

Existence and uniqueness

Let f_1, f_2, ..., f_n be continuous functions with continuous partial derivatives $\frac{\partial f_1}{\partial y_1}$, $\frac{\partial f_1}{\partial y_2}$, ..., $\frac{\partial f_n}{\partial y_n}$ in some domain R of the $ty_1y_2...y_n$-space containing the point $(t_0, K_1, ..., K_n)$. Then the ODE system has a solution on some interval $t_0 - \alpha < t < t_0 + \alpha$ satisfying the initial conditions, and this solution is unique.

Basic theory

Linear systems

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \ldots + a_{1n}(t)y_n + g_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \ldots + a_{2n}(t)y_n + g_2(t) \\ &\ldots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \ldots + a_{nn}(t)y_n + g_n(t) \end{aligned} \quad \Rightarrow \mathbf{y}' = A\mathbf{y} + \mathbf{g}$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

If $\mathbf{g} = \mathbf{0}$, the system is homogeneous.

Existence and uniqueness

Let the a_{ij}'s andg_i's functions be continuous functions of t in an open interval I containing t_0, then there exists a solution satisfying the initial conditions, and this solution is unique.

Basic theory

Superposition principle

The linear combination of any two solutions, $\mathbf{y}_1$ and $\mathbf{y}_2$, of the H problem is also a solution of the H problem.

Proof:

$$\begin{aligned}
\mathbf{y} &= c_1\mathbf{y}_1 + c_2\mathbf{y}_2 \\
\mathbf{y}' &= c_1\mathbf{y}_1' + c_2\mathbf{y}_2' \\
&= c_1(A\mathbf{y}_1) + c_2(A\mathbf{y}_2) \\
&= A(c_1\mathbf{y}_1 + c_2\mathbf{y}_2) \\
&= A\mathbf{y}
\end{aligned}$$

Basic theory

General solution. Wronskian

If the a_{ij}'s functions are continuous, then the general solution of the H problem can be written as

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + ... + c_n\mathbf{y}_n$$

where $\mathbf{y}_1$, $\mathbf{y}_2$, ..., $\mathbf{y}_n$ constitute a basis or fundamental system of solutions, and there is no singular solution. We can write the n basis functions as the columns of a matrix Y

$$Y = \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_2 & ... & \mathbf{y}_n \end{pmatrix}$$

and write the general solution as

$$\mathbf{y} = Y\mathbf{c}$$

The Wronskian is the determinant of Y

$$W = |Y|$$

Outline

Constant coefficients systems

Constant coefficients systems

$$\mathbf{y}' = A\mathbf{y}$$

We try a function of the form

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$\mathbf{y}' = \lambda \mathbf{x}e^{\lambda t}$$

And substitute it in the ODE

$$\lambda \mathbf{x}e^{\lambda t} = A\mathbf{x}e^{\lambda t}$$

$$\lambda \mathbf{x} = A\mathbf{x}$$

That is $\mathbf{y}$ is a solution if $\mathbf{x}$ is an eigenvector of A. If A has n distinct eigenvalues, then the general solution is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + ... + c_n\mathbf{x}_n e^{\lambda_n t}$$

Constant coefficients systems

Constant coefficients systems

The Wronskian of the basis of solutions is

$$\begin{aligned} W &= \begin{vmatrix} \mathbf{x}_1 e^{\lambda_1 t} & \ldots & \mathbf{x}_n e^{\lambda_n t} \end{vmatrix} \\ &= e^{\lambda_1 t + \ldots + \lambda_n t} \begin{vmatrix} \mathbf{x}_1 & \ldots & \mathbf{x}_n \end{vmatrix} \end{aligned}$$

The exponential term cannot be 0, and the determinant of the matrix of eigenvectors cannot be 0 because they are linearly independent vectors since they correspond to distinct eigenvalues. This proves that there is no singular solution if

all eigenvalues are distinct.

Phase-plane trajectories

Example

$$\mathbf{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \mathbf{y}$$

Solution:

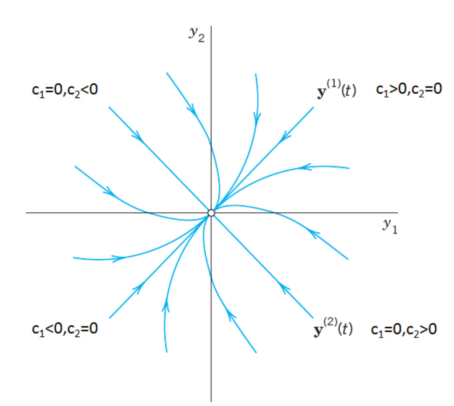
$$A \Rightarrow \begin{cases} \lambda_1 = -2, \mathbf{x}_1 = (1,1)^T \\ \lambda_2 = -4, \mathbf{x}_2 = (1,-1)^T \end{cases}$$

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$

Phase-plane trajectories

Example (continued)

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$



Phase-plane trajectories

Critical points

A **critical point** is a point at which $\mathbf{y}' = \mathbf{0}$, they are also called **equilibrium solutions**. Let us analyze the system

$$\mathbf{y}' = A\mathbf{y}$$

and the slope of trajectories in the phase plane at a given point (y_1, y_2)

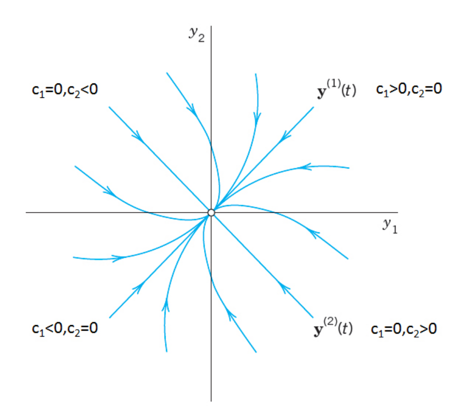
$$\frac{dy_2}{dy_1} = \frac{y_2' dt}{y_1' dt} = \frac{y_2'}{y_1'}$$

At critical points, this ratio becomes undefined $\left(\frac{0}{0}\right)$. There are five types of critical points: **improper nodes**, **proper nodes**, **saddle points**, **centers**, and **spiral points**.

Phase-plane trajectories

Example (continued): Improper node

An **improper node** is a critical point at which all trajectories, except two of them, have the same limiting direction of the tangent. The two exceptional directions also have a limiting direction of the tangent which, however, is different.

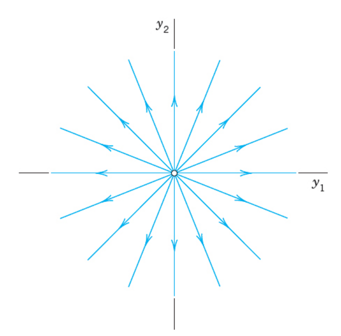


Phase-plane trajectories

Example: Proper node

$$\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y} \Rightarrow \mathbf{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$$

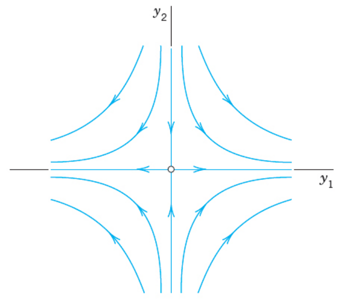
A **proper node** is a critical point at which every trajectory has a definite limiting direction and for any givend irection **d**, there is a trajectory having **d** as its limiting direction.



Example: Saddle point

$$\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} \Rightarrow \mathbf{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

A **saddle point** is a critical point at which there are two incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of the critical point bypass it.

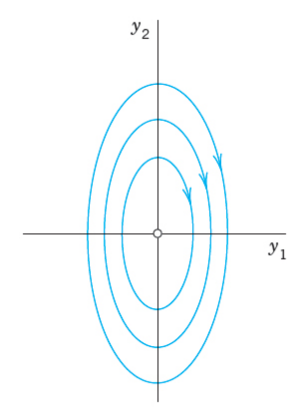


Phase-plane trajectories

Example: Center

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{y} \Rightarrow \mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{i2t} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-i2t}$$

A **center** is a critical point that is enclosed by infinitely many closed trajectories.

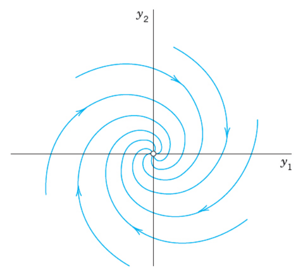


Phase-plane trajectories

Example: Spiral point

$$\mathbf{y}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y} \Rightarrow \mathbf{y} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-i)t}$$

A **spiral point** is a critical point about which trajectories spiral, approaching the critical point or going away from it, as $t \to \infty$.



Phase-plane trajectories

Example: Degenerate node

$$\mathbf{y}' = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \mathbf{y}$$

The problem is that A is not diagonalizable because it has a double eigenvalue at $\lambda = 3$ but only one associated eigenvector $\mathbf{x}_1 = (1, -1)^T$. One of the solutions is of the form:

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t}$$

For the second solution we look for solution of the type

$$\mathbf{y}_2 = t\mathbf{x}_1 e^{\lambda_1 t} + \mathbf{u} e^{\lambda_1 t}$$

with a constant **u** vector.

$$\mathbf{y}_2' = \mathbf{x}_1 e^{\lambda_1 t} + t\lambda_1 \mathbf{x}_1 e^{\lambda_1 t} + \lambda_1 \mathbf{u} e^{\lambda_1 t}$$

Phase-plane trajectories

Example: Degenerate node (continued)

We now substitute in the ODE

$$\mathbf{y}_2' = A\mathbf{y}_2$$

$$\mathbf{x}_1 e^{\lambda_1 t} + t\lambda_1 \mathbf{x}_1 e^{\lambda_1 t} + \lambda_1 \mathbf{u} e^{\lambda_1 t} = tA\mathbf{x}_1 e^{\lambda_1 t} + A\mathbf{u} e^{\lambda t}$$

$$\mathbf{x}_1 e^{\lambda_1 t} + t\lambda_1 \mathbf{x}_1 e^{\lambda_1 t} + \lambda_1 \mathbf{u} e^{\lambda_1 t} = t\lambda_1 \mathbf{x}_1 e^{\lambda_1 t} + A\mathbf{u} e^{\lambda_1 t}$$

$$\mathbf{x}_1 e^{\lambda_1 t} + \lambda_1 \mathbf{u} e^{\lambda_1 t} = A\mathbf{u} e^{\lambda_1 t}$$

$$\mathbf{x}_1 + \lambda_1 \mathbf{u} = A\mathbf{u}$$

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}_1 \Rightarrow \mathbf{u} = (0, 1)^T$$

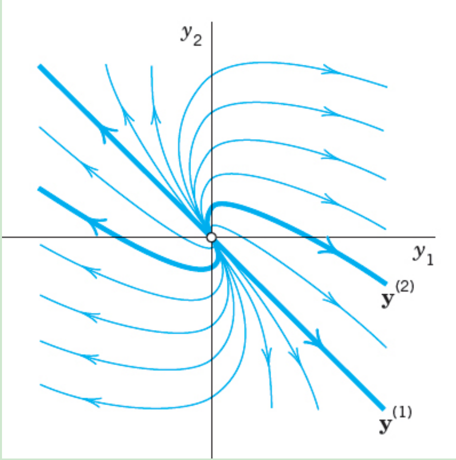
So the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} \right)$$

Phase-plane trajectories

Example: Degenerate node (continued)

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} \right)$$



Constant coefficients systems

Degenerate solutions

When the matrix A is not diagonalizable, then we may complete the fundamental system with solutions of the form

$$\mathbf{y}_2 = (t\mathbf{x}_1 + \mathbf{v}_1)\, e^{\lambda_1 t}$$

$$\mathbf{y}_3 = \left(\frac{1}{2}t^2\mathbf{x}_1 + t\mathbf{v}_1 + \mathbf{v}_2\right) e^{\lambda_1 t}$$

$$\mathbf{y}_4 = \left(\frac{1}{3}t^3\mathbf{x}_1 + \frac{1}{2}t^2\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right) e^{\lambda_1 t}$$

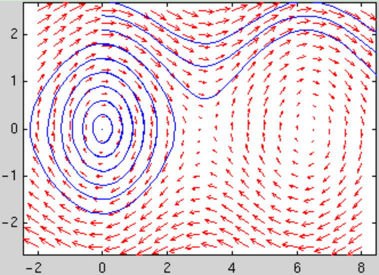
$$\dots$$

System ODEs

MATLAB

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -\sin(y_1) \end{pmatrix}$$

```
f = @(t,y) [y(2);-sin(y(1))]
vectfield(f,-2:.5:8,-2.5:.25:2.5)
hold on
for y20=0:0.3:2.7
 [ts,ys] = ode45(f,[0,10],[0;y20]);
 plot(ys(:,1),ys(:,2))
end
hold off
```



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 4, Section 3:

- 4.3.6
- 4.3.7
- 4.3.18

Outline

Criteria for critical points

Critical point classification

$$\mathbf{y}' = A\mathbf{y} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{y}$$

Let's analyze the characteristic polynomial and eigenvalues of A

$$\det\{A - \lambda I\} = \lambda^2 - \mathrm{Tr}\{A\}\lambda + \det\{A\} = 0$$

Let us define

$$\Delta = (\mathrm{Tr}\{A\} - 4(\det\{A\})^2)$$

The eigenvalues are

$$\lambda_{1,2} = \frac{\mathrm{Tr}\{A\} \pm \sqrt{\Delta}}{2}$$

Criteria for critical points

Critical point classification (continued)

$$\lambda_{1,2} = \frac{\mathrm{Tr}\{A\} \pm \sqrt{\Delta}}{2}$$

Type	$\mathrm{Tr}\{A\} = \lambda_1 + \lambda_2$	$\det\{A\} = \lambda_1\lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	**Comments**
Node		> 0	≥ 0	Real, same sign
Saddle point		< 0		Real, opposite signs
Center	$= 0$		< 0	Pure imaginary
Spiral point	$\neq 0$		< 0	Complex

Criteria for critical points

Stable critical point

A critical point P_0 is **stable** if all trajectories of the ODE that at some instant are close to P_0 remain close to P_0 at all future times; precisely: if for every disk D_ϵ of radius ϵ with center P_0 there is a disk D_δ of radius δ with center P_0 such that every trajectory of the ODE that has a point P_1 (corresponding to $t = t_1$, say) in D_δ has all its points corresponding to $t \geq t_1$ in D_ϵ. If a critical point is not stable, it is **unstable**.

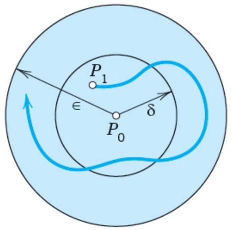


Fig. 90. Stable critical point P_0 of (1) (The trajectory initiating at P_1 stays in the disk of radius ϵ.)

Criteria for critical points

Asymptotically stable critical point

A critical point P_0 is **asymptotically stable** (stable and attractive) if P_0 is stable and every trajectory that has a point in D_δ approaches P_0 as $t \to \infty$.

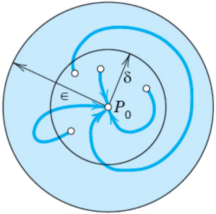


Fig. 91. Stable and attractive critical point P_0 of (1)

Criteria for critical points

Critical point classification (continued)

$$\det\{A - \lambda I\} = \lambda^2 - \mathrm{Tr}\{A\}\lambda + \det\{A\} = \lambda^2 - p\lambda + q = 0$$

$$\lambda_{1,2} = \frac{\mathrm{Tr}\{A\} \pm \sqrt{\Delta}}{2}$$

Type	$p = \mathrm{Tr}\{A\} = \lambda_1 + \lambda_2$	$q = \det\{A\} = \lambda_1\lambda_2$
Asymptotically stable	< 0	0
Stable	≤ 0	> 0
Unstable	> 0	or < 0

Criteria for critical points

Critical point classification (continued)

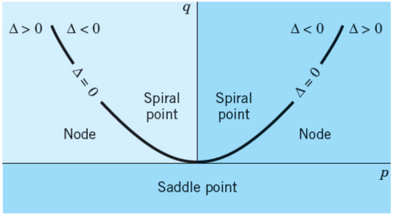


Fig. 92. Stability chart of the system (1) with p, q, Δ defined in (5).
Stable and attractive: The second quadrant without the q-axis.
Stability also on the positive q-axis (which corresponds to centers).
Unstable: Dark blue region

Criteria for critical points

Critical point classification (continued)

eigenvalues			linear system		nonlinear system	
real	both pos.	equal	proper or improper node	unstable	*similar to* node or spiral point	unstable
		different	node	unstable	*same*	
	both neg.	equal	proper or improper node	as. stable	*similar to* node or spiral point	as. stable
		different	node	as. stable	*same*	
	pos. and neg.		saddle point	unstable	*same*	
complex not real	real part pos.		spiral point	unstable	*same*	
	real part neg.		spiral point	as. stable	*same*	
	real part zero		center	stable	*similar to* center or spiral point	?

Criteria for critical points

Example

$$my'' + cy' + ky = 0$$

Solution:

$$y'' = -\frac{k}{m}y - \frac{c}{m}y'$$

We convert it to a system ODE with

$$y_1 = y, y_1' = y_2$$

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \mathbf{y}$$

$$\det(A - \lambda I) = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}$$

From where

$$p = -\frac{c}{m}, q = \frac{k}{m}, \Delta = \left(\frac{c}{m}\right)^2 - 4\frac{k}{m}$$

Criteria for critical points

Example (continued)

$$p = -\frac{c}{m}, q = \frac{k}{m}, \Delta = \left(\frac{c}{m}\right)^2 - 4\frac{k}{m}$$

No damping. $c = 0, p = 0, q > 0$, a center.
Underdamping. $c^2 < 4mk, p < 0, q > 0, \Delta < 0$, a stable and attractive spiral point.
Critical damping. $c^2 = 4mk, p < 0, q > 0, \Delta = 0$, a stable and attractive node.
Overdamping. $c^2 > 4mk, p < 0, q > 0, \Delta > 0$, a stable and attractive node.

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 4, Section 4:

- 4.4.3
- 4.4.14
- 4.4.17

Outline

Autonomous nonlinear systems

Autonomous nonlinear systems

Qualitative methods allow analyzing a system without actually solving it. For autonomous nonlinear systems

$$\mathbf{y}' = \mathbf{f}(\mathbf{y})$$

with a critical point $\mathbf{y}_0$ we may shift the origin so that the $\mathbf{y}_0$ is centered

$$\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{y}_0$$

$$\tilde{\mathbf{y}}' = \mathbf{f}(\tilde{\mathbf{y}} + \mathbf{y}_0)$$
$$\tilde{\mathbf{y}}' = \tilde{\mathbf{f}}(\tilde{\mathbf{y}})$$

and study the local behaviour of the system ODE around $\mathbf{0}$ as we have already done. For doing so, we may need to linearize the ODE.

Autonomous nonlinear systems

Linearization of autonomous nonlinear systems

$$\tilde{\mathbf{y}}' = \tilde{\mathbf{f}}(\tilde{\mathbf{y}}) \approx A\tilde{\mathbf{y}}$$

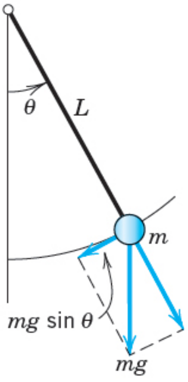
where A is the Jacobian of the function $\mathbf{f}$ evaluated at the origin $\mathbf{0}$:

$$A = \left. \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} & \dots & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_n} \\ \frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2} & \dots & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial \tilde{f}_n}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_n}{\partial \tilde{y}_2} & \dots & \frac{\partial \tilde{f}_n}{\partial \tilde{y}_n} \end{pmatrix} \right|_{\tilde{\mathbf{y}}=\mathbf{0}}$$

Theorem

If $\tilde{\mathbf{f}}$ has continuous components and continuous partial derivatives in a neighbourhood of the critical point $\mathbf{0}$ and $\det\{A\} \neq 0$, then the kind and stability of the critical point of the nonlinear system ODE is the same as those of the linearized system. Exceptions occur if A has equal or pure imaginary eigenvalues, then the nonlinear problem may have the same kind of critical point as the linearized system or a spiral point.

Example: Free undamped pendulum



(a) Pendulum

Gravity compensates the acceleration of the bob

$$mL\theta'' + mg\sin(\theta) = 0$$

$$\theta'' + k\sin(\theta) = 0 \quad k = \frac{g}{L}$$

To find the critical points we convert the equation into a system ODE

$$y_1 = \theta$$

$$y_2 = y_1'$$

$$y_2' + k\sin(y_1) = 0$$

Equivalently

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -k\sin(y_1) \end{pmatrix}$$

Autonomous nonlinear systems

Example (continued)

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -k\sin(y_1) \end{pmatrix}$$

The critical points are at $\mathbf{y} = (\pi n, 0)^T$ $(n \in \mathbb{Z})$. Let's analyze the one at $(0,0)$. Let's calculate the Jacobian of $\mathbf{f}$ at $(0,0)$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}\Bigg|_{\mathbf{y}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ -k\cos(y_1) & 0 \end{pmatrix}\Bigg|_{\mathbf{y}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}$$

To classify this critical point we note that

$$\mathrm{Tr}\{A\} = 0 \quad \det\{A\} = k > 0$$

So we conclude that $\mathbf{y} = \mathbf{0}$ is a center (always stable). The same happens to all points $(0, 2\pi n)$ since the sin function is periodic with period 2π.

Example (continued)

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -k\sin(y_1) \end{pmatrix}$$

Let's analyze the critical point at $(\pi, 0)$. We center the critical point by doing

$$\tilde{\mathbf{y}} = \mathbf{y} - \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

The system ODE becomes

$$\tilde{\mathbf{y}}' = \begin{pmatrix} \tilde{y}_2 \\ -k\sin(\tilde{y}_1 + \pi) \end{pmatrix}$$

Let's calculate the Jacobian of $\tilde{\mathbf{f}}$ at $(0, 0)$

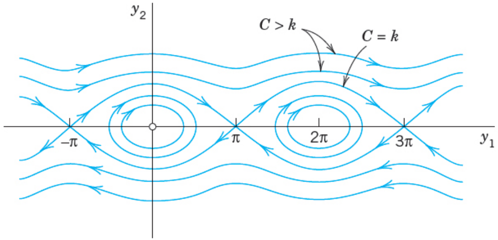
$$A = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial y_1} & \frac{\partial \tilde{f}_1}{\partial y_2} \\ \frac{\partial \tilde{f}_2}{\partial y_1} & \frac{\partial \tilde{f}_2}{\partial y_2} \end{pmatrix}\Bigg|_{\tilde{\mathbf{y}}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ -k\cos(\tilde{y}_1 + \pi) & 0 \end{pmatrix}\Bigg|_{\mathbf{y}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}$$

Example (continued)

To classify this critical point we note that

$$\mathrm{Tr}\{A\} = 0 \quad \det\{A\} = -k < 0$$

So we conclude that $\mathbf{y} = (\pi, 0)^T$ is a saddle point (unstable). The same happens to all points $(0, \pi + 2\pi n)$ since the sin function is periodic with period 2π.



(b) Solution curves $y_2(y_1)$ of (4) in the phase plane

Example 1 (C will be explained in Example 4.)

Autonomous nonlinear systems

Example: Damped pendulum

$$\theta'' + c\theta' + k\sin(\theta) = 0$$

To find the critical points we convert the equation into a system ODE

$$y_1 = \theta$$

$$y_2 = y_1'$$

$$y_2' + cy_2 + k\sin(y_1) = 0$$

Equivalently

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -k\sin(y_1) - cy_2 \end{pmatrix}$$

Example (continued)

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -k\sin(y_1) - cy_2 \end{pmatrix}$$

Critical points are at the same location as in the free undamped pendulum $\mathbf{y} = (\pi n, 0)$. Let's study the critical point at $(0, 0)$.

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}\right|_{\mathbf{y}=\mathbf{0}} = \left.\begin{pmatrix} 0 & 1 \\ -k\cos(y_1) & -c \end{pmatrix}\right|_{\mathbf{y}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix}$$

$$\mathrm{Tr}\{A\} = -c < 0 \quad \det\{A\} = k > 0 \quad \Delta = -c + 4k^2$$

If $\Delta < 0$, then we have a stable and attractive spiral point. If $\Delta > 0$, then it is a stable and attractive node.

Autonomous nonlinear systems

Example (continued)

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ -k\sin(y_1) - cy_2 \end{pmatrix}$$

Let's analyze the critical point at $(\pi, 0)$. We center the critical point by doing

$$\tilde{\mathbf{y}} = \mathbf{y} - \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

The system ODE becomes

$$\tilde{\mathbf{y}}' = \begin{pmatrix} \tilde{y}_2 \\ -k\sin(\tilde{y}_1 + \pi) - cy_2 \end{pmatrix}$$

Let's calculate the Jacobian of $\tilde{\mathbf{f}}$ at $(0, 0)$

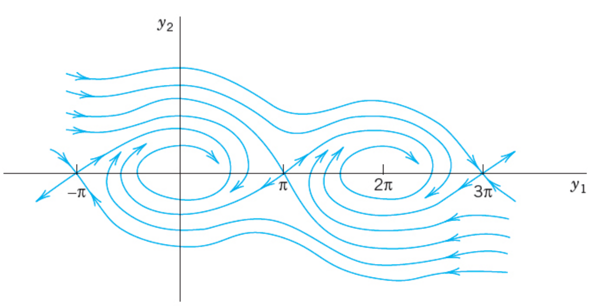
$$A = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} \\ \frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2} \end{pmatrix}\Bigg|_{\tilde{\mathbf{y}}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ -k\cos(\tilde{y}_1 + \pi) & -c \end{pmatrix}\Bigg|_{\mathbf{y}=\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ k & -c \end{pmatrix}$$

Example (continued)

To classify this critical point we note that

$$\mathrm{Tr}\{A\} = -c \quad \det\{A\} = -k < 0$$

So we conclude that $\mathbf{y} = (\pi, 0)^T$ is a saddle point (unstable). The same happens to all points $(0, \pi + 2\pi n)$ since the sin function is periodic with period 2π.



Autonomous nonlinear systems

Example: Lotka-Volterra population model

1. Rabbits have unlimited food supply. Hence, if there were no foxes, their number $y_1(t)$ would grow exponentially, $y_1' = ay_1$.
2. Actually, y_1 is decreased because of the kill by foxes, say, at a rate proportional to $y_1 y_2$, where $y_2(t)$ is the number of foxes. Hence $y_1' = ay_1 - by_1 y_2$, where $a > 0$ and $b > 0$.
3. If there were no rabbits, then $y_2(t)$ would exponentially decrease to zero, $y_2' = -ly_2$. However, y_2 is increased by a rate proportional to the number of encounters between predator and prey; together we have $y_2' = -ly_2 + ky_1 y_2$, where $k > 0$ and $l > 0$.

Solution:

$$y_1' = ay_1 - by_1 y_2$$

$$y_2' = ky_1 y_2 - ly_2$$

Example: Lotka-Volterra population model (continued)

Critical points are the solutions of:

$$0 = y_1' = y_1(a - by_2)$$

$$0 = y_2' = (ky_1 - l)y_2$$

That is $(0,0)$ or $\left(\frac{l}{k}\,\frac{a}{b}\right)$. Let's analyze $(0,0)$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}\Bigg|_{\mathbf{y}=\mathbf{0}} = \begin{pmatrix} a & -by_1 \\ ky_2 & -l \end{pmatrix}\Bigg|_{\mathbf{y}=\mathbf{0}} = \begin{pmatrix} a & 0 \\ 0 & -l \end{pmatrix}$$

Eigenvalues are $\lambda_1 = a$, $\lambda_2 = -l$. They have different signs, so we have a saddle point.

Autonomous nonlinear systems

Example: Lotka-Volterra population model (continued)

For the critical point $\left(\frac{l}{k}, \frac{a}{b}\right)$ we make the change of variables

$$\tilde{\mathbf{y}} = \mathbf{y} - \begin{pmatrix} \frac{l}{k} \\ \frac{a}{b} \end{pmatrix}$$

The system ODE becomes

$$\tilde{\mathbf{y}}' = \begin{pmatrix} \left(\tilde{y}_1 + \frac{l}{k}\right)\left(a - b\left(\tilde{y}_2 + \frac{a}{b}\right)\right) \\ \left(k\left(\tilde{y}_1 + \frac{l}{k}\right) - l\right)\left(\tilde{y}_2 + \frac{a}{b}\right) \end{pmatrix} = \begin{pmatrix} \left(\tilde{y}_1 + \frac{l}{k}\right)\left(-b\tilde{y}_2\right) \\ k\tilde{y}_1\left(\tilde{y}_2 + \frac{a}{b}\right) \end{pmatrix}$$

Let's calculate the Jacobian of $\tilde{\mathbf{f}}$ at $(0,0)$

$$A = \left.\begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial y_1} & \frac{\partial \tilde{f}_1}{\partial y_2} \\ \frac{\partial \tilde{f}_2}{\partial y_1} & \frac{\partial \tilde{f}_2}{\partial y_2} \end{pmatrix}\right|_{\tilde{\mathbf{y}}=\mathbf{0}} = \left.\begin{pmatrix} -b\tilde{y}_2 & \left(\tilde{y}_1 + \frac{l}{k}\right)b \\ k\left(\tilde{y}_2 + \frac{a}{b}\right) & k\tilde{y}_1 \end{pmatrix}\right|_{\mathbf{y}=\mathbf{0}} = \begin{pmatrix} 0 & -\frac{l}{k}b \\ k\frac{a}{b} & 0 \end{pmatrix}$$

Example: Lotka-Volterra population model (continued)

$$A = \begin{pmatrix} 0 & -\frac{l}{k}b \\ k\frac{a}{b} & 0 \end{pmatrix}$$

We observe that

$$\mathrm{Tr}\{A\} = 0 \quad \det\{A\} = al > 0$$

So the critical point is a stable center. Let's solve the equation around this critical point

$$y_1' = -\frac{l}{k}b\tilde{y}_2$$

$$y_2' = k\frac{a}{b}\tilde{y}_1$$

We rewrite the equation system as

$$y_1' = -\frac{l}{k}b\tilde{y}_2$$

$$k\frac{a}{b}\tilde{y}_1 = y_2'$$

Example: Lotka-Volterra population model (continued)

$$y_1' = -\frac{l}{k} b \tilde{y}_2$$

$$k \frac{a}{b} \tilde{y}_1 = y_2'$$

and multiply both equations

$$k \frac{a}{b} \tilde{y}_1 y_1' = -\frac{l}{k} b \tilde{y}_2 y_2'$$

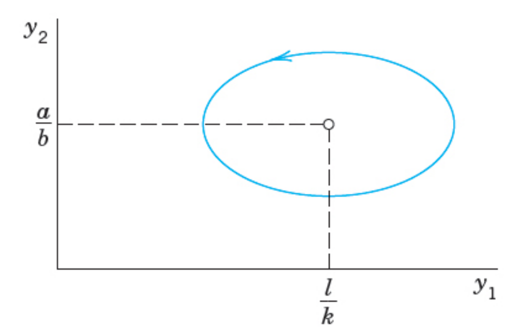
Integrating

$$k \frac{a}{2b} \tilde{y}_1^2 = -\frac{l}{2k} b \tilde{y}_2^2 + C$$

$$\frac{ak}{b} \tilde{y}_1^2 + \frac{bl}{k} \tilde{y}_2^2 = C$$

Example: Lotka-Volterra population model (continued)

$$\frac{ak}{b}\tilde{y}_1^2 + \frac{bl}{k}\tilde{y}_2^2 = C \Rightarrow \boxed{\frac{ak}{b}\left(y_1 - \frac{l}{k}\right)^2 + \frac{bl}{k}\left(y_2 - \frac{a}{b}\right)^2 = C}$$



Transformation to a first-order equation in the phase plane

Transformation to a first-order equation in the phase plane

Consider a second-order autonomous ODE

$$F(y, y', y'')$$

We make the change of variables

$$y_1 = y$$

$$y_2 = y_1'$$

And find y'' using the chain rule

$$y'' = y_2' = \frac{dy_2}{dt} = \frac{dy_2}{dy_1}\frac{dy_1}{dt} = \frac{dy_2}{dy_1}y_2$$

The ODE becomes

$$\boxed{F\left(y_1, y_2, \frac{dy_2}{dy_1}y_2\right) = 0}$$

Example: Free undamped pendulum

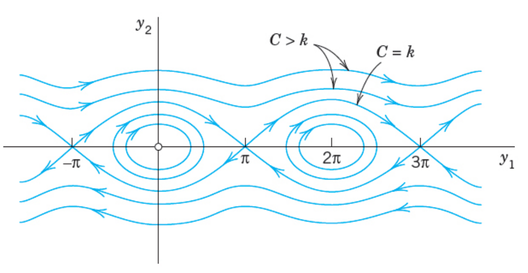
$$\theta'' + k\sin(\theta) = 0$$

Making the substitutions suggested by the method we get

$$\frac{dy_2}{dy_1} y_2 + k\sin(y_1) = 0$$

$$y_2 dy_2 = -k\sin(y_1) dy_1$$

$$\frac{1}{2} y_2^2 = k\cos(y_1) + C$$



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 4, Section 5:

- 4.5.5

Outline

Nonhomogeneous linear systems of ODEs

Nonhomogeneous linear systems of ODEs

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{g}(t)$$

If the entries of the A matrix and $\mathbf{g}$ vector are continuous, then the general solution can be expressed as

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$$

Method of undetermined coefficients

Valid for constant matrix A and $\mathbf{g}$ that is a sum of constant, powers, exponentials or sine/cosine functions.

Method of undetermined coefficients

Example

$$\mathbf{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t}$$

Solution:

The general solution of the H problem is

$$\mathbf{y}_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$

Since the excitation signal e^{-2t} is also a solution of the H problem we try a particular solution of the form

$$\mathbf{y}_p = (t\mathbf{u} + \mathbf{v})e^{-2t}$$

$$\mathbf{y}_p' = (-2t\mathbf{u} + \mathbf{u} - 2\mathbf{v})e^{-2t}$$

Method of undetermined coefficients

Example (continued)

Substituting in the ODE

$$(-2t\mathbf{u} + \mathbf{u} - 2\mathbf{v})e^{-2t} = A(t\mathbf{u} + \mathbf{v})e^{-2t} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t}$$

$$-2t\mathbf{u} + \mathbf{u} - 2\mathbf{v} = tA\mathbf{u} + A\mathbf{v} + \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

Identifying the coefficients of t

$$-2\mathbf{u} = A\mathbf{u}$$

That is $\mathbf{u}$ is an eigenvector of A associated to $\lambda = -2$

$$\mathbf{u} = a(1, 1)^T$$

Method of undetermined coefficients

Example (continued)

$$-2t\mathbf{u} + \mathbf{u} - 2\mathbf{v} = tA\mathbf{u} + A\mathbf{v} + \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

Identifying the coefficients without t

$$\mathbf{u} - 2\mathbf{v} = A\mathbf{v} + \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

$$(A + 2I)\mathbf{v} = \mathbf{u} - \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

We cannot solve as $(A + 2I)^{-1}(...)$ because -2 is an eigenvalue of A and $A + 2I$ is not invertible. Then

$$\left(\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} - \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

Method of undetermined coefficients

Example (continued)

$$\left(\begin{pmatrix}-3 & 1\\ 1 & -3\end{pmatrix} + \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\right)\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}a\\ a\end{pmatrix} - \begin{pmatrix}-6\\ 2\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}a+6\\ a-2\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}a+6\\ 2a+4\end{pmatrix}$$

For this system being compatible we need

$$2a + 4 = 0 \Rightarrow a = -2$$

Then

$$v_2 = v_1 + (-2+6) = v_1 + 4$$

We may simply take $v_1 = 0$

$$\mathbf{v} = \begin{pmatrix}0\\ 4\end{pmatrix}$$

Method of undetermined coefficients

Example (continued)

Finally

$$\mathbf{y}_p = \left(t\begin{pmatrix}-2 & -2\end{pmatrix} + \begin{pmatrix}0 & 4\end{pmatrix}\right)e^{-2t}$$

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + \left(t\begin{pmatrix}-2\\-2\end{pmatrix} + \begin{pmatrix}0\\4\end{pmatrix}\right)e^{-2t}$$

$$\boxed{\mathbf{y} = \begin{pmatrix}c_1 - 2t\\ c_1 - 2t + 4\end{pmatrix}e^{-2t} + \begin{pmatrix}c_2\\ -c_2\end{pmatrix}e^{-4t}}$$

Method of variation of parameters

Method of variation parameters

This is valid for non-constant A and arbitrary $\mathbf{g}$

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{g}(t)$$

If the general solution of the H problem is of the form

$$\mathbf{y}_h = \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \dots & \mathbf{y}_n \end{pmatrix} \mathbf{c} = Y(t)\mathbf{c}$$

Then we look for a solution of the form

$$\mathbf{y}_p = Y(t)\mathbf{u}(t)$$

$$\mathbf{y}_p' = Y'\mathbf{u} + Y\mathbf{u}'$$

And substitute in the ODE

$$Y'\mathbf{u} + Y\mathbf{u}' = AY\mathbf{u} + \mathbf{g}$$

Method of variation of parameters

Method of variation parameters (continued)

$$Y'\mathbf{u} + Y\mathbf{u}' = AY\mathbf{u} + \mathbf{g}$$

Since the columns of Y are solutions of the H problem we have

$$Y' = AY$$

Then

$$AY\mathbf{u} + Y\mathbf{u}' = AY\mathbf{u} + \mathbf{g}$$

$$Y\mathbf{u}' = \mathbf{g}$$

$$\boxed{\mathbf{u}' = Y^{-1}\mathbf{g}}$$

Method of variation of parameters

Example (same as for undetermined coefficients)

$$\mathbf{y}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t}$$

Solution:
The general solution of the H problem is

$$\mathbf{y}_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Y\mathbf{c}$$

$$Y^{-1} = \frac{1}{-2e^{-6t}} \begin{pmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{pmatrix}$$

$$\mathbf{u}' = Y^{-1}\mathbf{g} = \frac{1}{2} \begin{pmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{pmatrix} \begin{pmatrix} -6e^{-2t} \\ 2e^{-2t} \end{pmatrix} = \begin{pmatrix} -2 \\ -4e^{2t} \end{pmatrix}$$

$$\mathbf{u} = \int \begin{pmatrix} -2 \\ -4e^{2t} \end{pmatrix} dt = \begin{pmatrix} -2t \\ -2e^{2t} \end{pmatrix}$$

Method of variation of parameters

Example

$$\mathbf{y}_p = Y\mathbf{u} = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} -2t \\ -2e^{2t} \end{pmatrix} = \begin{pmatrix} -2-2t \\ 2-2t \end{pmatrix} e^{-2t}$$

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} + \begin{pmatrix} -2-2t \\ 2-2t \end{pmatrix} e^{-2t}$$

$$\boxed{\mathbf{y} = \begin{pmatrix} c_1 - 2 - 2t \\ c_1 + 2 - 2t \end{pmatrix} e^{-2t} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} e^{-4t}}$$

We may compare to the previous solution

$$\mathbf{y} = \begin{pmatrix} c_1 - 2t \\ c_1 - 2t + 4 \end{pmatrix} e^{-2t} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} e^{-4t}$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 4, Section 6:

- 4.6.5

Outline