Chapter 5. Series solutions of ODEs. Special functions

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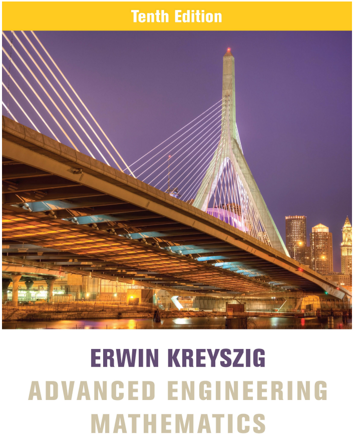
Outline

1. Series solutions of ODEs. Special functions
   - Power series methods
   - Legendre's equation. Legendre polynomials
   - Extended power series method: Frobenius method
   - Bessel's equation. Bessel functions $J_\nu(x)$
   - Bessel's equation. Bessel functions $Y_\nu(x)$

References



E. Kreyszig. Advanced Engineering Mathematics. John Wiley & sons. Chapter 5.

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Power series methods

Power series methods

This is the standard method to solve linear ODEs with variable coefficients. A power series is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m(x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

Taylor Series: $f(x_0) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1, \textbf{geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

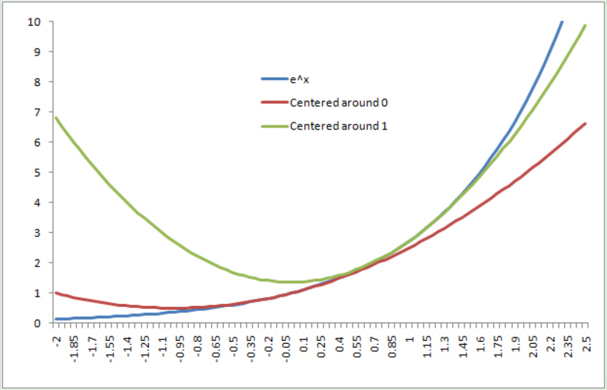
$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots.$$

Power series methods

Example

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

$$e^x = e + e(x-1) + e\frac{1}{2!}(x-1)^2 + e\frac{1}{3!}(x-1)^3 + e\frac{1}{4!}(x-1)^4 + \dots$$



Example

$$y' - y = 0$$

Solution:

We look for a solution of the form

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

And susbtitute it in the ODE

$$\left(a_1 + 2a_2 x + 3a_3 x^2 + \dots\right) - \left(a_0 + a_1 x + a_2 x^2 + \dots\right) = 0$$

$$(a_1 - a_0) + (2a_2 - a_1) + (3a_3 - a_2) + \dots = 0$$

Power series methods

Example (continued)

$$(a_1 - a_0) + (2a_2 - a_1) + (3a_3 - a_2) + ... = 0$$

$$\Rightarrow \begin{cases} a_1 - a_0 = 0 \Rightarrow a_1 = a_0 \\ 2a_2 - a_1 = 0 \Rightarrow a_2 = \frac{1}{2}a_1 = \frac{1}{2}a_0 \\ 3a_3 - a_2 = 0 \Rightarrow a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2}a_0 \\ ... \end{cases}$$

And in general

$$a_k = \frac{1}{k(k-1)...2}a_0 = \frac{1}{k!}a_0$$

So the solution of the ODE is

$$y = a_0(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...) = a_0 e^x$$

Power series methods

Power series methods

$$y'' + p(x)y' + q(x)y = 0$$

This method is applied to linear ODEs with variable coefficients because the coefficients, p and q, can also be substituted by a power series.

Example: A special case of Legendre equation

It occurs in problems with spherical symmetry

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

Solution:
Let's look for a solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \ldots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \ldots$$

Example (continued)

$$(1-x^2)y'' - 2xy' + 2y = 0$$

We now compute each one of the terms appearing in the ODE

$$\begin{array}{rcccccccc}
y'' = & 2a_2 & +6a_3x & +12a_4x^2 & +20a_5x^3 & +30a_6x^4 & +\dots & & \\
-x^2y'' = & & & -2a_2x^2 & -6a_3x^3 & -12a_4x^4 & -20a_5x^5 & -30a_6x^6 & +\dots \\
-2xy' = & & -2a_1x & -4a_2x^2 & -6a_3x^3 & -8a_4x^4 & -10a_5x^5 & -12a_6x^6 & +\dots \\
2y' = & 2a_0 & +2a_1x & +2a_2x^2 & +2a_3x^3 & +2a_4x^4 & +2a_5x^5 & +2a_6x^6 & +\dots
\end{array}$$

And solve for the coefficients of each power

$$\begin{array}{lll}
m=0: & 2a_2+2a_0=0 & \Rightarrow \quad a_2=-a_0 \\
m=1: & 6a_3=0 & \Rightarrow \quad a_3=0 \\
m=2: & 12a_4-4a_2=0 & \Rightarrow \quad a_4=\frac{1}{3}a_2=-\frac{1}{3}a_0 \\
m=3: & 20a_5-10a_3=0 & \Rightarrow \quad a_5=0 \\
m=4: & 30a_6-18a_4=0 & \Rightarrow \quad a_6=\frac{3}{5}a_4=\frac{3}{5}\left(-\frac{1}{3}a_0\right)=-\frac{1}{5}a_0 \\
\dots & &
\end{array}$$

Power series methods

Example (continued)

The general solution of the equation is

$$y = a_1 x + a_0 (1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \ldots$$

$$y = a_1 x + a_0 \left(1 - \sum_{m=1}^{\infty} \frac{1}{2m-1} x^{2m} \right)$$

Power series methods

Convergence

Operations (derivatives and integrals) with a power series are valid within its region of convergence

$$S = \lim_{n\to\infty} \sum_{m=0}^{n} a_m (x - x_0)^m = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

The region of convergence is a property of the a_m coefficients and its radius can be determined with

$$R = \frac{1}{\lim_{m\to\infty} |a_m|^{\frac{1}{m}}}$$

or

$$R = \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}$$

The series is valid in the interval

$$(x_0 - R, x_0 + R)$$

Power series methods

Example

$$e^x = \sum_{m=0}^{\infty} \frac{1}{m!} x^m \quad \Rightarrow \quad \left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{\frac{1}{(m+1)!}}{\frac{1}{m!}} \right| = \frac{1}{m+1} \quad \Rightarrow \quad R = \frac{1}{\lim\limits_{m\to\infty} \frac{1}{m+1}} = \frac{1}{0} = \infty$$

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m \quad \Rightarrow \quad \left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{1}{1} \right| = 1 \quad \Rightarrow \quad R = \frac{1}{\lim\limits_{m\to\infty} 1} = \frac{1}{1} = 1$$

$$\sum_{m=0}^{\infty} m! x^m \quad \Rightarrow \quad \left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{(m+1)!}{m!} \right| = m+1 \quad \Rightarrow \quad R = \frac{1}{\lim\limits_{m\to\infty} m+1} = \frac{1}{\infty} = 0$$

Existence of power series solutions

Given the ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

with p, q, and r analytic at x_0, then every solution of the ODE is analytic at x_0 and can be represented by a power series in terms of $(x - x_0)$ with a radius of convergence $R > 0$.

Power series methods

Derivative of a power series

If the power series

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

converges in $|x - x_0| < R$ (with $R > 0$), then

$$y'(x) = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1}$$

also converges at least in the region $|x - x_0| < R$.

Power series methods

Addition of two power series

If the power series

$$y_1(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

converges in $|x - x_0| < R_1$ and

$$y_2(x) = \sum_{m=0}^{\infty} b_m (x - x_0)^m$$

converges in $|x - x_0| < R_2$, then

$$(y_1 + y_2)(x) = \sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

converges at least in $|x - x_0| < \min(R_1, R_2)$.

Power series methods

Multiplication of two power series

Given the two power series

$$y_1(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m \quad y_2(x) = \sum_{l=0}^{\infty} b_l (x - x_0)^l$$

its multiplication is given

$$\begin{aligned} y_1(x) y_2(x) &= \left(\sum_{m=0}^{\infty} a_m (x - x_0)^m \right) \left(\sum_{l=0}^{\infty} b_l (x - x_0)^l \right) \\ &= \sum_{m=0}^{\infty} \left(a_m (x - x_0)^m \left(\sum_{l=0}^{\infty} b_l (x - x_0)^l \right) \right) \\ &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} a_m b_l (x - x_0)^{m+l} \\ &= \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} a_l b_{m-l} \right) (x - x_0)^m \end{aligned}$$

Vanishing of all coefficients

If a power series has a positive radius of convergence, R, and a sum that is 0 throughout its interval, then each coefficient of the series must be 0. That is, if

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = 0 \quad \forall x \in \mathbb{R} \text{ such that } |x - x_0| < R,$$

then

$$\forall m \quad a_m = 0$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 5, Section 1:

- 5.1.7
- 5.1.20

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Legendre's equation

Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

It appears in physical problems with spherical symmetry. We transform it into

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

The coefficients p and q are analytic at $x = 0$ (but they are not at $x = \pm 1$). Then, we can use a power series around 0 as a solution

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Let's call $k = n(n+1)$

Legendre's equation (continued)

Subtituting into the ODE

$$(1-x^2)\left(\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}\right) - 2x\left(\sum_{m=1}^{\infty} m a_m x^{m-1}\right) + k\left(\sum_{m=0}^{\infty} a_m x^m\right) = 0$$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

$$(2a_2 + ka_0) + (6a_3 + (k-2)a_1)x + \sum_{m=2}^{\infty} ((m+2)(m+1)a_{m+2} - (m^2+m-k)a_m)x^m = 0$$

Legendre's equation

Legendre's equation (continued)

$$(2a_2 + ka_0) + (6a_3 + (k-2)a_1)x + \sum_{m=2}^{\infty} ((m+2)(m+1)a_{m+2} - (m^2 + m - k)a_m)x^m = 0$$

$$\begin{aligned}
m = 0: \quad & 2a_2 + ka_0 = 0 \\
& a_2 = -\tfrac{k}{2} a_0 = -\tfrac{n(n+1)}{2!} a_0 \\
m = 1: \quad & 6a_3 + (k-2)a_1 = 0 \\
& a_3 = -\tfrac{k-2}{6} a_1 = -\tfrac{n(n+1)-2}{3!} a_1 = -\tfrac{(n-1)(n+2)}{3!} a_1 \\
m \geq 2: \quad & (m+2)(m+1)a_{m+2} - (m^2 + m - k)a_m \\
& a_{m+2} = \tfrac{m^2+m-k}{(m+2)(m+1)} a_m = \tfrac{m^2+m-n(n+1)}{(m+2)(m+1)} a_m = -\tfrac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m
\end{aligned}$$

$$\begin{aligned}
a_4 &= -\tfrac{(n-2)(n+3)}{4\cdot 3} a_2 \\
&= \left(-\tfrac{(n-2)(n+3)}{4\cdot 3}\right)\left(-\tfrac{n(n+1)}{2!} a_0\right) \\
&= \tfrac{(n-2)n(n+1)(n+3)}{4!} a_0
\end{aligned}
\qquad
\begin{aligned}
a_5 &= -\tfrac{(n-3)(n+4)}{5\cdot 4} a_3 \\
&= \left(-\tfrac{(n-3)(n+4)}{5\cdot 4} a_3\right)\left(-\tfrac{(n-1)(n+2)}{3!} a_1\right) \\
&= \tfrac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1
\end{aligned}$$

Legendre's equation

Legendre's equation (continued)

There actually two independent solutions and the general solution in $(-1,1)$ is a linear combination of the two:

$$y = a_0 y_1 + a_1 y_2$$

$$\begin{aligned} \boxed{y_1} &= a_0 + a_2 x^2 + a_4 x^4 + \ldots \\ &= \boxed{1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots} \end{aligned}$$

$$\begin{aligned} \boxed{y_2} &= a_1 x + a_3 x^3 + a_5 x^5 + \ldots \\ &= \boxed{x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots} \end{aligned}$$

Legendre's equation

Polynomial solutions

Consider the recursion

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m$$

If n is a positive integer, for $m = n$ we have

$$a_{n+2} = -\frac{(n-n)(n+n+1)}{(n+2)(n+1)} a_n = 0$$

and from this point on

$$0 = a_{n+2} = a_{n+4} = a_{n+6} = \dots$$

If n is even, y_1 is a polynomial of degree n.
If n is odd, y_2 is a polynomial of degree n.
These polynomials will be referred to as $P_n(x)$.

Legendre's equation

Polynomial solutions (continued)

We choose the highest coefficient (of degree n) to be

$$a_n = \frac{(2n)!}{2^n (n!)^2}$$

This choice will make $P_n(1) = 1$. Now we need to go back to calculate a_0 using

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m$$

$$a_m = -\frac{(m+2)(m+1)}{(n-m)(n+m+1)} a_{m+2}$$

$$a_{m-2} = -\frac{m(m-1)}{(n-m+2)(n+m-1)} a_m$$

Polynomial solutions (continued)

$$
\begin{aligned}
a_{n-2} &= -\frac{n(n-1)}{(n-n+2)(n+n-1)} a_n = -\frac{n(n-1)}{2(2n-1)} \left(\frac{(2n)!}{2^n (n!)^2} \right) \\
&= -\frac{n(n-1)[(2n)(2n-1)(2n-2)!}{2(2n-1)2^n[n(n-1)!][n(n-1)(n-2)!]} \\
&= -\frac{\cancel{n}\cancel{(n-1)}[(\cancel{2}\cancel{n})\cancel{(2n-1)}(2n-2)!]}{\cancel{2}\cancel{(2n-1)}2^n[\cancel{n}(n-1)!][\cancel{n}\cancel{(n-1)}(n-2)!]} \\
&= -\frac{(2n-2)!}{2^n(n-1)!(n-2)!} \\
&= (-1)^1 \frac{(2n-2\cdot 1)!}{2^n(1!)(n-1)!(n-2\cdot 1)!} \\
a_{n-4} &= -\frac{(n-2)((n-2)-1)}{(n-(n-2)+2)(n+(n-2)-1)} a_{n-2} = \dots \\
&= -\frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!} \\
&= (-1)^2 \frac{(2n-2\cdot 2)!}{2^n(2!)(n-2)!(n-2\cdot 2)!}
\end{aligned}
$$

and, in general,

$$
\boxed{a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}}
$$

Legendre's equation

Polynomial solutions (continued)

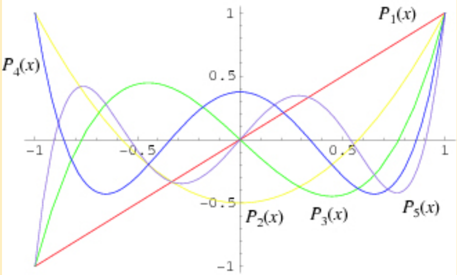
$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

The polynomial is finally

$$\boxed{P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^m}$$

$M = \frac{n}{2}$ if n is even
$M = \frac{n-1}{2}$ if n is odd.
Legendre's polynomials are orthogonal in $[-1, 1]$

$$\int_{-1}^{1} P_i(x) P_j(x) dx = \frac{2}{2i+1} \delta_{ij}$$



Legendre's equation

Polynomial solutions (continued)

$$P_0(x) = 1, \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Fortunately, they can be calculated recursively (Bonnet's recursion formula)

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Exercises

From Kreyszig (10th ed.), Chapter 5, Section 2:

- 5.2.2

10. TEAM PROJECT. Generating Functions. Generating functions play a significant role in modern applied mathematics (see [GenRef5]). The idea is simple. If we want to study a certain sequence $(f_n(x))$ and can find a function

$$G(u, x) = \sum_{n=0}^{\infty} f_n(x) u^n,$$

we may obtain properties of $(f_n(x))$ from those of G, which "generates" this sequence and is called a **generating function** of the sequence.

Exercises

(a) Legendre polynomials. Show that

$$\text{(12)} \quad G(u, x) = \frac{1}{\sqrt{1 - 2xu + u^2}} = \sum_{n=0}^{\infty} P_n(x) u^n$$

is a generating function of the Legendre polynomials. *Hint:* Start from the binomial expansion of $1/\sqrt{1-v}$, then set $v = 2xu - u^2$, multiply the powers of $2xu - u^2$ out, collect all the terms involving u^n, and verify that the sum of these terms is $P_n(x)u^n$.

(b) Potential theory. Let A_1 and A_2 be two points in space (Fig. 108, $r_2 > 0$). Using (12), show that

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}} = \frac{1}{r_2} \sum_{m=0}^{\infty} P_m(\cos\theta) \left(\frac{r_1}{r_2}\right)^m.$$

This formula has applications in potential theory. (Q/r is the electrostatic potential at A_2 due to a charge Q located at A_1. And the series expresses $1/r$ in terms of the distances of A_1 and A_2 from any origin O and the angle θ between the segments OA_1 and OA_2.)

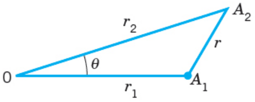


Fig. 108. Team Project 10

(c) Further applications of (12). Show that $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $P_{2n+1}(0) = 0$, and $P_{2n}(0) = (-1)^n \cdot 1 \cdot 3 \cdots (2n-1)/[2 \cdot 4 \cdots (2n)]$.

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Frobenius method

Frobenius method

Let $b(x)$ and $c(x)$ be analytic functions at $x = 0$. Then the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

has at least one solution that can be represented in the form

$$y = x^r \sum_{m=0}^{\infty} a_m x^m \quad (a_0 \neq 0)$$

where r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$). The ODE also has a second solution that may be similar to the previous one (with different r and coefficients) or may contain a logarithmic term.

Bessel's equation

$$y'' + \frac{1}{x} y' + \frac{x^2 - \nu^2}{x^2} y = 0$$

Frobenius method

Indicial equation

If b and c are not polynomials, let us expand them as

$$b = \sum_{m=0}^{\infty} b_m x^m \quad \sum_{m=0}^{\infty} c_m x^m$$

Let us calculate also the derivatives of the solution

$$\begin{aligned}
y &= x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1 x + \ldots) \\
y' &= \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1} \sum_{m=0}^{\infty} (m+r) a_m x^m \\
&= x^{r-1}(r a_0 + (r-1) a_1 + \ldots) \\
y'' &= \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} = x^{r-2} \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^m \\
&= x^{r-2}(r(r-1) a_0 + (r+1) r a_1 + \ldots)
\end{aligned}$$

Frobenius method

Indicial equation (continued)

Let us rewrite the ODE as

$$x^2 y'' + xby' + cy = 0$$

And substitute the solution

$$x^r(r(r-1)a_0 + ...) + (b_0 + ...)x^r(ra_0 + ...) + (c_0 + ...)x^r(a_0 + ...) = 0$$

Consider the coefficient of x^r

$$(r(r-1) + b_0 r + c_0)a_0 = 0$$

Since $a_0 \neq 0$

$$\boxed{r(r-1) + b_0 r + c_0 = 0}$$

This is the indicial equation and provides the r of one of the solutions, and determines the form of the other solution.

Frobenius method

Indicial equation (continued)

$$r(r-1)+b_0 r+c_0=0$$

One of the elements of the basis is

$$y_1 = x^{r_1}(a_0 + a_1 x + ...)$$

The other is

- Distinct roots (including complexes) not differing by an integer 1,2,...

$$y_2 = x^{r_2}(A_0 + A_1 x + ...)$$

- A double root:

$$y_2 = y_1 \log(x) + x^{r_1}(A_1 x + ...)$$

- Roots differing by an integer 1,2,... $(r_1 > r_2)$

$$y_2 = k y_1 \log(x) + x^{r_2}(A_0 + A_1 x + ...)$$

Frobenius method

Example: Euler-Cauchy equation

$$x^2 y'' + b_0 x y' + c_0 y = 0$$

Solution:

The indicial equation

$$r(r-1) + b_0 r + c_0 = 0 \Rightarrow r_1, r_2$$

If $r_1 \neq r_2$

$$y_1 = x^{r_1}$$
$$y_2 = x^{r_2}$$

If $r_1 = r_2$

$$y_1 = x^{r_1}$$
$$y_2 = x^{r_1} \log(x)$$

Frobenius method

Example

$$x(x-1)y'' + (3x-1)y' + y = 0$$

Solution:
We rewrite it as

$$y'' + \frac{3x-1}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0$$

$$y'' + \frac{\frac{3x-1}{x-1}}{x}y' + \frac{\frac{x}{x-1}}{x^2}y = 0$$

The functions $b = \frac{3x-1}{x-1}$ and $c = \frac{x}{x-1}$ are analytic around $x = 0$, so we can apply Frobenius method. Actually, we do not need the expansion of b and c

Frobenius method

Example (continued)

Let us substitute the Frobenius solution into the ODE

$$x(x-1)y'' + (3x-1)y' + y = 0$$

$$x^2y'' - xy'' + 3xy' - y' + y = 0$$

$$x^2\left(x^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m\right) - x\left(x^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m\right)$$

$$+3x\left(x^{r-1}\sum_{m=0}^{\infty}(m+r)a_m x^m\right) - \left(x^{r-1}\sum_{m=0}^{\infty}(m+r)a_m x^m\right) + \left(x^{r}\sum_{m=0}^{\infty}a_m x^m\right) = 0$$

Frobenius method

Example (continued)

This can be rewritten as

$$x^r \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m - x^{r-1} \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m$$

$$+3x^r \sum_{m=0}^{\infty} (m+r)a_m x^m - x^{r-1} \sum_{m=0}^{\infty} (m+r)a_m x^m + x^r \sum_{m=0}^{\infty} a_m x^m = 0$$

The smallest power is x^{r-1} and its coefficient gives the indicial equation

$$-r(r-1)a_0 - ra_0 = 0$$

$$-r^2 = 0$$

So, we have a double root at $r = 0$.

Frobenius method

Example (continued)

First solution:
We substitute $r = 0$ in the equation

$$x^r \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m - x^{r-1} \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m$$

$$+3x^r \sum_{m=0}^{\infty} (m+r)a_m x^m - x^{r-1} \sum_{m=0}^{\infty} (m+r)a_m x^m + x^r \sum_{m=0}^{\infty} a_m x^m = 0$$

That is

$$\sum_{m=0}^{\infty} m(m-1)a_m x^m - x^{-1} \sum_{m=0}^{\infty} m(m+1)a_m x^m$$

$$+3 \sum_{m=0}^{\infty} m a_m x^m - x^{-1} \sum_{m=0}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

Frobenius method

Example (continued)

First solution: (continued)

$$\sum_{m=0}^{\infty} m(m-1)a_m x^m - x^{-1}\sum_{m=0}^{\infty} m(m+1)a_m x^m$$

$$+3\sum_{m=0}^{\infty} m a_m x^m - x^{-1}\sum_{m=0}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m(m-1)+3m+1)a_m x^m - \sum_{m=0}^{\infty} (m(m+1)-m)a_m x^{m-1} = 0$$

$$\sum_{m=0}^{\infty} (m^2+2m+1)a_m x^m - \sum_{m=0}^{\infty} m^2 a_m x^{m-1} = 0$$

$$\sum_{m=0}^{\infty} (m^2+2m+1)a_m x^m - \sum_{m'=-1}^{\infty} (m'+1)^2 a_{m'+1} x^{m'} = 0$$

Frobenius method

Example (continued)

First solution: (continued)

$$\sum_{m=0}^{\infty}(m^2+2m+1)a_m x^m - \sum_{m'=-1}^{\infty}(m'+1)^2 a_{m'+1}x^{m'} = 0$$

$$\sum_{m=0}^{\infty}(m^2+2m+1)a_m x^m - \sum_{m=0}^{\infty}(m+1)^2 a_{m+1}x^{m} = 0$$

$$\sum_{m=0}^{\infty}[(m^2+2m+1)a_m - (m+1)^2 a_{m+1}]x^{m} = 0$$

$$\sum_{m=0}^{\infty}(m+1)^2(a_m - a_{m+1})x^{m} = 0$$

$$(m+1)^2(a_m - a_{m+1}) = 0$$

$$a_{m+1} = a_m$$

Frobenius method

Example (continued)

First solution: (continued)

$$a_{m+1} = a_m$$

Hence

$$a_0 = a_1 = a_2 = \ldots$$

We may choose $a_0 = 1$. The first solution is

$$y_1 = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} \quad |x| < 1$$

Frobenius method

Example (continued)

Second solution:

We apply reduction of order

$$\left[y'' + py' + qy = 0 \quad U = \frac{1}{y_1^2} e^{-\int p dx} \quad u = \int U dx \right]$$

$$y'' + \frac{\frac{3x-1}{x-1}}{x} y' + \frac{\frac{x}{x-1}}{x^2} y = 0$$

$$-\int \frac{3x-1}{x(x-1)} dx = -\int \left(\frac{2}{x-1} + \frac{1}{x} \right) dx = -2\log|x-1| - \log|x| = \log \frac{1}{x(x-1)^2}$$

$$U = \frac{1}{\left(\frac{1}{1-x}\right)^2} e^{\log \frac{1}{x(x-1)^2}} = (1-x)^2 \frac{1}{x(x-1)^2} = \frac{1}{x}$$

$$u = \int \frac{1}{x} dx = \log(x) \Rightarrow y_2 = u y_1 = \frac{\log(x)}{1-x}$$

Frobenius method

Example (continued)

The general solution of

$$x(x-1)y'' + (3x-1)y' + y = 0$$

is

$$y = c_1 \frac{1}{1-x} + c_2 \frac{\log(x)}{1-x}$$

Frobenius method

Example

$$(x^2 - x)y'' - xy' + y = 0$$

Solution

Substituting the Frobenius solution in the ODE we get

$$(x^2 - x)\left(x^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m\right) - x\left(x^{r-1}\sum_{m=0}^{\infty}(m+r)a_m x^m\right)$$

$$+\left(x^r\sum_{m=0}^{\infty}a_m x^m\right) = 0$$

$$x^r\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m - x^{r-1}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m$$

$$-x^r\sum_{m=0}^{\infty}(m+r)a_m x^m + x^r\sum_{m=0}^{\infty}a_m x^m = 0$$

Frobenius method

Example (continued)

$$x^r \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m - x^{r-1} \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m$$

$$-x^r \sum_{m=0}^{\infty} (m+r)a_m x^m + x^r \sum_{m=0}^{\infty} a_m x^m = 0$$

The smallest power is $r-1$ whose coefficient is

$$-r(r-1)a_0 = 0 \Rightarrow r_1 = 1, r_2 = 0$$

We have two roots, whose difference is an integer.

Frobenius method

Example (continued)

First solution: Substitute $r = 1$ in

$$x^r \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m - x^{r-1} \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^m$$

$$-x^r \sum_{m=0}^{\infty} (m+r)a_m x^m + x^r \sum_{m=0}^{\infty} a_m x^m = 0$$

$$x \sum_{m=0}^{\infty} (m+1)m a_m x^m - \sum_{m=0}^{\infty} (m+1)m a_m x^m - x \sum_{m=0}^{\infty} (m+1)a_m x^m + x \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+1)m a_m x^{m+1} - \sum_{m=0}^{\infty} (m+1)m a_m x^m - \sum_{m=0}^{\infty} (m+1)a_m x^{m+1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0$$

$$\sum_{m=0}^{\infty} (m+1)m a_m x^{m+1} - \sum_{m=1}^{\infty} (m+1)m a_m x^m - \sum_{m=0}^{\infty} (m+1)a_m x^{m+1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0$$

Frobenius method

Example (continued)

First solution: (continued)

$$\sum_{m=0}^{\infty}(m+1)ma_m x^{m+1} - \sum_{m=1}^{\infty}(m+1)ma_m x^{m} - \sum_{m=0}^{\infty}(m+1)a_m x^{m+1} + \sum_{m=0}^{\infty}a_m x^{m+1} = 0$$

$$\sum_{m=0}^{\infty}(m+1)ma_m x^{m+1} - \sum_{m'=0}^{\infty}(m'+2)(m'+1)a_{m'+1}x^{m'+1}$$

$$-\sum_{m=0}^{\infty}(m+1)a_m x^{m+1} + \sum_{m=0}^{\infty}a_m x^{m+1} = 0$$

$$\sum_{m=0}^{\infty}\left([(m+1)m-(m+1)+1]a_m-(m+2)(m+1)a_{m+1}\right)x^{m+1} = 0$$

$$\sum_{m=0}^{\infty}\left(m^2 a_m-(m+2)(m+1)a_{m+1}\right)x^{m+1} = 0$$

Frobenius method

Example (continued)

First solution: (continued)

$$\sum_{m=0}^{\infty} \left(m^2 a_m - (m+2)(m+1)a_{m+1}\right) x^{m+1} = 0$$

$$m^2 a_m - (m+2)(m+1)a_{m+1} = 0$$

$$a_{m+1} = \frac{m^2}{(m+2)(m+1)} a_m$$

If we choose $a_0 = 1$, then

$$a_1 = \frac{0^2}{(0+2)(0+1)} a_0 = 0 = a_2 = a_3 = \dots$$

So

$$y_1 = x^{r_1} a_0 = x^1 \cdot 1 = x$$

Frobenius method

Example (continued)

Second solution: Let's apply a reduction of order:

$$y_2 = uy_1 = ux$$

$$y_2' = u + u'x$$

$$y_2'' = u' + u'' + u' = u'' + 2u'$$

And substitute in the ODE

$$(x^2 - x)y'' - xy' + y = 0$$

$$(x^2 - x)(u'' + 2u') - x(u + u'x) + ux = 0$$

$$(x^2 - x)u'' + (x - 2)u' = 0$$

$$\frac{u''}{u'} = -\frac{x-2}{x^2-x} = -\frac{2}{x} + \frac{1}{1-x}$$

Frobenius method

Example (continued)

Second solution: (continued)

$$\frac{u''}{u'} = -\frac{x-2}{x^2-x} = -\frac{2}{x} + \frac{1}{1-x}$$

$$\log(u') = -2\log|x| + \log|x-1| = \log\frac{x-1}{x^2}$$

$$u' = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$$

$$u = \log(x) + \frac{1}{x}$$

$$y_2 = uy_1 = \left(\log(x) + \frac{1}{x}\right)x = x\log(x) - 1$$

The general solution is

$$\boxed{y = c_1 x + c_2(x\log(x) - 1)}$$

Exercises

14. TEAM PROJECT. Hypergeometric Equation, Series, and Function. Gauss's hypergeometric ODE[5] is

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0. \tag{15}$$

Here, a, b, c are constants. This ODE is of the form $p_2 y'' + p_1 y' + p_0 y = 0$, where p_2, p_1, p_0 are polynomials of degree 2, 1, 0, respectively. These polynomials are written so that the series solution takes a most practical form, namely,

$$\begin{aligned} y_1(x) &= 1 + \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 \\ &\quad + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\,c(c+1)(c+2)}x^3 + \cdots. \end{aligned} \tag{16}$$

This series is called the **hypergeometric series**. Its sum $y_1(x)$ is called the **hypergeometric function** and is denoted by $F(a, b, c; x)$. Here, $c \neq 0, -1, -2, \cdots$. By choosing specific values of a, b, c we can obtain an incredibly large number of special functions as solutions of (15) [see the small sample of elementary functions in part (c)]. This accounts for the importance of (15).

(a) Hypergeometric series and function. Show that the indicial equation of (15) has the roots $r_1 = 0$ and $r_2 = 1 - c$. Show that for $r_1 = 0$ the Frobenius method gives (16). Motivate the name for (16) by showing that

$$F(1, 1, 1; x) = F(1, b, b; x) = F(a, 1, a; x) = \frac{1}{1-x}.$$

(b) Convergence. For what a or b will (16) reduce to a polynomial? Show that for any other a, b, c ($c \neq 0, -1, -2, \cdots$) the series (16) converges when $|x| < 1$.

(c) Special cases. Show that

$$\begin{aligned} (1+x)^n &= F(-n, b, b; -x), \\ (1-x)^n &= 1 - nxF(1-n, 1, 2; x), \\ \arctan x &= xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; -x^2) \\ \arcsin x &= xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}; x^2), \\ \ln(1+x) &= xF(1, 1, 2; -x), \\ \ln\frac{1+x}{1-x} &= 2xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; x^2). \end{aligned}$$

Exercises

Exercises

Find more such relations from the literature on special functions, for instance, from [GenRef1] in App. 1.

(d) Second solution. Show that for $r_2 = 1 - c$ the Frobenius method yields the following solution (where $c \neq 2, 3, 4, \cdots$):

$$\text{(17)} \quad y_2(x) = x^{1-c}\left(1 + \frac{(a-c+1)(b-c+1)}{1!\,(-c+2)}x + \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2!\,(-c+2)(-c+3)}x^2 + \cdots\right).$$

Show that

$$y_2(x) = x^{1-c}F(a-c+1, b-c+1, 2-c; x).$$

(e) On the generality of the hypergeometric equation. Show that

$$\text{(18)} \quad (t^2 + At + B)\ddot{y} + (Ct + D)\dot{y} + Ky = 0$$

with $\dot{y} = dy/dt$, etc., constant A, B, C, D, K, and $t^2 + At + B = (t - t_1)(t - t_2)$, $t_1 \neq t_2$, can be reduced to the hypergeometric equation with independent variable

$$x = \frac{t - t_1}{t_2 - t_1}$$

and parameters related by $Ct_1 + D = -c(t_2 - t_1)$, $C = a + b + 1$, $K = ab$. From this you see that (15) is a "normalized form" of the more general (18) and that various cases of (18) can thus be solved in terms of hypergeometric functions.

Outline

1. Series solutions of ODEs. Special functions
   - Power series methods
   - Legendre's equation. Legendre polynomials
   - Extended power series method: Frobenius method
   - Bessel's equation. Bessel functions $J_\nu(x)$
   - Bessel's equation. Bessel functions $Y_\nu(x)$

Bessel's equation

Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad \nu \geq 0$$

It appears in physical problems with cylindrical symmetry. We may transform it into

$$y'' + \frac{1}{x}y' + \frac{x^2 - \nu^2}{x^2}y = 0$$

The functions b and c (see Frobenius method) are analytic at $x = 0$ so we can try a Frobenius solution:

$$x^2\left(x^{r-2}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m\right) + x\left(x^{r-1}\sum_{m=0}^{\infty}(m+r)a_m x^m\right)$$

$$+x^2\left(x^r\sum_{m=0}^{\infty}a_m x^m\right) - \nu^2\left(x^r\sum_{m=0}^{\infty}a_m x^m\right) = 0$$

Bessel's equation

$$x^r\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^m + x^r\sum_{m=0}^{\infty}(m+r)a_m x^m$$

$$+x^{r+2}\sum_{m=0}^{\infty}a_m x^m - \nu^2 x^r\sum_{m=0}^{\infty}a_m x^m = 0$$

The smallest power is r and its coefficient gives the indicial equation

$$r(r-1)a_0 + ra_0 - \nu^2 a_0 = (r^2-\nu^2)a_0 = 0 \Rightarrow r_1, r_2 = \pm\nu$$

Substituting $r = r_1 = \nu$ in the equation above we get

$$x^\nu\sum_{m=0}^{\infty}(m+\nu)(m+\nu-1)a_m x^m + x^\nu\sum_{m=0}^{\infty}(m+\nu)a_m x^m$$

$$+x^{\nu+2}\sum_{m=0}^{\infty}a_m x^m - \nu^2 x^\nu\sum_{m=0}^{\infty}a_m x^m = 0$$

Bessel's equation

Bessel's equation

$$\sum_{m=0}^{\infty} m(m+2\nu)a_m x^{m+\nu} + \sum_{m=0}^{\infty} a_m x^{m+\nu+2} = 0$$

$$\sum_{m=0}^{\infty} m(m+2\nu)a_m x^{m+\nu} + \sum_{m'=2}^{\infty} a_{m'-2} x^{m'+\nu} = 0$$

$$(1+2\nu)a_1 x^{1+\nu} + \sum_{m=2}^{\infty} [m(m+2\nu)a_m + a_{m-2}] x^{m+\nu} = 0$$

$$(1+2\nu)a_1 = 0 \Rightarrow a_1 = 0$$

$$m(m+2\nu)a_m + a_{m-2} = 0 \Rightarrow a_m = -\frac{1}{m(2\nu+m)} a_{m-2}$$

For odd m's, a_3 is a function of a_1 (that is 0), a_5 a function of a_3, ...

$$0 = a_1 = a_3 = a_5 = \dots$$

Bessel's equation

Bessel's equation

$$a_m = -\frac{1}{m(2\nu + m)} a_{m-2}$$

For even m's, we can write $m = 2k$

$$a_{2k} = -\frac{1}{2k(2\nu + 2k)} a_{2k-2} = -\frac{1}{2^2 k(\nu + k)} a_{2k-2} \quad k = 1, 2, ...$$

That is

$$a_2 = -\frac{1}{2^2(\nu + 1)} a_0$$

$$a_4 = -\frac{1}{2^2 2(\nu + 2)} a_2 = -\frac{1}{2^2 2(\nu + 2)} \left(-\frac{1}{2^2(\nu + 1)} a_0 \right) = \frac{1}{2^4 2!(\nu + 1)(\nu + 2)} a_0$$

In general,

$$a_{2k} = \frac{(-1)^m}{2^{2k} k!(\nu + 1)(\nu + 2)...(\nu + k)} a_0$$

Bessel's equation

Bessel's equation: Bessel's functions of first kind $J_\nu(x)$

The first solution of Bessel's equation is

$$y_1 = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! (\nu+1)(\nu+2)\ldots(\nu+k)} a_0 x^{2k}$$

If $\nu \in \mathbb{Z}$, $\nu = n$ let us choose

$$a_0 = \frac{1}{2^n n!}$$

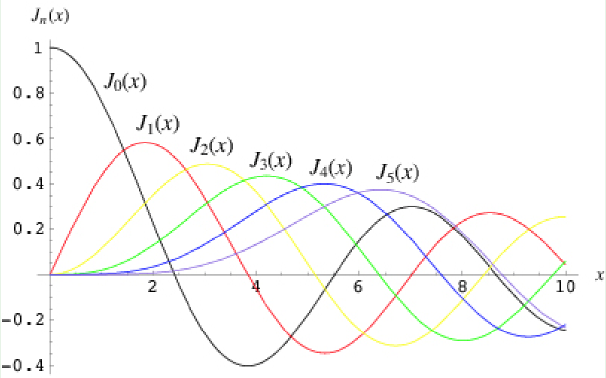
Then

$$y_1 = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! (n+1)(n+2)\ldots(n+k)} \frac{1}{2^n n!} x^{2k}$$

$$y_1 = \boxed{J_n = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n} k! (n+k)!} x^{2k}}$$

This **Bessel's function of the first kind and order** n.

Bessel's functions of first kind $J_\nu(x)$



For large x, they fulfill

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

The Gamma function

Let us define the Gamma function as

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt$$

Integrating by parts ($u = t^x$, $dv = e^{-t}dt$) we get

$$\Gamma(x+1) = -e^{-t}t^x\Big|_0^\infty + \int_0^\infty x t^{x-1} e^{-t} dt = 0 + x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x)$$

Additionally

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

Bessel's equation

The Gamma function

$$
\begin{array}{rcl}
\Gamma(x+1) & = & x\Gamma(x) \\
\hline
\Gamma(1) & = & 1 \\
\Gamma(2) & = & 1\Gamma(1) = 1 \cdot 1 \\
\Gamma(3) & = & 2\Gamma(2) = 2 \cdot 1 \\
\Gamma(4) & = & 3\Gamma(3) = 3 \cdot 2 \cdot 1 \\
\ldots & & \\
\Gamma(n+1) & = & n!
\end{array}
$$

Another interesting result

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Bessel's equation: Bessel's functions of first kind $J_\nu(x)$ (continued)

The first solution of Bessel's equation is

$$y_1 = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(\nu+1)(\nu+2)...(\nu+k)} a_0 x^{2k}$$

If $\nu \notin \mathbb{Z}$, let us choose

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$$

Then

$$y_1 = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(\nu+1)(\nu+2)...(\nu+k)} \frac{1}{2^\nu \Gamma(\nu+1)} x^{2k}$$

$$y_1 = \boxed{J_\nu = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu} k!\Gamma(\nu+k+1)} x^{2k}}$$

Bessel's equation: Bessel's functions of first kind $J_\nu(x)$ (continued)

$$y_1 = J_\nu = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu} k! \Gamma(\nu + k + 1)} x^{2k}$$

An interesting result is that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

The general solution if $\nu \notin \mathbb{Z}$ is

$$\boxed{y = c_1 J_\nu + c_2 J_{-\nu}}$$

where

$$J_{-\nu} = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k-\nu} k! \Gamma(-\nu + k + 1)} x^{2k}$$

Bessel's equation: Bessel's functions of first kind $J_\nu(x)$ (continued)

$$y_1 = J_\nu = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu} k! \Gamma(\nu + k + 1)} x^{2k}$$

An interesting result is that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

The general solution if $\nu \notin \mathbb{Z}$ is

$$\boxed{y = c_1 J_\nu + c_2 J_{-\nu}}$$

where

$$J_{-\nu} = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k-\nu} k! \Gamma(-\nu + k + 1)} x^{2k}$$

Bessel's equation

Bessel's equation: Bessel's functions of first kind $J_\nu(x)$ (continued)

For $\nu \in \mathbb{Z}$ there is problem because

$$J_{-n} = (-1)^n J_n$$

that is, there is a linear dependence between the two solutions that will be solved by Bessel's functions of second kind.

Some useful properties

Derivatives

$$(x^\nu J_\nu(x))' = x^\nu J_{\nu-1}(x)$$

$$(x^{-\nu} J_\nu(x))' = -x^{-\nu} J_{\nu+1}(x)$$

Recursion

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 5, Section 5:

- 5.4.3
- 5.4.6

Outline

1. Series solutions of ODEs. Special functions
   - Power series methods
   - Legendre's equation. Legendre polynomials
   - Extended power series method: Frobenius method
   - Bessel's equation. Bessel functions $J_\nu(x)$
   - Bessel's equation. Bessel functions $Y_\nu(x)$

Bessel's equation

Let's look for a second solution of Bessel's equation in the case of $\nu \in \mathbb{Z}$. For simplicity we will start with $n = 0$. The ODE is then

$$x^2 y'' + xy' + x^2 y = 0$$

$$xy'' + y' + xy = 0$$

We know from the previous section that the indicial equation has a double root at $r = 0$. The first solution is

$$y_1 = J_0$$

The second solution according to Frobenius method must be of the form

$$y_2 = y_1 \log(x) + x^r \sum_{m=1}^{\infty} A_m x^m$$

$$y_2 = J_0 \log(x) + \sum_{m=1}^{\infty} A_m x^m$$

Bessel's equation

Bessel's equation

$$y_2 = J_0 \log(x) + \sum_{m=1}^{\infty} A_m x^m$$

$$y_2' = J_0' \log(x) + J_0 x^{-1} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2'' = J_0'' \log(x) + 2J_0' x^{-1} - J_0 x^{-2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

We now substitute in the ODE

$$xy'' + y' + xy = 0$$

Bessel's equation

Bessel's equation

$$x\left(J_0''\log(x) + 2J_0'x^{-1} - J_0x^{-2} + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-2}\right)$$

$$+\left(J_0'\log(x) + J_0x^{-1} + \sum_{m=1}^{\infty} mA_m x^{m-1}\right) + x\left(J_0\log(x) + \sum_{m=1}^{\infty} A_m x^m\right) = 0$$

$$\cancel{(xJ_0'' + J_0' + xJ_0)}\log(x) + 2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$2J_0' + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

Bessel's equation

We know that

$$J_n = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n} k!(n+k)!} x^{2k}$$

In particular

$$J_0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} x^{2m}$$

$$J_0' = \sum_{m=1}^{\infty} \frac{(-1)^m 2m}{2^{2m}(m!)^2} x^{2m-1} = \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-1} m!(m-1)!} x^{2m-1}$$

Substituting in the solution of Bessel's equation

$$2\left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-1} m!(m-1)!} x^{2m-1}\right) + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-2} m!(m-1)!} x^{2m-1} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

Bessel's equation

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-2} m!(m-1)!} x^{2m-1} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$\left(-x + \frac{1}{2^2 2!} x^3 - \ldots\right) + (A_1 + 4A_2 x + \ldots) + (A_1 x^2 + A_2 x^3 + \ldots) = 0$$

The only term in x^0 comes from the second series, so it must be

$$A_1 = 0$$

Let's consider now the even terms. There is none in the first series. In the second and third series the corresponding terms are

$$\begin{array}{lll} x^{m-1} = x^{2s} & \Rightarrow m = 2s+1 & \Rightarrow (2s+1)A_{2s+1}x^{2s} \\ x^{m+1} = x^{2s} & \Rightarrow m = 2s-1 & \Rightarrow A_{2s-1}x^{2s} \end{array}$$

Bessel's equation

Their sum gives

$$(2s+1)A_{2s+1} + A_{2s-1} = 0 \Rightarrow A_{2s+1} = -\frac{1}{2s+1}A_{2s-1}$$

Consequently, if $A_1 = 0$, so are A_3, A_5, ... Let's go back to

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-2}m!(m-1)!}x^{2m-1} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$\left(-x + \frac{1}{2^2 2!}x^3 - \dots\right) + (4A_2 x + 16A_4 x^3 + \dots) + (A_2 x^3 + A_4 x^5 + \dots) = 0$$

For the power x^1, we have

$$-1 + 4A_2 = 0 \Rightarrow A_2 = \frac{1}{4}$$

Bessel's equation

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m-2} m!(m-1)!} x^{2m-1} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

For the rest of powers x^{2s+1} $(s = 1, 2, \ldots)$ we have

1st series	$2m - 1 = 2s + 1 \Rightarrow m = s + 1$	$\frac{(-1)^{s+1}}{2^{2(s+1)-2}(s+1)!((s+1)-1)!} = \frac{(-1)^{s+1}}{2^{2s}(s+1)!s!}$
2nd series	$m - 1 = 2s + 1 \Rightarrow m = 2s + 2$	$(2s+2)^2 A_{2s+2}$
3rd series	$m + 1 = 2s + 1 \Rightarrow m = 2s$	A_{2s}

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$$

$$A_{2s+2} = -\frac{1}{(2s+2)^2} A_{2s} - \frac{(-1)^{s+1}}{2^{2s}(s+1)!s!(2s+2)^2}$$

Bessel's equation

After some manipulation we obtain the general term

$$A_{2s} = \frac{(-1)^{s-1}}{2^{2s}(s!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{s}\right)$$

Let us call

$$h_s = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{s}$$

Finally, the second solution is

$$y_2 = J_0\log(x)+\sum_{s=1}^{\infty}\frac{(-1)^{s-1}h_s}{2^{2s}(s!)^2}x^{2s}$$

y_2 is independent of $y_1 = J_0$, so both functions together are a basis of solutions.

Bessel's equation

It is customary to use a different basis

$$\begin{aligned} Y_0 &= \tfrac{2}{\pi}(y_2 + (\gamma - \log(2))J_0) \\ &= \tfrac{2}{\pi}\left(J_0(x)\left(\log \tfrac{x}{2} + \gamma\right) + \sum_{s=1}^{\infty} \tfrac{(-1)^{s-1} h_s}{2^{2s}(s!)^2} x^{2s}\right) \end{aligned}$$

where γ is Euler's constant. The set $\{J_0, Y_0\}$ is also a basis. In general

$$\boxed{Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin \nu\pi}}$$

and the general solution of Bessel's equation

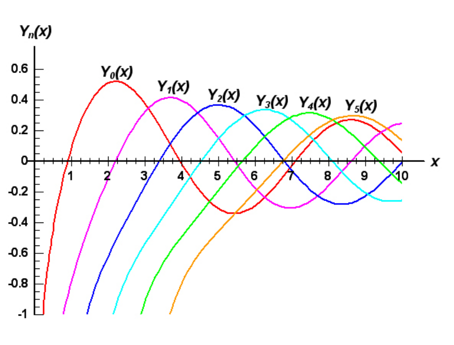
$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \quad \nu \geq 0$$

is

$$\boxed{y = c_1 J_\nu + c_2 Y_\nu}$$

Bessel's equation

Bessel's functions of the second kind



For large x, they fulfill

$$Y_n(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

Bessel's equation

Recurrence formulas

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x) \qquad Y_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x) - Y_{\nu-1}(x)$$

$$J'_{\nu+1}(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right] \qquad Y'_{\nu+1}(x) = \frac{1}{2}\left[Y_{\nu-1}(x) - Y_{\nu+1}(x)\right]$$

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) \qquad Y'_\nu(x) = Y_{\nu-1}(x) - \frac{\nu}{x} Y_\nu(x)$$

$$J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) \qquad Y'_\nu(x) = \frac{\nu}{x} Y_\nu(x) - Y_{\nu+1}(x)$$

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x) \qquad \frac{d}{dx}\left[x^\nu Y_\nu(x)\right] = x^\nu Y_{\nu-1}(x)$$

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x) \qquad \frac{d}{dx}\left[x^{-\nu} Y_\nu(x)\right] = -x^{-\nu} Y_{\nu+1}(x)$$

Modified Bessel's equation

Modified Bessel's equation

Bessel's equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

Modified Bessel's equation $x^2y'' + xy' - (x^2 + \nu^2)y = 0$

The general solution of the Modified Bessel's equation is of the form

$$y = c_1 I_\nu + c_2 K_\nu$$

where I_ν is the modified Bessel function of first kind:

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = x^\nu \sum_{k=0}^{\infty} \frac{1}{2^{2k+\nu} k! \Gamma(k + \nu + 1)} x^{2k}$$

compare it to

$$J_\nu = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu} k! \Gamma(\nu + k + 1)} x^{2k}$$

Modified Bessel's equation

Modified Bessel's equation (continued)

and K_ν is the modified Bessel function of second kind:

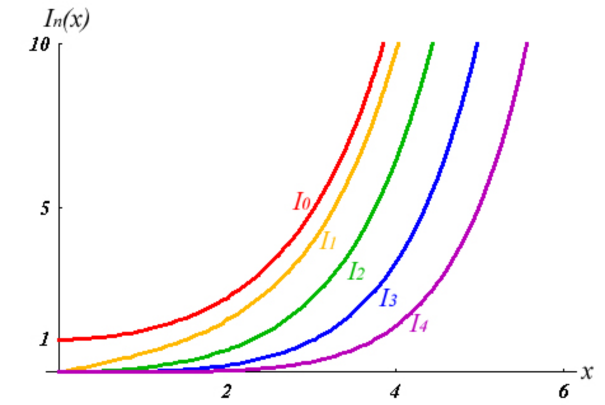
$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu(x)} - I_\nu(x)}{\sin \nu\pi}$$

compare it to

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin \nu\pi}$$

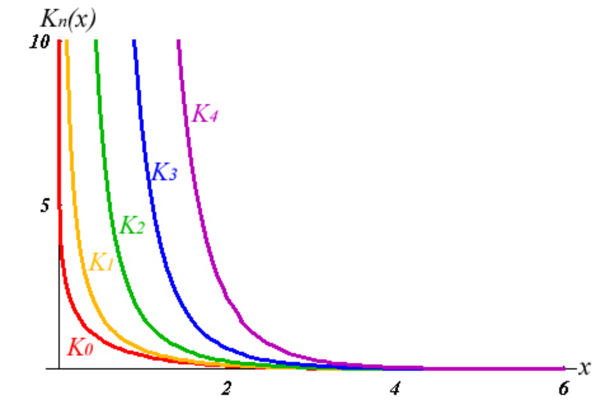
Modified Bessel's equation

Modified Bessel's function of first kind



Modified Bessel's equation

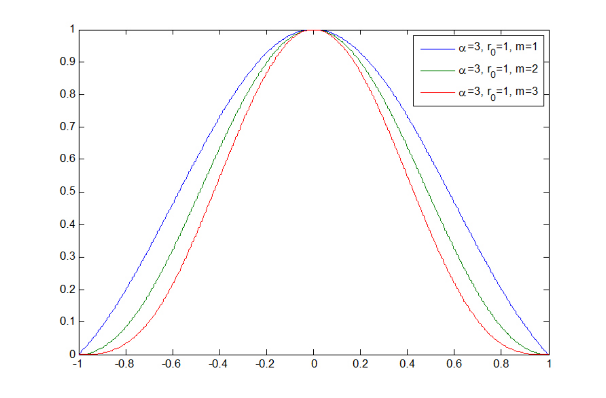
Modified Bessel's function of second kind



Modified Bessel's equation

Blobs

$$B(r) = \begin{cases} \dfrac{\left(\sqrt{1-\left(\frac{r}{r_0}\right)^2}\right)^m I_m\left(\alpha\sqrt{1-\left(\frac{r}{r_0}\right)^2}\right)}{I_m(\alpha)} & 0 \leq r \leq r_0 \\ 0 & r > r_0 \end{cases}$$



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 5, Section 5:

- 5.5.1
- 5.5.2

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