Chapter 6. Laplace transforms

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Biomedical Engineering

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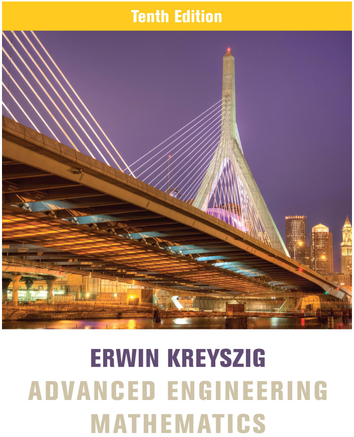
Outline

1. Laplace transforms
   - Laplace transform. Linearity. s-shifting theorem
   - Transforms of derivatives and integrals
   - Unit step function (Heaviside function), t-shifting theorem
   - Short impulses. Dirac's δ function. Partial fractions
   - Convolution. Integral equations
   - Differentiation and integration of transforms. ODEs with variable coefficients
   - Systems of ODEs

References



E. Kreyszig. Advanced Engineering Mathematics. John Wiley & sons. Chapter 6.

Outline

Laplace transform

Laplace transform

Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f\} = \int_0^{\infty} e^{-st} f(t) dt$$

In fact this is a particular case of a more general family of transforms in which a generic kernel (with variabkes in both spaces) is used

$$F(s) = \int_0^{\infty} k(s,t) f(t) dt$$

The inverse Laplace transform is denoted as

$$f(t) = \mathcal{L}^{-1}\{F\}$$

Laplace transform

Laplace transform

This operator fulfills

$$\mathcal{L}^{-1}\{\mathcal{L}\{f\}\} = f \quad \mathcal{L}\{\mathcal{L}^{-1}\{F\}\} = F$$

Example

Calculate the Laplace transform of the function $f(t) = 1$, and $f(t) = e^{at}$ when $t \geq 0$.

Solution:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} 1 dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s} \\ F(s) &= \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt - \frac{1}{s-a} e^{-(s-a)t}\Big|_0^\infty = \frac{1}{s-a} \end{aligned}$$

Laplace transform

Linearity

$$\mathcal{L}\{af + bg\} = aF + bG$$

Proof:

$$\int\limits_0^\infty e^{-st}(af + bg)dt = a\int\limits_0^\infty e^{-st} f dt + b\int\limits_0^\infty e^{-st} g dt = aF + bG$$

Example

$$\mathcal{L}\left\{\cosh(at)\right\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\left\{e^{at}\right\} + \frac{1}{2}\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$$

$$\mathcal{L}\left\{\sinh(at)\right\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\left\{e^{at}\right\} - \frac{1}{2}\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2}$$

Laplace transform

Example

Let's calculate the Laplace transforms of $\cos(\omega t)$ and $\sin(\omega t)$

$$L_c = \int_0^\infty e^{-st}\cos(\omega t)dt = \left.\frac{e^{-st}}{-s}\cos(\omega t)\right|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)dt = \frac{1}{s} - \frac{\omega}{s}L_s$$

$$L_s = \int_0^\infty e^{-st}\sin(\omega t)dt = \left.\frac{e^{-st}}{-s}\sin(\omega t)\right|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos(\omega t)dt = \frac{\omega}{s}L_c$$

We have an equation system with 2 equations and two unknowns

$$\left.\begin{array}{c} L_c = \frac{1}{s} - \frac{\omega}{s}L_s \\ L_s = \frac{\omega}{s}L_c \end{array}\right\} \Rightarrow L_c = \frac{s}{s^2+\omega^2}, L_s = \frac{\omega}{s^2+\omega^2}$$

Laplace transform

Example

$$
\begin{aligned}
\mathcal{L}\left\{t^{n+1}\right\} &= \int_0^\infty e^{-st} t^{n+1} dt = \frac{e^{-st}}{-s} t^{n+1}\Big|_0^\infty + \frac{n+1}{s}\int_0^\infty e^{-st} t^n dt = \frac{n+1}{s}\mathcal{L}\left\{t^n\right\} \\
&= \frac{n+1}{s}\frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\} \\
&= \frac{n+1}{s}\frac{n}{s}\dots\frac{2}{s}\frac{1}{s}\mathcal{L}\left\{t^0\right\} \\
&= \frac{n+1}{s}\frac{n}{s}\dots\frac{2}{s}\frac{1}{s}\frac{1}{s} \\
&= \frac{(n+1)!}{s^{n+2}}
\end{aligned}
$$

Laplace transform pairs

Table 6.1 **Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$**

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a + 1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s - a)^2 + \omega^2}$

Laplace transform

s-shifting

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

where $s - a > k$.

Proof:

$$F(s-a) = \int_0^\infty e^{-(s-a)t}f(t)dt = \int_0^\infty e^{-st}(e^{-at}f(t))dt = \mathcal{L}\{e^{at}f(t)\}$$

Example

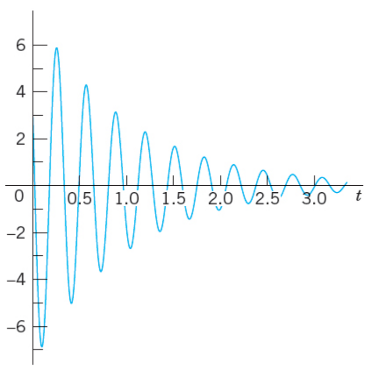
$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{at}\cos(\omega t)\} = \frac{s-a}{(s-a)^2+\omega^2}$$

Laplace transform

Example

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3s-137}{s^2+2s+401}\right\} &= \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\}\\ &= 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+400}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+400}\right\}\\ &= 3e^{-t}\cos(20t) - 7e^{-t}\sin(20t)\end{aligned}$$



Laplace transform

Existence and uniqueness

If $f(t)$ is defined and is piecewise continuous on every finite interval on $t \geq 0$ and

$$|f(t)| \leq Me^{kt}$$

for some constants M and k (this is called the growth condition), then the Laplace transform $\mathcal{L}\{f\}$ exists for all $s > k$. And if it exists, it is unique. What is more, if two functions have the same transform, they can only differ in a finite set of isolated points.

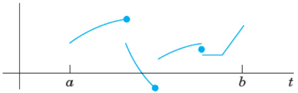


Fig. 115. Example of a piecewise continuous function $f(t)$. (The dots mark the function values at the jumps.)

Example

Functions like $\cosh(t) < e^t$, $t^n < n!e^t$ fulfill the growth condition, while e^{t^2} does not.

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 6, Section 1:

- 6.1.4
- 6.1.20
- 6.1.21
- 6.1.22
- 6.1.26
- 6.1.33
- 6.1.39

Outline

1. Laplace transforms
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Laplace transform of first derivative

Laplace transform of first derivative

If f is continuous for all $t \geq 0$, satisfies the growth condition and f' is piecewise continuous on every finite interval on the semiaxis $t \geq 0$, then

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

<u>Proof:</u>

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t) dt = e^{-st} f(t) \Big|_0^\infty + s \int_0^\infty e^{-st} f(t) dt = -f(0) + s\mathcal{L}\{f\}$$

Laplace transform of derivatives

Laplace transform of second derivative

If f and f' are continuous for all $t \geq 0$, satisfies the growth condition and f'' is piecewise continuous on every finite interval on the semiaxis $t \geq 0$, then

$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

Proof:
Applying the previous theorem

$$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\} - f'(0) = s(s\mathcal{L}\{f\} - f(0)) - f'(0) = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

Laplace transform of derivatives

If f, f', ... $f^{(n-1)}$ are continuous for all $t \geq 0$, satisfies the growth condition an $f^{(n)}$ is piecewise continuous on every finite interval on the semiaxis $t \geq 0$, then

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

Laplace transform of derivatives

Example

$$
\begin{aligned}
&f(t) = t\sin(\omega t) \Rightarrow f(0) = 0\\
&f'(t) = \sin(\omega t) + t\omega\cos(\omega t) \Rightarrow f'(0) = 0\\
&f''(t) = 2\omega\cos(\omega t) - \omega^2 t\sin(\omega t)
\end{aligned}
$$

Then,

$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0) = s^2\mathcal{L}\{f\}$$

But it is also

$$\mathcal{L}\{f''\} = \mathcal{L}\left\{2\omega\cos(\omega t) - \omega^2 t\sin(\omega t)\right\} = 2\omega\frac{s}{s^2+\omega^2} - \omega^2\mathcal{L}\{f\}$$

That is

$$s^2\mathcal{L}\{f\} = 2\omega\frac{s}{s^2+\omega^2} - \omega^2\mathcal{L}\{f\} \Rightarrow \boxed{\mathcal{L}\{f\} = \frac{2\omega s}{(s^2+\omega^2)^2}}$$

Laplace transform of the integral

Laplace transform of the integral

If f is piecewise continuous for all $t \geq 0$, and satisfies the growth condition, then for $s > k > 0$ and $t > 0$ we have

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

Proof: Since $f(t)$ is piecewise continuous, its integral is continuous and it fulfills the growth condition because

$$\left|\int_0^t f(\tau)d\tau\right| \leq \int_0^t |f(\tau)|d\tau \leq \int_0^t Me^{k\tau}d\tau = M\frac{1}{k}(e^{kt}-1) \leq \frac{M}{k}e^{kt}$$

Laplace transform of the integral

Laplace transform of the integral

Then, we can apply the first derivative theorem to the integral of f

$$\mathcal{L}\left\{\left(\int_0^t f(\tau)d\tau\right)'\right\} = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} - \int_0^0 f(\tau)d\tau$$

$$\mathcal{L}\{f(t)\} = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

Laplace transform of the integral

Example

We know that $\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin(\omega t)}{\omega}$, then

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^t \frac{\sin(\omega\tau)}{\omega} d\tau = -\frac{\cos(\omega\tau)}{\omega^2}\Big|_0^t = \frac{1}{\omega^2}(1-\cos(\omega t))$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} &= \int_0^t \frac{1}{\omega^2}(1-\cos(\omega\tau))d\tau = \left(\frac{\tau}{\omega^2} - \frac{\sin(\omega\tau)}{\omega^3}\right)\Big|_0^t \\ &= \frac{t}{\omega^2} - \frac{\sin(\omega t)}{\omega^3}\end{aligned}$$

Differential equations, Initial Value Problems

Differential equations, Initial Value Problems

Given the IVP

$$y'' + ay' + by = r(x) \quad y(0) = K_0, y'(0) = K_1$$

Let us transform the equation

$$(s^2 Y - sy(0) - y'(0)) + a(sY - y(0)) + bY = R$$

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R \Rightarrow \boxed{Y = \frac{(s + a)y(0) + y'(0) + R}{s^2 + as + b}}$$

Let us define the **transfer function** as

$$Q = \frac{1}{s^2 + as + b}$$

If the system is originally at rest ($y(0) = y'(0) = 0$), then $\boxed{Y = QR}$.
Now we only need to perform $\mathcal{L}^{-1}$ to recover y.

Differential equations, Initial Value Problems

Example

$$y'' - y = t \quad y(0) = 1, y'(0) = 1$$

Solution:

$$(s^2 Y - sy(0) - y'(0)) - Y = \frac{1}{s^2}$$

$$(s^2 Y - s - 1) - Y = \frac{1}{s^2}$$

$$(s^2 - 1)Y = \frac{1}{s^2} + s + 1$$

$$Y = \frac{1}{s^2(s^2 - 1)} + \frac{s+1}{s^2 - 1}$$

$$Y = \frac{1}{s-1} + \frac{1}{s^2 - 1} - \frac{1}{s^2}$$

$$\boxed{y = e^t + \sinh(t) - t}$$

Differential equations, Initial Value Problems

Example (continued)

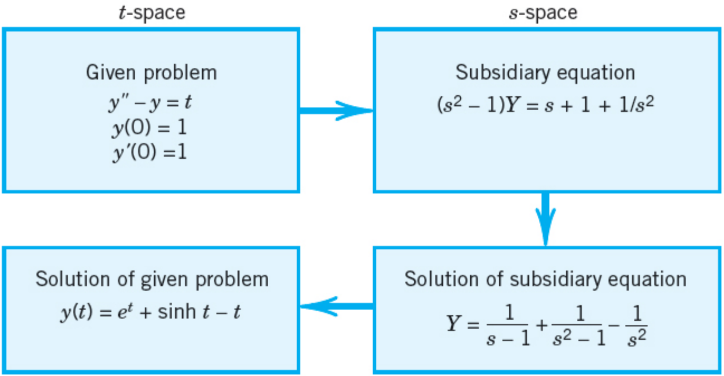

Fig. 116. Steps of the Laplace transform method

Differential equations, Initial Value Problems

Example

$$y'' + y = 2t \quad y\left(\frac{1}{4}\pi\right) = \frac{1}{2}\pi, y'\left(\frac{1}{4}\pi\right) = 2 - \sqrt{2}$$

Solution:

Let us shift the time so that $t = \frac{1}{4}\pi$ becomes $\tilde{t} = 0$. For doing so we do the change of variables

$$\tilde{t} = t - \frac{1}{4}\pi \Rightarrow t = \tilde{t} + \frac{1}{4}\pi$$

$$y' = \frac{dy}{dt} = \frac{dy}{d\tilde{t}}\frac{d\tilde{t}}{dt} = \frac{dy}{d\tilde{t}} = \tilde{y}'$$

$$y'' = \frac{d(y')}{dt} = \frac{d(\tilde{y}')}{dt} = \frac{d(\tilde{y}')}{d\tilde{t}}\frac{d\tilde{t}}{dt} = \tilde{y}''$$

So, the IVP becomes

$$\tilde{y}'' + \tilde{y} = 2\left(\tilde{t} + \frac{\pi}{4}\right) \quad \tilde{y}(0) = \frac{1}{2}\pi, \tilde{y}'(0) = 2 - \sqrt{2}$$

Differential equations, Initial Value Problems

Example (continued)

$$\tilde{y}'' + \tilde{y} = 2\left(\tilde{t} + \frac{\pi}{4}\right) \quad \tilde{y}(0) = \frac{1}{2}\pi, \tilde{y}'(0) = 2 - \sqrt{2}$$

$$(s^2\tilde{Y} - s\tilde{y}(0) - \tilde{y}'(0)) + \tilde{Y} = 2\left(\frac{1}{s^2} + \frac{\pi}{4}\frac{1}{s}\right)$$

$$(s^2 + 1)\tilde{Y} = s\frac{1}{2}\pi + (2 - \sqrt{2}) + \frac{2}{s^2} + \frac{\pi}{2s}$$

$$\tilde{Y} = \frac{\pi}{2}\frac{s}{s^2+1} + (2-\sqrt{2})\frac{1}{s^2+1} + 2\frac{1}{s^2(s^2+1)} + \frac{\pi}{2}\frac{1}{s(s^2+1)}$$

$$\tilde{y} = \frac{\pi}{2}\cos(\tilde{t}) + (2-\sqrt{2})\sin(\tilde{t}) + 2(\tilde{t} - \sin(\tilde{t})) + \frac{\pi}{2}(1 - \cos(\tilde{t}))$$

$$\tilde{y} = 2\tilde{t} + \frac{\pi}{2} - \sqrt{2}\sin(\tilde{t})$$

$$y = 2\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2} - \sqrt{2}\sin\left(t - \frac{\pi}{4}\right)$$

$$y = 2t - \sqrt{2}\left(\frac{1}{\sqrt{2}}(\sin(t) - \cos(t))\right) = 2t - \sin(t) + \cos(t)$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 6, Section 2:

- 6.2.3
- 6.2.12
- 6.2.16
- 6.2.24

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Unit step function (Heaviside function)

Unit step function (Heaviside function)

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

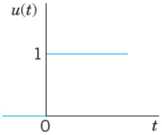


Fig. 118. Unit step function $u(t)$

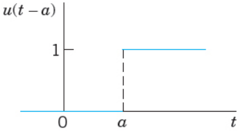


Fig. 119. Unit step function $u(t-a)$

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)dt = \int_a^\infty e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_a^\infty = \frac{e^{-as}}{s}$$

Unit step function (Heaviside function)

Example

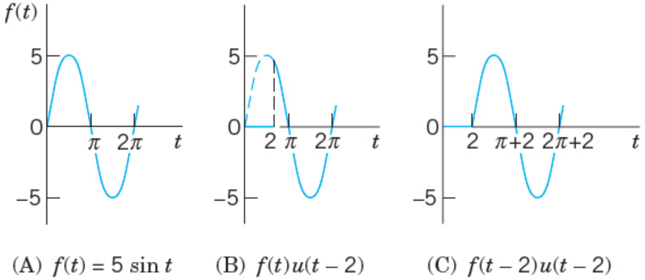
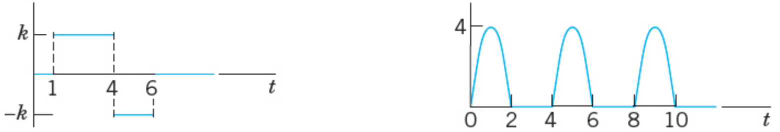


Fig. 120. Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.

Unit step function (Heaviside function)

Example



(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4\sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Fig. 121. Use of many unit step functions.

Unit step function (Heaviside function)

Time shifting theorem

If $f(t)$ has the transform $F(s)$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

Solution:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st}f(t-a)u(t-a)dt$$

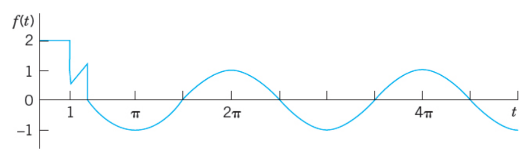
We do the change of variable $\tilde{t} = t - a$

$$\begin{aligned}
\mathcal{L}\{f(t-a)u(t-a)\} &= \int_{-a}^\infty e^{-s(\tilde{t}+a)}f(\tilde{t})u(\tilde{t})d\tilde{t} = e^{-as}\int_0^\infty e^{-s\tilde{t}}f(\tilde{t})d\tilde{t} = e^{-as}F(s) \\
\mathcal{L}\{f(t)u(t-a)\} &= \int_0^\infty e^{-st}f(t)u(t-a)dt = \int_{-a}^\infty e^{-s(\tilde{t}+a)}f(\tilde{t}+a)u(\tilde{t})d\tilde{t} \\
&= e^{-as}\int_0^\infty e^{-s\tilde{t}}f(\tilde{t}+a)d\tilde{t} = e^{-as}\mathcal{L}\{f(t+a)\}
\end{aligned}$$

Example

Compute the Laplace transform of the function

$$f(t) = \begin{cases} 2 & 0 < t < 1 \\ \frac{1}{2}t^2 & 1 < t < \frac{\pi}{2} \\ \cos(t) & t > \frac{\pi}{2} \end{cases}$$



Solution:

$$f(t) = [2(u(t) - u(t-1))] + \left[\frac{1}{2}t^2\left(u(t-1) - u\left(t - \frac{\pi}{2}\right)\right)\right] + \left[\cos(t)u\left(t - \frac{\pi}{2}\right)\right]$$

Unit step function (Heaviside function)

Example (continued)

$$f(t) = [2(u(t) - u(t-1))] + \left[\frac{1}{2}t^2\left(u(t-1) - u\left(t - \frac{\pi}{2}\right)\right)\right] + \left[\cos(t)u\left(t - \frac{\pi}{2}\right)\right]$$

$$F(s) = \left[2\left(\frac{1}{s} - \frac{e^{-s}}{s}\right)\right] + \left[\frac{1}{2}\frac{2}{s^3}\left(\frac{e^{-s}}{s} - \frac{e^{-\frac{\pi}{2}s}}{s}\right)\right] + \left[\frac{s}{s^2+1}\frac{e^{-\frac{\pi}{2}s}}{s}\right]$$

Example

$$\mathcal{L}\left\{\tfrac{1}{2}t^2u(t-1)\right\} \quad = \quad \tfrac{1}{2}e^{-s}\mathcal{L}\left\{(t+1)^2\right\} = \tfrac{e^{-s}}{2}\mathcal{L}\left\{t^2+2t+1\right\} = \tfrac{e^{-s}}{2}\left(\tfrac{2}{s^3} + 2\tfrac{1}{s^2} + \tfrac{1}{s}\right)$$

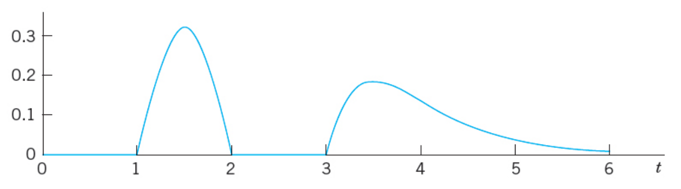
Unit step function (Heaviside function)

Example

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+\pi^2}+\frac{e^{-2s}}{s^2+\pi^2}+\frac{e^{-3s}}{(s+2)^2}\right\} = \frac{1}{\pi}\sin(\pi(t-1))u(t-1)+ \frac{1}{\pi}\sin(\pi(t-2))u(t-2)+(t-3)e^{-2(t-3)}u(t-3)$$

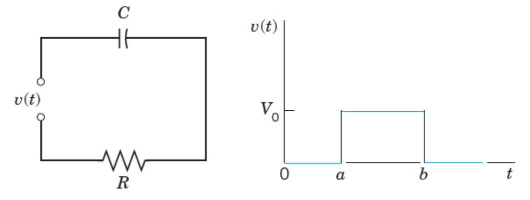
Note that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = tu(t)$$
$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t} = te^{-2t}u(t)$$
$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+2)^2}\right\} = (t-3)e^{-2(t-3)}u(t-3)$$



Unit step function (Heaviside function)

Example



Solution:

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = V_0(u(t-a) - u(t-b))$$

If we transform the equation we get

$$RI + \frac{1}{C}\frac{1}{s}I = V_0\left(\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right)$$

Unit step function (Heaviside function)

Example (continued)

$$RI + \frac{1}{C}\frac{1}{s}I = V_0\left(\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right)$$

$$\left(R + \frac{1}{Cs}\right)I = V_0\left(\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right)$$

$$I = \frac{Cs}{RCs+1}V_0\frac{1}{s}\left(e^{-sa} - e^{-sb}\right)$$

$$I = \frac{C}{RCs+1}V_0\left(e^{-sa} - e^{-sb}\right)$$

$$I = \frac{C}{s+\frac{1}{RC}}\frac{V_0}{RC}\left(e^{-sa} - e^{-sb}\right)$$

$$I = \frac{V_0}{R}\frac{1}{s+\frac{1}{RC}}\left(e^{-sa} - e^{-sb}\right)$$

Unit step function (Heaviside function)

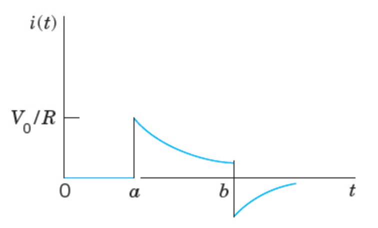
Example (continued)

$$\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\frac{V_0}{R}\right\} = \frac{V_0}{R}e^{-\frac{t}{RC}}$$

Now

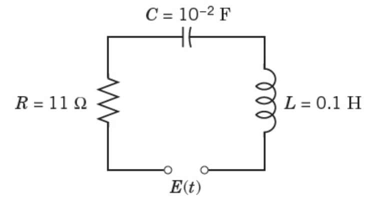
$$I = \frac{V_0}{R}\frac{1}{s+\frac{1}{RC}}e^{-sa} - \frac{V_0}{R}\frac{1}{s+\frac{1}{RC}}e^{-sb}$$

$$i = \frac{V_0}{R}\left(e^{-\frac{t-a}{RC}}u(t-a) - e^{-\frac{t-b}{RC}}u(t-b)\right)$$



Unit step function (Heaviside function)

Example



with $E(t) = 100\sin(400t)(u(t) - u(t-2\pi))$.
Solution:

$$Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$

If we transform the equation we get

$$LsI + RI + \frac{1}{Cs}I = 100\frac{400}{s^2+400^2}\left(\frac{1}{s} - \frac{e^{-s2\pi}}{s}\right)$$

Unit step function (Heaviside function)

Example (continued)

$$LsI + RI + \frac{1}{Cs}I = 100\frac{400}{s^2+400^2}\left(\frac{1}{s} - \frac{e^{-s2\pi}}{s}\right)$$

$$\left(Ls + R + \frac{1}{Cs}\right)I = 100\frac{400}{s^2+400^2}\left(\frac{1}{s} - \frac{e^{-s2\pi}}{s}\right)$$

$$I = \frac{Cs}{LCs^2+RCs+1}100\frac{400}{s^2+400^2}\left(\frac{1}{s} - \frac{e^{-s2\pi}}{s}\right)$$

$$I = \frac{0.01}{0.001s^2+0.11s+1}\frac{4\cdot 10^4}{s^2+400^2}(1-e^{-s2\pi})$$

$$I = \frac{1}{0.001s^2+0.11s+1}\frac{4\cdot 10^2}{s^2+400^2}(1-e^{-s2\pi})$$

$$I = \frac{1}{0.001}\frac{1}{s^2+110s+1000}\frac{4\cdot 10^2}{s^2+400^2}(1-e^{-s2\pi})$$

$$I = \frac{1}{(s-10)(s-100)}\frac{4\cdot 10^5}{s^2+400^2}(1-e^{-s2\pi})$$

Unit step function (Heaviside function)

Example (continued)

$$I = \frac{1}{(s-10)(s-100)}\frac{4\cdot 10^5}{s^2+400^2}(1-e^{-s2\pi})$$

Let's factorize

$$\frac{1}{(s-10)(s-100)}\frac{4\cdot 10^5}{s^2+400^2} = \frac{A}{s-10}+\frac{B}{s-100}+\frac{Cs+D}{s^2+400^2}$$

$$\begin{aligned}
4\cdot 10^5 &= A(s-100)(s^2+400^2)+B(s-10)(s^2+400^2)+(Cs+D)(s-10)(s-100)\\
&= (A+B+C)s^3+(-100A-10B-110C+D)s^2+\\
&\quad (1.6\cdot 10^5 A+1.6\cdot 10^5 B+10^3 C-110D)s+\\
&\quad (-1.6\cdot 10^7 A-1.6\cdot 10^5 B+10^3 D)
\end{aligned}$$

$$A=-0.0251, B=0.0237, C=0.0015, D=-2.1155$$

$$\frac{1}{(s-10)(s-100)}\frac{4\cdot 10^5}{s^2+400^2} = -\frac{0.0251}{s-10}+\frac{0.0237}{s-100}+\frac{0.0015s}{s^2+400^2}-\frac{2.1155}{s^2+400^2}$$

$$\frac{1}{(s-10)(s-100)}\frac{4\cdot 10^5}{s^2+400^2} = -\frac{0.0251}{s-10}+\frac{0.0237}{s-100}+\frac{3.66\cdot 10^{-6}400s}{s^2+400^2}-\frac{2.1155}{s^2+400^2}$$

Unit step function (Heaviside function)

Example (continued)

$$\frac{1}{(s-10)(s-100)}\frac{4\cdot 10^5}{s^2+400^2} = -\frac{0.0251}{s-10}+\frac{0.0237}{s-100}+\frac{3.66\cdot 10^{-6}400s}{s^2+400^2}-\frac{2.1155}{s^2+400^2}$$

$$i_1 = \mathcal{L}^{-1}\left\{\tfrac{1}{(s-10)(s-100)}\tfrac{4\cdot 10^5}{s^2+400^2}\right\} = -0.251e^{-10t}+0.0237e^{-100t}+ 3.66\cdot 10^{-6}\sin(400t)-\tfrac{2.1155}{\cos}(400t)$$

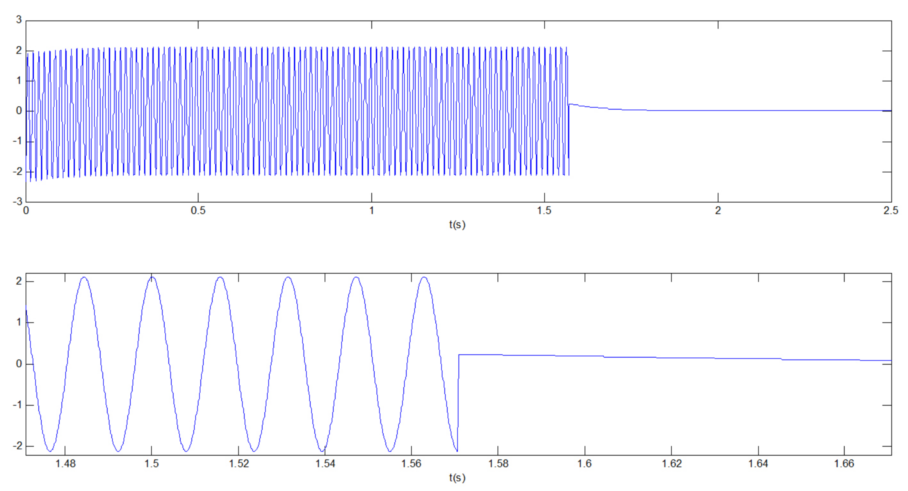
But the intensity was

$$I = \frac{1}{(s-10)(s-100)}\frac{4\cdot 10^5}{s^2+400^2}(1-e^{-s2\pi})$$

$$i = i_1(t) - i_1\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right)$$

Unit step function (Heaviside function)

Example (continued)



Unit step function (Heaviside function)

Example (continued)

RLC.m

```
function RLC
    t=0:0.0001:2.5;
    ii1=i1(t);
    ii2=i1(t-pi/2);
    subplot(211)
    plot(t,ii1-ii2)
    xlabel('t(s)')
    subplot(212)
    plot(t,ii1-ii2)
    axis([pi/2-0.1 pi/2+0.1 -2.2 2.2])
    xlabel('t(s)')
end

function I=i1(t)
 I=-0.251*exp(-10*t)+0.0237*exp(-100*t)+3.66e-6*sin(400*t)-2.1155*cos(400*t);
 I(t<0)=0;
end
```

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 6, Section 3:

- 6.3.8
- 6.3.9
- 6.3.17
- 6.3.19

Outline

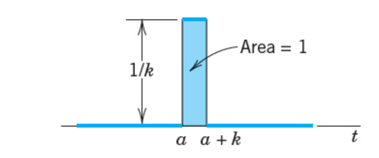
1. Laplace transforms
    - Laplace transform. Linearity. s-shifting theorem
    - Transforms of derivatives and integrals
    - Unit step function (Heaviside function), t-shifting theorem
    - Short impulses. Dirac's δ function. Partial fractions
    - Convolution. Integral equations
    - Differentiation and integration of transforms. ODEs with variable coefficients
    - Systems of ODEs

Dirac's δ function

Dirac's δ function

They represent short impulses and they can be seen as the limit of the function

$$f_k(t-a) = \begin{cases} 1/k & a \leq t \leq a+k \\ 0 & \text{otherwise} \end{cases}$$



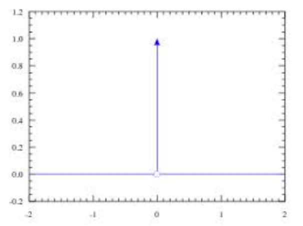
It must be noted that for any k the area under f_k is always 1

$$I_k = \int_0^{\infty} f_k(t-a)dt = \int_a^{a+k} \frac{1}{k} dt = \frac{1}{k}\ t\Big|_a^{a+k} = \frac{1}{k} k = 1$$

Dirac's δ function

Dirac's δ function (continued)

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a)$$



In fact δ is not a function (it is zero everywhere except at a point whose value is ∞ and its integral is 1). Technically, it is a distribution or a measure. It fulfills the sifiting property

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

Dirac's δ function

Dirac's δ function (continued)

Let us calculate the Laplace transform of f_k

$$\begin{aligned}\mathcal{L}\{f_k(t-a)\} &= \mathcal{L}\left\{\frac{1}{k}(u(t-a)-u(t-(a+k)))\right\} = \frac{1}{k}\left(\frac{e^{-sa}}{s} - \frac{e^{-s(a+k)}}{s}\right) \\ &= e^{-as}\frac{1-e^{-sk}}{sk}\end{aligned}$$

Now

$$\begin{aligned}\mathcal{L}\{\delta(t-a)\} &= \lim_{k\to 0}\mathcal{L}\{f_k(t-a)\} = \lim_{k\to 0} e^{-as}\frac{1-e^{-sk}}{sk} \\ &= e^{-as}\lim_{k\to 0}\frac{1-(1-sk)}{sk} = e^{-as}\lim_{k\to 0}\frac{sk}{sk} = e^{-as}\end{aligned}$$

where we have used L'Hopital to avoid the 0/0 ratio.

Dirac's δ function

Example: Hammerblow on a mass-spring system

Which is the impulse response of a mass-spring system to a blow at $t = 1$?

$$y'' + 3y' + 2y = \delta(t-1)$$

<u>Solution:</u>
We now transform the equation taking into account that the system is originally at rest ($y(0) = y'(0) = 0$):

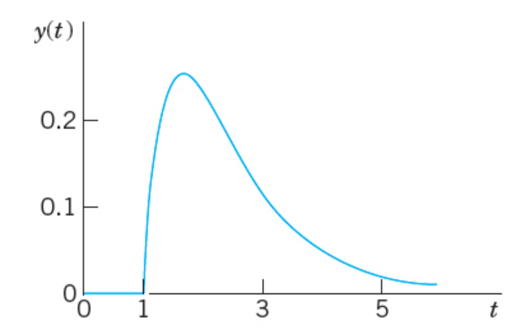
$$s^2 Y + 3sY + 2Y = e^{-s}$$

$$Y = \frac{1}{s^2+3s+2} e^{-s} = \frac{1}{(s+1)(s+2)} e^{-s} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right) e^{-s}$$

$$y = \left(e^{-(t-1)} - e^{-2(t-1)}\right) u(t-1)$$

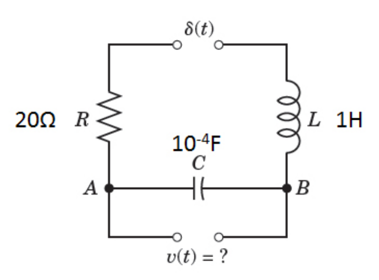
Example: Hammerblow on a mass-spring system (continued)

$$y = \left(e^{-(t-1)} - e^{-2(t-1)}\right) u(t-1)$$



Dirac's δ function

Example: RLC circuit



Solution:

$$Li' + Ri + \frac{q}{C} = \delta(t)$$

Where q is the charge at the capacitor.

Dirac's δ function

Example: RLC circuit (continued)

This charge and the intensity are related by

$$i = q'$$

and the relationship between this charge and the voltage at the capacitor is

$$v = \frac{q}{C}$$

So the ODE becomes

$$LCv'' + RCv' + v = \delta(t)$$

$$v'' + \frac{R}{L}v' + \frac{1}{LC}v = \frac{1}{LC}\delta(t)$$

Since the system is originally at rest, we have

$$s^2 V + \frac{R}{L}sV + \frac{1}{LC}V = \frac{1}{LC}$$

Example: RLC circuit (continued)

$$s^2 V + \frac{R}{L} sV + \frac{1}{LC} V = \frac{1}{LC}$$

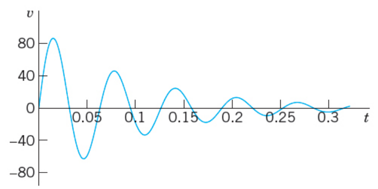
$$(s^2 + \frac{20}{1} s + \frac{1}{1 \cdot 10^{-4}}) V = \frac{1}{1 \cdot 10^{-4}}$$

$$(s^2 + 20s + 10^4) V = 10^4$$

$$((s+10)^2 + 9900) V = 10^4$$

$$V = \frac{10^4}{(s+10)^2 + 9900} = \frac{10000}{\sqrt{9900}} \frac{\sqrt{9900}}{(s+10)^2 + 9900}$$

$$v = 100.5 \sin(99.50t) e^{-10t}$$



Partial fractions

Partial fractions

Many response transforms are in the form of a ration of polynomials

$$Y(S) = \frac{B(s)}{A(s)}$$

Depending on the factorization of $A(s)$ we have a fraction expansion using

$\frac{1}{s-a}$	$\frac{A_1}{s-a}$	$A_1 e^{-at}$
$\frac{1}{(s-a)^2}$	$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2}$	$(A_1 + A_2 t)e^{-at}$
$\frac{1}{(s-a)^3}$	$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3}$	$(A_1 + A_2 t + A_3 \frac{1}{2} t^2)e^{-at}$
$\frac{1}{(s-s_0)(s-s_0^*)}$ $s_0 = \alpha + i\beta$	$\frac{A_1 s + A_2}{(s-\alpha)^2 + \beta^2}$ $\frac{A_1(s-\alpha) + A_2 + A_1\alpha}{(s-\alpha)^2 + \beta^2}$	$e^{-\alpha t}\left(A_1 \cos(\beta t) + \frac{A_2 + A_1\alpha}{\beta} \sin(\beta t)\right)$

Partial fractions

Example

$$y'' + 2y' + 2y = 10\sin(2t)(u(t) - u(t-\pi)) \quad y(0) = 1, y'(0) = -5$$

Solution:

$$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10\frac{2}{s^2+4}(1 - e^{-s\pi})$$

$$(s^2 + 2s + 2)Y = s - 3 + 10\frac{2}{s^2+4}(1 - e^{-s\pi})$$

$$Y = \frac{s-3}{s^2+2s+2} + \frac{20}{(s^2+2s+2)(s^2+4)}(1 - e^{-s\pi})$$

$$\mathcal{L}^{-1}\left\{\frac{s-3}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1)-4}{(s+1)^2+1}\right\} = e^{-t}(\cos(t) - 4\sin(t))$$

Partial fractions

Example (continued)

$$\frac{20}{(s^2+2s+2)(s^2+4)} = \frac{As+B}{(s+1)^2+1} + \frac{Ms+N}{s^2+4}$$

$$20 = (As+B)(s^2+4) + (Ms+N)(s^2+2s+2)$$

$$20 = (A+M)s^3 + (2A+B+N)s^2 + (2A+2B+4M)s + (2B+4N)$$

$$A=2, B=6, M=-2, N=-2$$

$$\begin{aligned}\tfrac{1}{(s^2+2s+2)(s^2+4)} &= \tfrac{2s+6}{(s+1)^2+1} - 2\tfrac{s+1}{s^2+4} \\ &= \tfrac{2(s+1)+4}{(s+1)^2+1} - 2\tfrac{s}{s^2+4} - 2\tfrac{1}{s^2+4} \\ &= 2\tfrac{(s+1)}{(s+1)^2+1} + 4\tfrac{1}{(s+1)^2+1} - 2\tfrac{s}{s^2+4} - 2\tfrac{1}{s^2+4}\end{aligned}$$

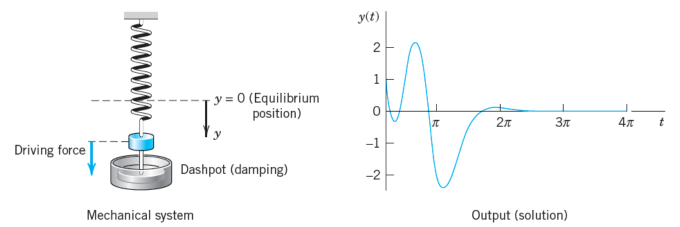
$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+2)(s^2+4)}\right\} = e^{-t}(2\cos(t) + 4\sin(t)) - 2\cos(2t) - 2\sin(2t)$$

Example (continued)

Finally,

$$Y = \frac{s-3}{s^2+2s+2} + \frac{20}{(s^2+2s+2)(s^2+4)}(1-e^{-s\pi})$$

$$\begin{aligned} y &= e^{-t}(\cos(t) - 4\sin(t)) + \\ &\quad e^{-t}(2\cos(t) + 4\sin(t)) - 2\cos(2t) - 2\sin(2t) + \\ &\quad e^{-(t-\frac{\pi}{2})}(2\cos(t-\tfrac{\pi}{2}) + 4\sin(t-\tfrac{\pi}{2})) - 2\cos(2(t-\tfrac{\pi}{2})) - 2\sin(2(t-\tfrac{\pi}{2})) \\ &= e^{-t}3\cos(t) - 2\cos(2t) - 2\sin(2t) + \\ &\quad e^{-(t-\frac{\pi}{2})}(2\sin(t) - 4\cos(t)) + 2\cos(2t) + 2\sin(2t) \end{aligned}$$



Exercises

From Kreyszig (10th ed.), Chapter 6, Section 4:

13. PROJECT. Heaviside Formulas. (a) Show that for a simple root a and fraction $A/(s-a)$ in $F(s)/G(s)$ we have the *Heaviside formula*

$$A = \lim_{s \to a} \frac{(s-a)F(s)}{G(s)}.$$

(b) Similarly, show that for a root a of order m and fractions in

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{s-a} + \text{further fractions}$$

we have the *Heaviside formulas* for the first coefficient

$$A_m = \lim_{s \to a} \frac{(s-a)^m F(s)}{G(s)}$$

and for the other coefficients

$$A_k = \frac{1}{(m-k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s-a)^m F(s)}{G(s)} \right], \quad k = 1, \cdots, m-1.$$

Exercises

From Kreyszig (10th ed.), Chapter 6, Section 4:

14. TEAM PROJECT. Laplace Transform of Periodic Functions

(a) Theorem. *The Laplace transform of a piecewise continuous function $f(t)$ with period p is*

$$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt \qquad (s > 0). \tag{11}$$

Prove this theorem. *Hint:* Write $\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$. Set $t = (n-1)p$ in the nth integral. Take out $e^{-(n-1)p}$ from under the integral sign. Use the sum formula for the geometric series.

(b) Half-wave rectifier. Using (11), show that the half-wave rectification of $\sin \omega t$ in Fig. 137 has the Laplace transform

$$\mathcal{L}(f) = \frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})} = \frac{\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}.$$

(A *half-wave rectifier* clips the negative portions of the curve. A *full-wave rectifier* converts them to positive; see Fig. 138.)

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 6, Section 4:

(c) Full-wave rectifier. Show that the Laplace transform of the full-wave rectification of $\sin \omega t$ is

$$\frac{\omega}{s^2 + \omega^2} \coth \frac{\pi s}{2\omega}.$$

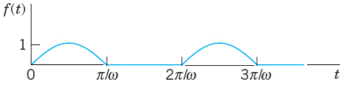


Fig. 137. Half-wave rectification

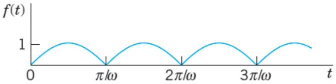


Fig. 138. Full-wave rectification

(d) Saw-tooth wave. Find the Laplace transform of the saw-tooth wave in Fig. 139.

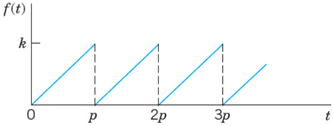


Fig. 139. Saw-tooth wave

Outline

1. Laplace transforms
   - Laplace transform. Linearity. s-shifting theorem
   - Transforms of derivatives and integrals
   - Unit step function (Heaviside function), t-shifting theorem
   - Short impulses. Dirac's δ function. Partial fractions
   - Convolution. Integral equations
   - Differentiation and integration of transforms. ODEs with variable coefficients
   - Systems of ODEs

Convolution

Convolution

If two functions f and g satisfy the assumptions of the existence theorem so that their transforms F and G exist, then

$$\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = F(s)G(s)$$

<u>Proof</u>

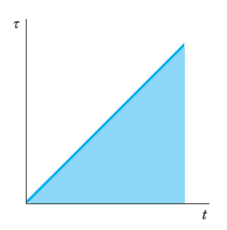
$$\begin{aligned}
F(s)G(s) &= \left(\int_0^\infty e^{-s\sigma}f(\sigma)d\sigma\right)\left(\int_0^\infty e^{-s\tau}g(\tau)d\tau\right) \\
&= \int_0^\infty\left(\int_0^\infty e^{-s(\sigma+\tau)}f(\sigma)d\sigma\right)g(\tau)d\tau \quad [t=\sigma+\tau] \\
&= \int_0^\infty\left(\int_\tau^\infty e^{-st}f(t-\tau)dt\right)g(\tau)d\tau \\
&= \int_0^\infty\left(\int_\tau^\infty e^{-st}f(t-\tau)g(\tau)dt\right)d\tau
\end{aligned}$$

Convolution

Convolution (continued)

$$F(s)G(s) \quad = \quad \int_0^\infty \left(\int_\tau^\infty e^{-st} f(t-\tau) g(\tau) dt \right) d\tau$$

We are integrating in the shaded region



Now we change the order of integration

$$F(s)G(s) \quad = \quad \int_0^\infty \left(\int_0^t e^{-st} f(t-\tau) g(\tau) d\tau \right) dt$$

Convolution

Convolution (continued)

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \left(\int_0^t e^{-st} f(t-\tau) g(\tau) d\tau \right) dt \\ &= \int_0^\infty e^{-st} \left(\int_0^t f(t-\tau) g(\tau) d\tau \right) dt \\ &= \mathcal{L}\left\{ \int_0^t f(t-\tau) g(\tau) d\tau \right\} \\ &= \mathcal{L}\left\{ f(t) \star g(t) \right\} \end{aligned}$$

Example

$$\begin{aligned} \mathcal{L}^{-1}\left\{ \frac{1}{s(s-a)} \right\} &= \mathcal{L}^{-1}\left\{ \frac{1}{s} \frac{1}{s-a} \right\} = 1 \star e^{at} \\ &= \int_0^t e^{a\tau} 1 d\tau = \frac{1}{a}\ e^{a\tau}\Big|_0^t = \frac{1}{a}(e^{at} - 1) \end{aligned}$$

Convolution

Example

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin(\omega t) \star \frac{1}{\omega}\sin(\omega t) \\
&= \frac{1}{\omega^2}\int_0^t \sin(\omega\tau)\sin(\omega(t-\tau))\tau \\
&= \frac{1}{2\omega^2}\int_0^t \left(-\cos(\omega t) + \cos(2\omega\tau - \omega t)\right) d\tau \\
&= \frac{1}{2\omega^2}\left(-\tau\cos(\omega t) - \frac{\sin(\omega(t-2\tau))}{2\omega}\right)_0^t \\
&= \frac{1}{2\omega^2}\left[\left(-t\cos(\omega t) + \frac{\sin(\omega t)}{2\omega}\right) - \left(-\frac{\sin(\omega t)}{2\omega}\right)\right] \\
&= \frac{1}{2\omega^2}\left[-t\cos(\omega t) + \frac{\sin(\omega t)}{\omega}\right]
\end{aligned}$$

Convolution

Example

$$y'' + \omega_0^2 y = K\sin(\omega_0 t) \quad y(0) = y'(0) = 0$$

Solution:

$$s^2 Y + \omega_0^2 Y = K\frac{\omega_0}{s^2+\omega_0^2}$$

$$Y = K\omega_0 \frac{1}{(s^2+\omega_0^2)^2}$$

This is a case of resonance (the input signal is of the same frequency as the natural frequency of the system). This implies a multiple complex root of the Laplace transform. The solution is given by the previous example

$$y = K\omega_0 \frac{1}{2\omega_0^2}\left[-t\cos(\omega_0 t) + \frac{\sin(\omega_0 t)}{\omega_0}\right] = \frac{K}{2\omega_0}\left[-t\cos(\omega_0 t) + \frac{\sin(\omega_0 t)}{\omega_0}\right]$$

Convolution

Properties

- Commutative: $f \star g = g \star f$
- Distributive: $f \star (g_1 + g_2) = f \star g_1 + f \star g_2$
- Associative: $(f \star g) \star h = f \star (g \star h)$
- Zero element: $f \star 0 = 0$
- ~~Identity element~~: $f \star 1 \neq f$
- ~~Positivity~~: $f \star f \not\geq 0$

Convolution

Response of systems at rest

$$y'' + ay' + by = r(t) \quad y(0) = y'(0) = 0$$

$$s^2 Y + asY + bY = R$$

$$Y = \frac{1}{s^2 + as + b} R = QR$$

$$\boxed{y = q \star r}$$

being q the inverse Laplace transform of Q. Q is called the transfer function of the system.

Convolution

Example

$$y'' + 3y' + 2y = u(t) - u(t-1) \quad y(0) = y'(0) = 0$$

<u>Solution:</u>

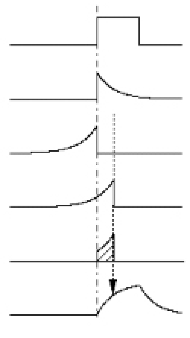
$$Q = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$q = e^{-t} - e^{-2t}$$

$$y = (e^{-t} - e^{-2t}) \star (u(t) - u(t-1))$$

If $0 < t < 1$

$$\begin{aligned} y &= \int_0^t (e^{-(t-\tau)} - e^{-2(t-\tau)}) d\tau \\ &= \left(e^{-(t-\tau)} - \tfrac{1}{2} e^{-2(t-\tau)}\right)_0^t \\ &= \tfrac{1}{2} - e^{-t} + \tfrac{1}{2} e^{-2t} \end{aligned}$$

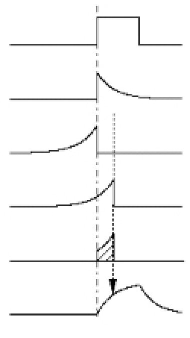


Convolution

Example (continued)

If $1 \leq t$

$$\begin{aligned} y &= \int_0^1 \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau \\ &= \left(e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}\right)_0^1 \\ &= \left(e^{-(t-1)} - \frac{1}{2}e^{-2(t-1)}\right) - \left(e^{-t} - \frac{1}{2}e^{-2t}\right) \end{aligned}$$



Convolution

Example: Volterra integral equation of the second kind

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)d\tau = t$$

Solution:

$$y - y \star (\sin(t)) = t$$

$$Y - Y\frac{1}{s^2+1} = \frac{1}{s^2}$$

$$Y\frac{s^2}{s^2+1} = \frac{1}{s^2}$$

$$Y = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$y = t + \frac{t^3}{6}$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 6, Section 5:

- 6.5.6
- 6.5.12
- 6.5.17

Outline

1. Laplace transforms
   - Laplace transform. Linearity. s-shifting theorem
   - Transforms of derivatives and integrals
   - Unit step function (Heaviside function), t-shifting theorem
   - Short impulses. Dirac's δ function. Partial fractions
   - Convolution. Integral equations
   - Differentiation and integration of transforms. ODEs with variable coefficients
   - Systems of ODEs

Differentiation of transforms

Differentiation of transforms

If f satisfies the assumptions of the existence theorem, then

$$\mathcal{L}^{-1}\left\{F'(s)\right\} = -tf(t)$$

Proof

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t) dt \\
F'(s) &= \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t) dt\right) \\
&= \int_0^\infty \frac{d}{ds}(e^{-st}) f(t) dt \\
&= \int_0^\infty \left(-t(e^{-st}) f(t)\right) dt \\
&= \int_0^\infty \left(e^{-st}(-tf(t))\right) dt \\
&= \mathcal{L}\{-tf(t)\}
\end{aligned}$$

Differentiation of transforms

Example

$$\mathcal{L}\left\{\sin(\omega t)\right\} = \frac{\omega}{s^2+\omega^2} = F(s)$$

$$\mathcal{L}\left\{-t\sin(\omega t)\right\} = -\frac{2s\omega}{(s^2+\omega^2)^2} = F'(s)$$

$$\mathcal{L}\left\{t\sin(\omega t)\right\} = \frac{2s\omega}{(s^2+\omega^2)^2} = -F'(s)$$

Integration of transforms

If f satisfies the assumptions of the existence theorem and the limit of $\frac{f(t)}{t}$ exists when t approaches 0 from the right, then

$$\mathcal{L}^{-1}\left\{\int_s^{\infty} F(\tilde{s})d\tilde{s}\right\} = \frac{f(t)}{t}$$

<u>Proof</u>

$$\begin{aligned}
\int_s^{\infty} F(\tilde{s})d\tilde{s} &= \int_s^{\infty}\left(\int_0^{\infty} e^{-\tilde{s}t}f(t)dt\right)d\tilde{s} = \int_0^{\infty}\left(\int_s^{\infty} e^{-\tilde{s}t}f(t)d\tilde{s}\right)dt \\
&= \int_0^{\infty} f(t)\left(\int_s^{\infty} e^{-\tilde{s}t}d\tilde{s}\right)dt = \int_0^{\infty} f(t)\left(-\tfrac{1}{t}e^{-\tilde{s}t}\Big|_s^{\infty}\right)dt \\
&= \int_0^{\infty} f(t)\tfrac{1}{t}e^{-st}dt = \int_0^{\infty} e^{-st}\tfrac{f(t)}{t}dt \\
&= \mathcal{L}\left\{\tfrac{f(t)}{t}\right\}
\end{aligned}$$

Integration of transforms

Example

Calculate $\mathcal{L}^{-1}\left\{\log \frac{s^2+\omega^2}{s^2}\right\}$

<u>Solution:</u> Let us define

$$G = F' = \frac{d}{ds}(\log(s^2+\omega^2) - \log(s^2)) = \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$

$$g = 2\cos(\omega t) - 2 = 2(\cos(\omega t) - 1)$$

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{-\int_s^\infty G(s)ds\right\} = -\frac{g(t)}{t} = \frac{2}{t}(1-\cos(\omega t))$$

ODEs with variable coefficients

ODEs with variable coefficients

Given that

$$\mathcal{L}\{y'\} = sY - y(0)$$
$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$$

Then

$$\mathcal{L}\{ty'\} = -\frac{d}{ds}(sY - y(0)) = -Y - sY'$$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}\left(s^2Y - sy(0) - y'(0)\right) = -2sY - s^2Y' + y(0)$$

So we can transform an ODE in y into another ODE in Y' that may be simpler. This only works with some ODEs (e.g. it does not work with t^2y' (it would give terms with Y'').

ODEs with variable coefficients

Example: Laguerre polynomials

$$ty'' + (1-t)y' + ny = 0$$

Solution:

$$(-2sY - s^2Y' + y(0)) + (sY - y(0)) - (-Y - sY') + nY = 0$$

$$(s - s^2)Y' + (n+1-s)Y = 0$$

$$\frac{Y'}{Y} = -\frac{n+1-s}{s-s^2} = \frac{n}{s-1} - \frac{n+1}{s}$$

$$\frac{dY}{Y} = \left(\frac{n}{s-1} - \frac{n+1}{s}\right) ds$$

$$\log(Y) = n\log(s-1) - (n+1)\log(s)$$

$$Y = \frac{(s-1)^n}{s^{n+1}}$$

ODEs with variable coefficients

Example: Laguerre polynomials (continued)

$$Y = \frac{(s-1)^n}{s^{n+1}}$$

We note that

$$\mathcal{L}\{t^n e^{-t}\} = \frac{n!}{(s+1)^{n+1}}$$

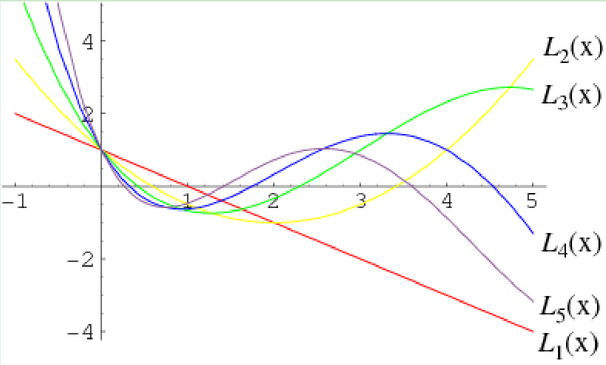
$$\mathcal{L}^{-1}\left\{\frac{n!s^n}{(s+1)^{n+1}}\right\} = \frac{d^n(t^n e^{-t})}{dt^n}$$

$$\mathcal{L}^{-1}\left\{\frac{s^n}{(s+1)^{n+1}}\right\} = \frac{1}{n!}\frac{d^n(t^n e^{-t})}{dt^n}$$

$$\mathcal{L}^{-1}\left\{\frac{(s-1)^n}{s^{n+1}}\right\} = e^t\frac{1}{n!}\frac{d^n(t^n e^{-t})}{dt^n} = \frac{1}{n!}(D-1)^n t^n$$

ODEs with variable coefficients

Example: Laguerre polynomials (continued)



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 6, Section 6:

- 6.6.2
- 6.6.20

Outline

1. Laplace transforms
   - Laplace transform. Linearity. s-shifting theorem
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   - Systems of ODEs

Systems of ODEs

Systems of ODEs

$$y_1' = a_{11}y_1 + a_{12}y_2 + g_1$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + g_2$$

If we transform it

$$sY_1 - y_1(0) = a_{11}Y_1 + a_{12}Y_2 + G_1$$
$$sY_2 - y_2(0) = a_{21}Y_1 + a_{22}Y_2 + G_2$$

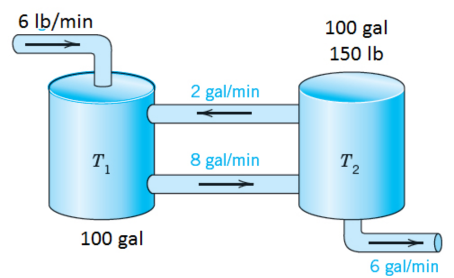
Equivalently

$$s\mathbf{Y} - \mathbf{y}(0) = A\mathbf{Y} + \mathbf{G}$$

$$(A - sI)\mathbf{Y} = -\mathbf{y}(0) - \mathbf{G}$$

Systems of ODEs

Example



Solution:

$$y_1' = -\frac{y_1}{100}8 + \frac{y_2}{100}2 + 6$$
$$y_2' = \frac{y_1}{100}8 - \frac{y_2}{100}2$$

$$s\mathbf{Y} - \begin{pmatrix} 0 \\ 150 \end{pmatrix} = \begin{pmatrix} -0.08 & 0.02 \\ 0.08 & -0.02 \end{pmatrix} \mathbf{Y} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

Systems of ODEs

Example (continued)

$$sY - \begin{pmatrix} 0 \\ 150 \end{pmatrix} = \begin{pmatrix} -0.08 & 0.02 \\ 0.08 & -0.02 \end{pmatrix} Y + \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.08 - s & 0.02 \\ 0.08 & -0.02 - s \end{pmatrix} Y = \begin{pmatrix} -6 \\ -150 \end{pmatrix}$$

$$Y_1 = \frac{\begin{vmatrix} -6 & 0.02 \\ -150 & -0.02 - s \end{vmatrix}}{\begin{vmatrix} -0.08 - s & 0.02 \\ 0.08 & -0.02 - s \end{vmatrix}} = \frac{9s + 0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04}$$

$$Y_2 = \frac{\begin{vmatrix} -0.08 - s & -6 \\ 0.08 & -150 \end{vmatrix}}{\begin{vmatrix} -0.08 - s & 0.02 \\ 0.08 & -0.02 - s \end{vmatrix}} = \frac{150s^2 + 12s + 0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}$$

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$

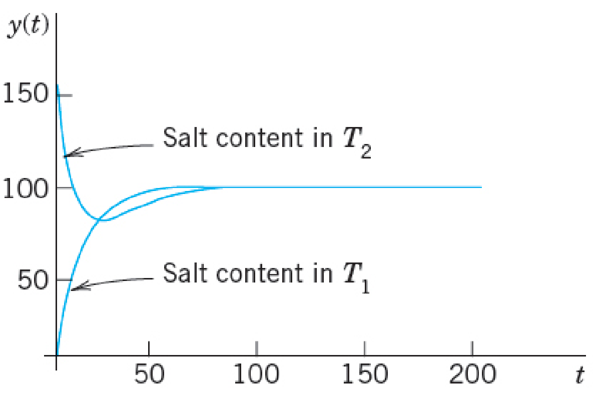
$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}$$

Systems of ODEs

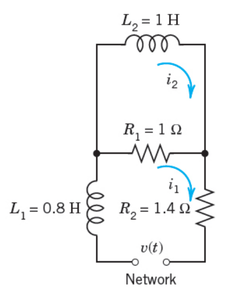
Example (continued)

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$

$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}$$



Example



$$0.8i_1' + 1(i_1 - i_2) + 1.4i_1 = 100(u(t) - u(t-0.5))$$
$$1i_2' + 1(i_2 - i_1) = 0$$

Transforming it

$$0.8sI_1 + (I_1 - I_2) + 1.4I_1 = 100\left(\frac{1}{s} - \frac{e^{-0.5s}}{s}\right)$$
$$sI_2 + (I_2 - I_1) = 0$$

And solving

$$v(t) = 100(u(t) - u(t-0.5))$$
$$i_1(0) = i_2(0) = 0$$

$$I_1 = \left(\frac{500}{7s} - \frac{125}{3(s+0.5)} - \frac{625}{21(s+3.5)}\right)(1 - e^{-0.5s})$$
$$I_2 = \left(\frac{500}{7s} - \frac{250}{3(s+0.5)} + \frac{250}{21(s+3.5)}\right)(1 - e^{-0.5s})$$

Example (continued)

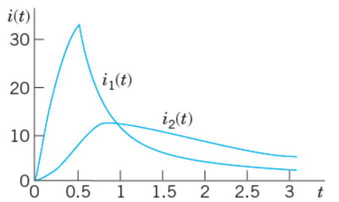
$$I_1 = \left(\frac{500}{7s} - \frac{125}{3(s+0.5)} - \frac{625}{21(s+3.5)}\right)(1 - e^{-0.5s})$$
$$I_2 = \left(\frac{500}{7s} - \frac{250}{3(s+0.5)} + \frac{250}{21(s+3.5)}\right)(1 - e^{-0.5s})$$

Let us define

$$\tilde{I}_1 = \frac{500}{7s} - \frac{125}{3(s+0.5)} - \frac{625}{21(s+3.5)} \Rightarrow \tilde{i}_1 = \frac{500}{7} - \frac{125}{3}e^{-0.5t} + \frac{625}{21}e^{-3.5t}$$
$$\tilde{I}_2 = \frac{500}{7s} - \frac{250}{3(s+0.5)} + \frac{250}{21(s+3.5)} \Rightarrow \tilde{i}_2 = \frac{500}{7} - \frac{250}{3}e^{-0.5t} + \frac{250}{21}e^{-3.5t}$$

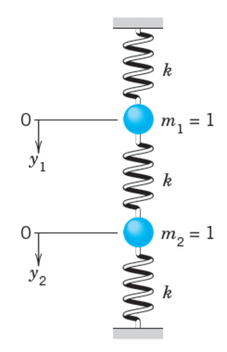
Finally

$$i_1(t) = \tilde{i}_1(t) - \tilde{i}_1(t - 0.5)$$
$$i_2(t) = \tilde{i}_2(t) - \tilde{i}_2(t - 0.5)$$



Currents

Example



$$y_1(0) = y_2(0) = 1$$
$$y_1'(0) = -y_2'(0) = \sqrt{3k}$$

$$y_1'' = -ky_1 + k(y_2 - y_1)$$
$$y_1'' = -k(y_2 - y_1) - ky_2$$

Transforming it

$$s^2 Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1)$$
$$s^2 Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2$$

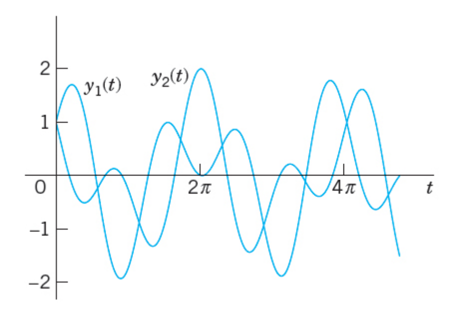
And solving

$$Y_1 = \frac{s}{s^2+k} + \frac{\sqrt{3k}}{s^2+3k}$$
$$Y_2 = \frac{s}{s^2+k} - \frac{\sqrt{3k}}{s^2+3k}$$

$$y_1 = \cos(\sqrt{k}t) + \sin(\sqrt{3k}t)$$
$$y_2 = \cos(\sqrt{k}t) - \sin(\sqrt{3k}t)$$

Example (continued)

$$y_1 = \cos(\sqrt{k}t) + \sin(\sqrt{3k}t)$$
$$y_2 = \cos(\sqrt{k}t) - \sin(\sqrt{3k}t)$$



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 6, Section 7:

- 6.7.2

Systems of ODEs

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	*s*-Shifting (First Shifting Theorem)	6.1
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function	6.2

Systems of ODEs

$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$ $= \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$	*t*-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	f Periodic with Period p	6.4 Project 16

Systems of ODEs

	$F(s) = \mathscr{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\dfrac{1}{s-a}$	e^{at}	6.1
8	$\dfrac{1}{(s-a)^2}$	te^{at}	
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!} t^{n-1} e^{at}$	
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)} t^{k-1} e^{at}$	
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{a-b}(e^{at} - e^{bt})$	
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{a-b}(ae^{at} - be^{bt})$	

Systems of ODEs

13	$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\sin\omega t$	6.1
14	$\dfrac{s}{s^2+\omega^2}$	$\cos\omega t$	
15	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$	
16	$\dfrac{s}{s^2-a^2}$	$\cosh at$	
17	$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega}e^{at}\sinh\omega t$	
18	$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$	
19	$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1-\cos\omega t)$	6.2
20	$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}(\omega t-\sin\omega t)$	

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
21	$\dfrac{1}{(s^2 + \omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	6.6
22	$\dfrac{s}{(s^2 + \omega^2)^2}$	$\dfrac{t}{2\omega} \sin \omega t$	6.6
23	$\dfrac{s^2}{(s^2 + \omega^2)^2}$	$\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	6.6
24	$\dfrac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2 - a^2}(\cos at - \cos bt)$	
25	$\dfrac{1}{s^4 + 4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\dfrac{s}{s^4 + 4k^4}$	$\dfrac{1}{2k^2} \sin kt \sinh kt$	
27	$\dfrac{1}{s^4 - k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$	
28	$\dfrac{s}{s^4 - k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$	

Systems of ODEs

29	$\sqrt{s-a} - \sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$	
30	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\dfrac{a-b}{2}t\right)$	I 5.5
31	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	J 5.4
32	$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}} e^{at}(1+2at)$	
33	$\dfrac{1}{(s^2-a^2)^k} \quad (k>0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	I 5.5
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\dfrac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$	J 5.4
37	$\dfrac{1}{\sqrt{s}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
38	$\dfrac{1}{s^{3/2}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k>0)$	$\dfrac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$	

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
40	$\frac{1}{s}\ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	γ 5.5
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \frac{s^2+\omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \frac{s^2-a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$	
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t}\sin \omega t$	
45	$\frac{1}{s}\operatorname{arccot} s$	$\operatorname{Si}(t)$	App. A3.1

Outline

1. Laplace transforms
   - Laplace transform. Linearity. s-shifting theorem
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   - Short impulses. Dirac's δ function. Partial fractions
   - Convolution. Integral equations
   - Differentiation and integration of transforms. ODEs with variable coefficients
   - Systems of ODEs