Chapter 7. Fourier analysis

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Biomedical Engineering

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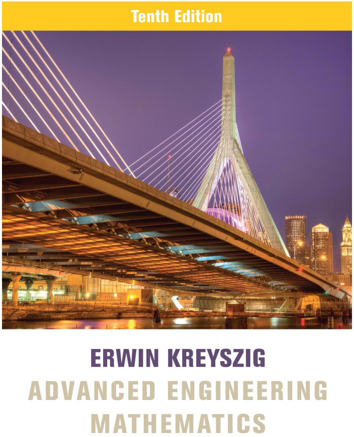
Outline

1. Fourier Analysis
   - Fourier series
   - Arbitrary period. Even and odd functions. Half-range expansions
   - Forced oscillations
   - Approximation by trigonometric polynomials
   - Sturm-Liouville problems. Orthogonal functions
   - Orthogonal series. Generalized Fourier series
   - Fourier integral
   - Fourier transform

References



E. Kreyszig. Advanced Engineering Mathematics. John Wiley & sons. Chapter 11.

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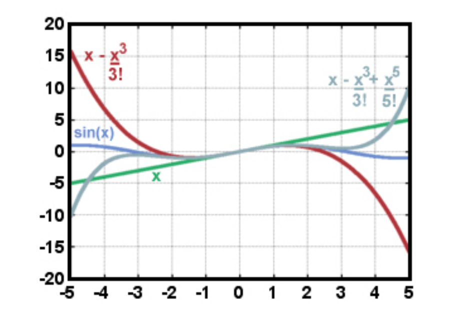
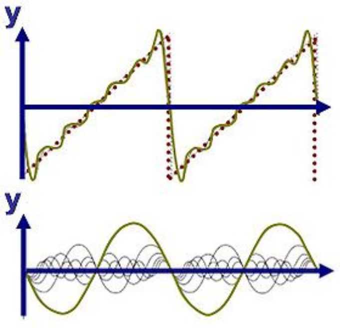
Fourier series

Fourier series





$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \qquad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Periodic functions

Periodic functions

A function is periodic with period p if

$$f(x) = f(x + p)$$

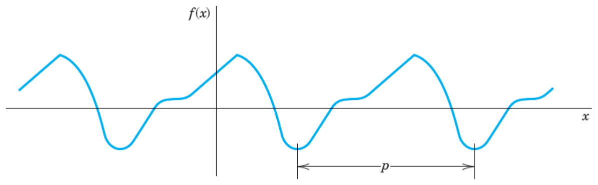


Fig. 258. Periodic function of period p

If it is periodic with period p, it is also periodic with period $2p$, $3p$, ... The smallest period is called the **fundamental period**.

Periodic functions

The basis functions of the Fourier series (1, cos(x), sin(x), cos($2x$), sin($2x$), ...) are periodic with period 2π

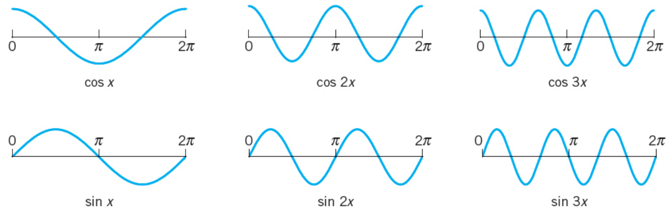


Fig. 259. Cosine and sine functions having the period 2π (the first few members of the trigonometric system (3), except for the constant 1)

If the series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

it is also periodic with period 2π.

Fourier series

Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Example

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases} \qquad f(x+2\pi) = f(x)$$

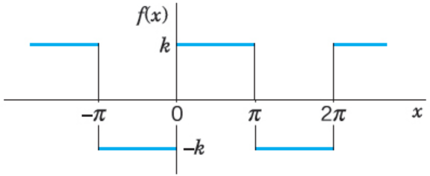


Fig. 260. Given function $f(x)$ (Periodic reactangular wave)

$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-k)dx + \int_{0}^{\pi} kdx\right) = \frac{1}{2\pi}(-k\pi + k\pi) = 0$$

Fourier series

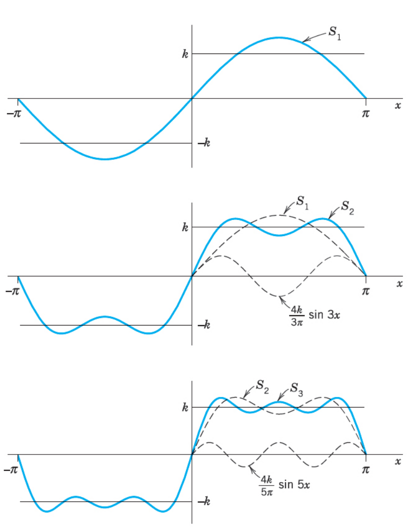
Example (continued)

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-k)\cos(nx)dx + \int_{0}^{\pi}k\cos(nx)dx\right)\\
&= \frac{1}{\pi}\left(\left[-k\frac{\sin(nx)}{n}\right]_{-\pi}^{0} + \left[k\frac{\sin(nx)}{n}\right]_{0}^{\pi}\right) = 0\\
b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-k)\sin(nx)dx + \int_{0}^{\pi}k\sin(nx)dx\right)\\
&= \frac{1}{\pi}\left(\left[k\frac{\cos(nx)}{n}\right]_{-\pi}^{0} - \left[k\frac{\cos(nx)}{n}\right]_{0}^{\pi}\right)\\
&= \frac{k}{n\pi}\left(\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0)\right)\\
&= \frac{k}{n\pi}\left(2 - \cos(-n\pi)\right)\\
&= \frac{2k}{n\pi}\left(1 - (-1)^n\right)
\end{aligned}
$$

$$
b_1 = \frac{4k}{\pi} \quad b_2 = 0 \quad b_3 = \frac{4k}{3\pi} \quad b_4 = 0 \quad b_5 = \frac{4k}{5\pi} \quad \dots
$$

Example (continued)

$$f(x) = \sum_{n=1}^{\infty} \frac{2k}{n\pi} (1 - (-1)^n) \sin(nx)$$
$$S_1 = \frac{4k}{\pi} \sin(x)$$
$$S_2 = \frac{4k}{\pi} \sin(x) + \frac{4k}{3\pi} \sin(3x)$$
$$S_3 = \frac{4k}{\pi} \sin(x) + \frac{4k}{3\pi} \sin(3x) + \frac{4k}{5\pi} \sin(5x)$$



Fourier series

Fourier basis is orthogonal

$$
\begin{aligned}
\langle 1, \cos(nx) \rangle &= \int_{-\pi}^{\pi} \cos(nx) dx = 0 \\
\langle 1, \sin(nx) \rangle &= \int_{-\pi}^{\pi} \sin(nx) dx = 0 \\
\langle \cos(nx), \cos(mx) \rangle &= \int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = 0 \quad (n \neq m) \\
\langle \sin(nx), \sin(mx) \rangle &= \int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx = 0 \quad (n \neq m) \\
\langle \cos(nx), \sin(mx) \rangle &= \int_{-\pi}^{\pi} \cos(nx) \sin(mx) dx = 0
\end{aligned}
$$

But they are not orthonormal

$$
\begin{aligned}
\langle 1, 1 \rangle &= \|1\|^2 = 2\pi \\
\langle \cos(nx), \cos(nx) \rangle &= \|\cos(nx)\|^2 = \pi \\
\langle \sin(nx), \sin(nx) \rangle &= \|\sin(nx)\|^2 = \pi
\end{aligned}
$$

Orthogonal decomposition theorem (Algebra)

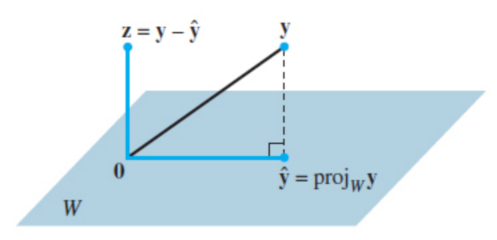
Orthogonal decomposition theorem

Let W be a vector subspace of a vector space V. Then, any vector $\mathbf{y} \in V$ can be written uniquely as

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

with $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$. In fact, if $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_p\}$ is an orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 + ... + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\|\mathbf{u}_p\|^2}\mathbf{u}_p$$



Fourier series

Fourier series as an orthogonal projection

$$\hat{\mathbf{y}} = \frac{\langle \mathbf{y}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\langle \mathbf{y}, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 + \ldots + \frac{\langle \mathbf{y}, \mathbf{u}_p \rangle}{\|\mathbf{u}_p\|^2}\mathbf{u}_p$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{\langle f(x), 1 \rangle}{\|1\|^2} \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{\langle f(x), \cos(x) \rangle}{\|\cos(x)\|^2} \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{\langle f(x), \sin(x) \rangle}{\|\sin(x)\|^2} \end{aligned}$$

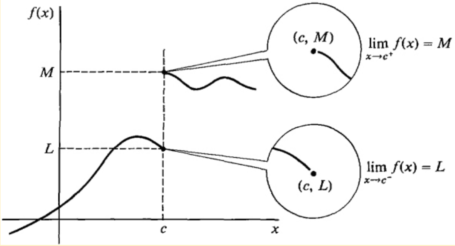
Fourier series

Class of functions that can be represented

Let f be periodic with period 2π and piecewise continuous in the interval $[-\pi, \pi]$. Furthermore, let f have a left-hand derivative and a right-hand derivative at each point of that interval. Then, the Fourier series converges. Its sum is $f(x)$ except at points x_0 where $f(x)$ is discontinuous. There the sum o the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Left- and right-hand limits and derivatives

$$
\begin{aligned}
f(x_0 - 0) &= \lim_{\substack{h\to 0\\ h>0}} f(x_0 - h)\\
f(x_0 + 0) &= \lim_{\substack{h\to 0\\ h>0}} f(x_0 + h)\\
f'(x_0 - 0) &= \lim_{\substack{h\to 0\\ h>0}} \frac{f(x_0-h)-f(x_0-0)}{-h}\\
f'(x_0 + 0) &= \lim_{\substack{h\to 0\\ h>0}} \frac{f(x_0+h)-f(x_0+0)}{-h}
\end{aligned}
$$



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 11, Section 1:

- 11.1.14
- 11.1.15

Outline

Arbitrary period

Arbitrary period

Assume f is periodic with period $p = 2L$. We do the change of variable

$$v = \frac{2\pi}{p}x = \frac{\pi}{L}x \Rightarrow x = \frac{L}{\pi}v$$

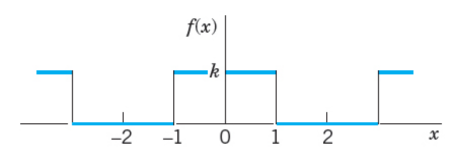
Then $f(v)$ becomes of period 2π

$$\begin{aligned}
f(x) &= f\left(\tfrac{L}{\pi}v\right) = a_0 + \textstyle\sum_{n=1}^{\infty} a_n \cos(nv) + b_n \sin(nv) \\
&= \boxed{a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)} \\
a_0 &= \tfrac{1}{2L} \int_{-L}^{L} f(x)dx \\
a_n &= \tfrac{1}{L} \int_{-L}^{L} f(x) \cos\left(\tfrac{n\pi}{L}x\right) dx \\
b_n &= \tfrac{1}{L} \int_{-L}^{L} f(x) \sin\left(\tfrac{n\pi}{L}x\right) dx
\end{aligned}$$

Arbitrary period

Example

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases} \qquad p = 2L = 4, L = 2$$



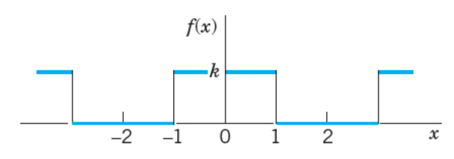
$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{4}\int_{-1}^{1} kdx = \frac{k}{2} \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{1}{4}\int_{-1}^{1} k\cos\left(\frac{n\pi}{2}x\right)dx = \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right) \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx = 0 \end{aligned}$$

Arbitrary period

Example (continued)

$$
\begin{aligned}
f(x) &= \frac{k}{2} + \sum_{n=1}^{\infty} \left(\frac{2k}{n\pi} \sin\left(\frac{n\pi}{2}\right)\right) \cos\left(\frac{n\pi}{L} x\right) \\
&= \frac{k}{2} + \frac{2k}{\pi}\left(\cos \frac{\pi}{2} x - \frac{1}{3} \cos \frac{3\pi}{2} x + \frac{1}{5} \cos \frac{5\pi}{2} x - \ldots\right)
\end{aligned}
$$

Since the function is even, it has a cosine only series.



Arbitrary period

Example: Change of scale

$$f(x) = \begin{cases} -k & -2 < x < 0 \\ k & 0 < x < 2 \end{cases} \qquad f(x+4) = f(x)$$

Solution:
We know from a previous example the Fourier series for a similar function with period 2π

$$\begin{aligned} g(v) &= \begin{cases} -k & -\pi < v < 0 \\ k & 0 < v < \pi \end{cases} \qquad g(v+2\pi) = g(v) \\ &= \sum_{n=1}^{\infty} \tfrac{2k}{n\pi}\left(1 - (-1)^n\right)\sin(nv) \\ &= \tfrac{4k}{\pi}\sin(v) + \tfrac{4k}{3\pi}\sin(3v) + \tfrac{4k}{5\pi}\sin(5v) + \ldots \end{aligned}$$

If we do the change of variable $v = \frac{\pi}{2}x$, then

$$f(x) = g\left(\frac{\pi}{2}x\right) = \frac{4k}{\pi}\sin\left(\frac{\pi}{2}x\right) + \frac{4k}{3\pi}\sin\left(\frac{3\pi}{2}x\right) + \frac{4k}{5\pi}\sin\left(\frac{5\pi}{2}x\right) + \ldots$$

Linearity.

The Fourier series is linear

Let FS be an operator that assigns to each function, its Fourier series. Then

$$FS(f_1 + f_2) = FS(f_1) + FS(f_2)$$

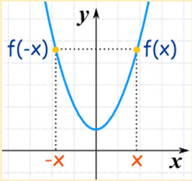
$$FS(cf) = cFS(f)$$

Even and odd functions.

Even functions

If the function is even, then the Fourier series simplifies to

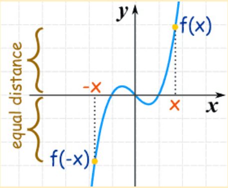
$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) \\
a_0 &= \frac{1}{L}\int_0^L f(x)dx \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx
\end{aligned}
$$



Odd functions

If the function is odd, then the Fourier series simplifies to

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \\
a_0 &= 0 \\
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx
\end{aligned}
$$



Half-range expansion

Half-range expansion

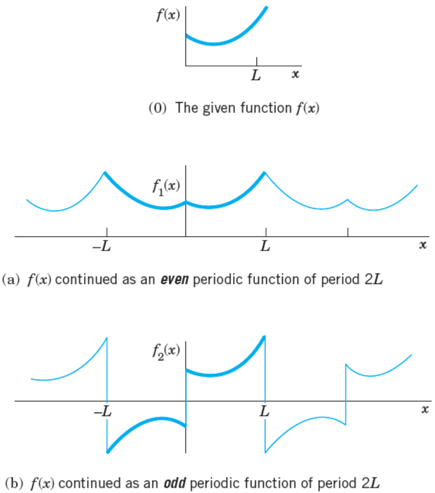
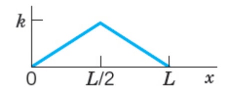


Fig. 270. Even and odd extensions of period $2L$

If only half of the range ($[0, L]$) is of interest, we may extend the function in an odd or even way, and then use the simplified Fourier series expresion for odd or even functions.

Half-range expansion

Example



$$f(x) = \begin{cases} \frac{2k}{L}x & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \frac{L}{2} < x < L \end{cases}$$

Solution: Even extension

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)dx = \frac{k}{2} \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{4k}{n^2\pi^2}\left(2\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) - 1\right) \\ \tilde{f}_e(x) &= \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\left(\frac{2\pi}{L}x\right) + \frac{1}{6^2}\cos\left(\frac{6\pi}{L}x\right) + \dots\right) \end{aligned}$$

Odd extension

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \\ \tilde{f}_o(x) &= \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\left(\frac{\pi}{L}x\right) - \frac{1}{3^2}\sin\left(\frac{3\pi}{L}x\right) + \frac{1}{5^2}\sin\left(\frac{5\pi}{L}x\right)\dots\right) \end{aligned}$$

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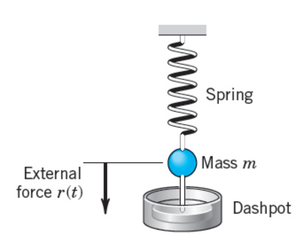
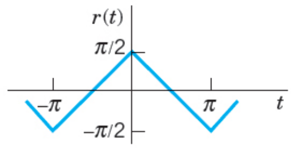
   - Fourier transform

Forced oscillations

Example block: undamped spring





$$my'' + cy' + ky = r(t)$$

$$m = 1 \text{ (g)},\ c = 0.05 \text{ (g/s)},\ k = 25 \text{ (g/s}^2\text{)}$$

$$y'' + 0.05y' + 25y = r(t)$$

$$r(t) = \begin{cases} t + \frac{\pi}{2} & -\pi < t < 0 \\ -t + \frac{\pi}{2} & 0 < t < \pi \end{cases}$$

Forced oscillations

Example block: undamped spring (continued)

Solution:

We expand the driving force in its Fourier series

$$r(t) = \frac{4}{\pi}\left(\cos(t) + \frac{1}{3^2}\cos(3t) + \frac{1}{5^2}\cos(5t) + ...\right)$$

Then we consider the ODE

$$y'' + 0.05y' + 25y = \frac{4}{\pi n^2}\cos(nt)$$

Its steady state solution is

$$y_n = \frac{4(25-n^2)}{n^2\pi D_n}\cos(nt) + \frac{0.2}{n\pi D_n}\sin(nt)$$

with $D_n = (25-n^2)^2 + (0.05n)^2$. We are interested in the steady state solution because $r(t)$ is periodic.

Forced oscillations

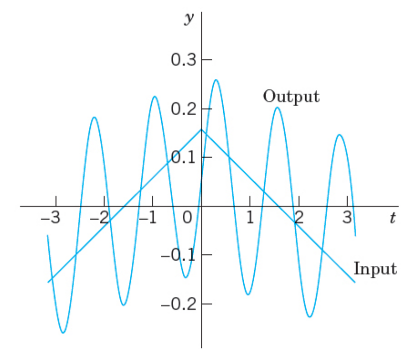
Example block: undamped spring (continued)

$$r(t) = \frac{4}{\pi}\left(\cos(t) + \frac{1}{3^2}\cos(3t) + \frac{1}{5^2}\cos(5t) + \dots\right)$$

$$y_n = \frac{4(25 - n^2)}{n^2 \pi D_n}\cos(nt) + \frac{0.2}{n\pi D_n}\sin(nt)$$

The steady state solution is

$$\boxed{y = y_1 + y_3 + y_5 + \dots}$$



Exercises

Exercises

From Kreyszig (10th ed.), Chapter 11, Section 3:

- 11.3.4

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Approximation by trigonometric polynomials

Let us consider a function f and its Fourier series

$$f(x) = a_0 + \sum_{0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Let us find the best trigonometric approximation of degree N

$$F(x) = A_0 + \sum_{n=0}^{N} A_n \cos(nx) + B_n \sin(nx)$$

such that the approximation error is minimized

$$E = \int_{-\pi}^{\pi} (f - F)^2 dx$$

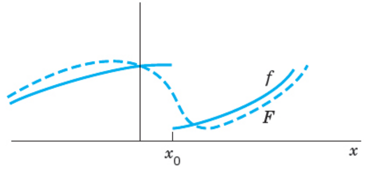


Fig. 278. Error of approximation

Approximation by trigonometric polynomials

Approximation by trigonometric polynomials

$$E = \int_{-\pi}^{\pi} f^2 dx + \int_{-\pi}^{\pi} F^2 dx - 2 \int_{-\pi}^{\pi} fF dx$$

Approximation by trigonometric polynomials

Let us calculate

$$
\begin{aligned}
\int_{-\pi}^{\pi} F^2 dx &= \int_{-\pi}^{\pi} \left(A_0 + \sum_{n=0}^{N} A_n \cos(nx) + B_n \sin(nx) \right)^2 dx \\
&= \int_{-\pi}^{\pi} A_0^2 dx + \sum_{n=0}^{N} \int_{-\pi}^{\pi} A_0 A_n \cos(nx) dx + \sum_{n=0}^{N} \int_{-\pi}^{\pi} A_0 B_n \sin(nx) dx \\
&\quad + \sum_{n=0}^{N} \int_{-\pi}^{\pi} A_0 A_n \cos(nx) dx + \sum_{n=0}^{N} \sum_{m=0}^{N} \int_{-\pi}^{\pi} A_n A_m \cos(nx) \cos(mx) dx \\
&\quad + \sum_{n=0}^{N} \sum_{m=0}^{N} \int_{-\pi}^{\pi} A_n B_m \cos(nx) \sin(mx) dx \\
&\quad + \sum_{n=0}^{N} \int_{-\pi}^{\pi} A_0 B_n \sin(nx) dx + \sum_{n=0}^{N} \sum_{m=0}^{N} \int_{-\pi}^{\pi} B_n A_m \sin(nx) \cos(mx) dx \\
&\quad + \sum_{n=0}^{N} \sum_{m=0}^{N} \int_{-\pi}^{\pi} B_n B_m \sin(nx) \sin(mx) dx \\
&= 2\pi A_0^2 + \sum_{n=0}^{N} (\pi A_n^2 + \pi B_n^2)
\end{aligned}
$$

Approximation by trigonometric polynomials

Similarly

$$\begin{aligned}\int_{-\pi}^{\pi} fF dx &= \int_{-\pi}^{\pi}\left(a_0 + \sum_{n=0}^{\infty} a_n\cos(nx) + b_n\sin(nx)\right)\left(A_0 + \sum_{n=0}^{N} A_n\cos(nx) + B_n\sin(nx)\right)\\ &= 2\pi a_0 A_0 + \sum_{n=0}^{N}(\pi a_n A_n + \pi b_n B_n)\end{aligned}$$

So that

$$\begin{aligned}E &= \int_{-\pi}^{\pi} f^2 dx + \pi\left(2A_0^2 + \sum_{n=0}^{N}(A_n^2 + B_n^2)\right) - 2\pi\left(2a_0A_0 + \sum_{n=0}^{N}(a_nA_n + b_nB_n)\right)\\ &= \int_{-\pi}^{\pi} f^2 dx + \pi\left(2(A_0^2 - 2a_0A_0) + \sum_{n=0}^{N}((A_n^2 - 2a_nA_n) + (B_n^2 - 2b_nB_n))\right)\end{aligned}$$

Approximation by trigonometric polynomials

$$E = \int_{-\pi}^{\pi} f^2 dx + \pi \left(2(A_0^2 - 2a_0 A_0) + \sum_{n=0}^{N} ((A_n^2 - 2a_n A_n) + (B_n^2 - 2b_n B_n)) \right)$$

Now, we optimize E with respect to the A_0, A_n and B_n coefficients

$$\begin{aligned} \frac{\partial E}{\partial A_0} &= 2\pi(2A_0 - 2a_0) = 0 \Rightarrow A_0 = a_0 \\ \frac{\partial E}{\partial A_n} &= \pi(2A_n - 2a_n) = 0 \Rightarrow A_n = a_n \\ \frac{\partial E}{\partial B_n} &= \pi(2B_n - 2b_n) = 0 \Rightarrow B_n = b_n \end{aligned}$$

That is, the partial sum of order N of the Fourier series is **the best trigonometric approximation** of order N to f, and the error becomes

$$E = \int_{-\pi}^{\pi} f^2 dx - \pi \left(2a_0^2 + \sum_{n=0}^{N} (a_n^2 + b_n^2) \right)$$

Approximation by trigonometric polynomials

Approximation by trigonometric polynomials

$$E = \int_{-\pi}^{\pi} f^2 dx - \pi \left(2a_0^2 + \sum_{n=0}^{N} (a_n^2 + b_n^2) \right)$$

Since $E \geq 0$ we have (**Bessel's inequality**)

$$2a_0^2 + \sum_{n=0}^{N} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 dx$$

In fact, if there is a Fourier series representation of f, there is no approximation error (**Parseval's identity**)

$$\boxed{2a_0^2 + \sum_{n=0}^{N} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 dx}$$

Example

$$f(x) = x + \pi \qquad -\pi < x < \pi$$

$$F(x) = \pi + 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \ldots + \frac{(-1)^{N+1}}{N}\sin(Nx)\right)$$

$$E = \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi\left(2\pi^2 + \sum_{n=0}^{N}\left(\frac{2(-1)^{N+1}}{N}\right)^2\right)$$

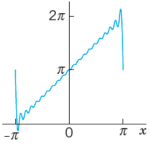


Fig. 279. F with $N = 20$ in Example 1

Numeric values are:

N	E^*	N	E^*	N	E^*	N	E^*
1	8.1045	6	1.9295	20	0.6129	70	0.1782
2	4.9629	7	1.6730	30	0.4120	80	0.1561
3	3.5666	8	1.4767	40	0.3103	90	0.1389
4	2.7812	9	1.3216	50	0.2488	100	0.1250
5	2.2786	10	1.1959	60	0.2077	1000	0.0126

Outline

Sturm-Liouville problems

Sturm-Liouville problems

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0$$

y is a solution in an interval $[a, b]$ satisfying boundary conditions of the form

$$k_1 y(a) + k_2 y'(a) = 0$$

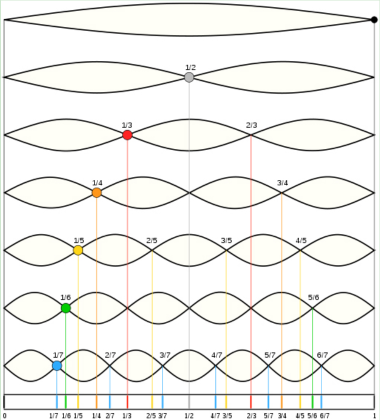
$$l_1 y(b) + l_2 y'(b) = 0$$

$y = 0$ is a trivial solution, the rest of solutions are called **eigenfunctions** and they are associated to specific values of λ (their **eigenvalue**). If p, q, r, and p' are real-valued and continuous in $[a, b]$, and r is positive throughout $[a, b]$ (or negative), then all eigenvalues of the Sturm-Liouville problem are real.

Sturm-Liouville problems

Example

Vibrating string



$$y'' + \lambda y = 0 \quad y(0) = 0, y(\pi) = 0$$

<u>Solution:</u>
We can reformulate the problem as a Sturm-Liouville problem as

$$(1y')' + (0 + \lambda 1)y = 0$$

$$1y(0) + 0y'(0) = 0$$

$$1y(\pi) + 0y'(\pi) = 0$$

Sturm-Liouville problems

Example (continued)

If $\lambda = -\nu^2$ is negative, the general solution is

$$y = c_1 e^{\nu x} + c_2 e^{-\nu x}$$

From the boundary conditions, we get $c_1 = c_2 = 0$.
If $\lambda = 0$, the general solution is

$$y = c_1 + c_2 x$$

and again from the boundary conditions $c_1 = c_2 = 0$.
Finally, if $\lambda = \nu^2$ is positive, the general solution is

$$y = c_1 \cos(\nu x) + c_2 \sin(\nu x)$$

From the boundary conditions, $c_1 = 0$ and

$$y(\pi) = c_2 \sin(\nu x) = 0 \Rightarrow \nu = \pm 1, \pm 2, ...$$

Example (continued)

$$y(\pi) = c_2 \sin(\nu x) = 0 \Rightarrow \nu = \pm 1, \pm 2, ...$$

That is, the functions

$$y_\nu = \sin(\nu x) \quad \nu = \sqrt{\lambda} = 1, 2, 3, ...$$

are eigenfunctions of the ODE and their associated eigenvalue is $\lambda = \nu^2$.

Orthgonal functions

Orthogonality

Let us define the **inner product** of two functions y_m and y_n with respect to the weight function $r(x)$ in the interval $[a, b]$ as

$$\boxed{\langle f, g \rangle_r = \int_a^b r(x) f(x) g(x) dx}$$

The **norm** of a function is defined as $\|f\| = \sqrt{\langle f, f \rangle_r}$.
Two functions are **orthogonal** if $\langle f, g \rangle_r = 0$.
A set of functions $\{y_1, y_2, ...\}$ is **orthonormal** if

$$\langle y_m, y_n \rangle_r = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Sturm-Liouville problems

Example (continued)

The set of functions $y_\nu = \sin(\nu x) \quad \nu = \sqrt{\lambda} = 1, 2, 3, ...\}$ are orthogonal $(\nu_1 \neq \nu_2)$ in the interval $[0, \pi]$

$$\begin{aligned}
\langle y_{\nu_1}, y_{\nu_2} \rangle &= \int_0^\pi \sin(\nu_1 x)\sin(\nu_2 x) dx \\
&= \frac{1}{2}\int_0^\pi \cos((\nu_1 - \nu_2)x) dx - \frac{1}{2}\int_0^\pi \cos((\nu_1 + \nu_2)x) dx \\
&= \frac{1}{2}\left[\frac{\sin((\nu_1 - \nu_2)x)}{\nu_1 - \nu_2}\right]_0^\pi - \frac{1}{2}\left[\frac{\sin((\nu_1 + \nu_2)x)}{\nu_1 + \nu_2}\right]_0^\pi \\
&= 0
\end{aligned}$$

but they are not orthonormal because

$$\|y_\nu\|^2 = \int_0^\pi \sin^2(\nu x) dx = \int_0^\pi \left(\frac{1}{2} - \frac{\cos(2\nu x)}{2}\right) dx = \frac{\pi}{2}$$

The set of functions $y_\nu = \sqrt{\frac{2}{\pi}}\sin(\nu x) \quad \nu = \sqrt{\lambda} = 1, 2, 3, ...\}$ is orthonormal.

Sturm-Liouville problems

Orthogonality of eigenfunctions

If p, q, r and p' are real-valued and continuous in the interval $[a, b]$ and $r > 0$. Let the function y_m and y_n be eigenfunctions associated to different eigenvalues λ_m and λ_n, then

$$\langle y_m, y_n \rangle_r = 0$$

Boundary conditions

Boundary conditions

Mixed Dirichlet-Neumann conditions:

$$k_1 y(a) + k_2 y'(a) = \alpha$$
$$l_1 y(b) + l_2 y'(b) = \beta$$

if $\alpha = \beta = 0$, the boundary conditions are said to be **homogeneous**. If $k_2 = l_2 = 0$ they are called **Dirichlet boundary conditions**. If $k_1 = l_1 = 0$, they are called **Neumann boundary conditions**. The conditions

$$y(a) = y(b) \quad y'(a) = y'(b)$$

are called **periodic boundary conditions**.

Singular Sturm-Liouville problem

Singular Sturm-Liouville problem

A Sturm-Liouville problem

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0$$

is called **singular** in any of the following cases:

1. $p(a) = 0$, BC at a is dropped, BC at b is homogeneous mixed.
2. $p(b) = 0$, BC at b is dropped, BC at a is homogeneous mixed.
3. $p(a) = p(b) = 0$, there is no BC.
4. The interval $[a, b]$ is infinite.

Otherwise, the problem is **regular**.

Singular Sturm-Liouville problem

Example: Legendre's equation and polynomials

Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

is a Sturm-Liouville problem

$$((1-x^2)y')' + n(n+1)y = 0$$

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0$$

with $p = 1 - x^2$, $q = 0$, $r = 1$.
$p(-1) = p(1) = 0$, so the Sturm-Liouville problem is singular, and we do not need boundary conditions. The Legendre polynomial $P_n(x)$ is a non-trivial solution of the problem associated to the eigenvalue $\lambda = n(n+1)$. By the previous theorem, Legendre polynomials are orthogonal in the interval $[-1, 1]$.

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 11, Section 5:

- 11.5.6
- 11.5.9
- 11.5.11

Exercises

- 11.5.14

(a) Chebyshev polynomials[6] of the first and second kind are defined by

$$T_n(x) = \cos(n \arccos x)$$
$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}$$

respectively, where $n = 0, 1, \cdots$. Show that

$$T_0 = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1.$$
$$T_3(x) = 4x^3 - 3x,$$
$$U_0 = 1, \qquad U_1(x) = 2x, \qquad U_2(x) = 4x^2 - 1,$$
$$U_3(x) = 8x^3 - 4x.$$

Show that the Chebyshev polynomials $T_n(x)$ are orthogonal on the interval $-1 \leqq x \leqq 1$ with respect to the weight function $r(x) = 1/\sqrt{1-x^2}$. (*Hint.* To evaluate the integral, set $\arccos x = \theta$.) Verify that $T_n(x)$, $n = 0, 1, 2, 3$, satisfy the **Chebyshev equation**

$$(1 - x^2)y'' - xy' + n^2 y = 0.$$

(b) Orthogonality on an infinite interval: Laguerre polynomials[7] are defined by $L_0 = 1$, and

$$L_n(x) = \frac{e^x}{n!}\frac{d^n(x^n e^{-x})}{dx^n}, \qquad n = 1, 2, \cdots.$$

Show that

$$L_n(x) = 1 - x, \qquad L_2(x) = 1 - 2x + x^2/2,$$
$$L_3(x) = 1 - 3x + 3x^2/2 - x^3/6.$$

Prove that the Laguerre polynomials are orthogonal on the positive axis $0 \leqq x < \infty$ with respect to the weight function $r(x) = e^{-x}$. *Hint.* Since the highest power in L_m is x^m, it suffices to show that $\int e^{-x}x^k L_n\,dx = 0$ for $k < n$. Do this by k integrations by parts.

Outline

1. Fourier Analysis
   - Fourier series
   - Arbitrary period. Even and odd functions. Half-range expansions
   - Forced oscillations
   - Approximation by trigonometric polynomials
   - Sturm-Liouville problems. Orthogonal functions
   - Orthogonal series. Generalized Fourier series
   - Fourier integral
   - Fourier transform

Generalized Fourier series

Generalized Fourier series

Let the set $\{y_1, y_2, ...\}$ be orthogonal with respect to the weight function r in an interval $[a, b]$. Let f be a function that we want to expand in this ortohogonal basis

$$f = \sum_{m=0}^{\infty} a_m y_m(x)$$

To find the Fourier coefficients, a_m we compute the inner product of f with y_n

$$\langle f, y_n \rangle_r = \left\langle \sum_{m=0}^{\infty} a_m y_m(x), y_n \right\rangle_r = \sum_{m=0}^{\infty} a_m \langle y_m(x), y_n(x) \rangle_r = a_n \|y_n\|^2$$

$$\boxed{a_n = \frac{\langle f, y_n \rangle_r}{\|y_n\|_r^2}} = \frac{\int_a^b r f y_n dx}{\int_a^b r y_n^2 dx}$$

Generalized Fourier series

Fourier-Legendre series

Legendre polynomials, $P_m(x)$, are orthogonal in $[-1,1]$ with respect to $r(x)=1$. In this interval we can perform an eigenfunction expansion of the form

$$f=\sum_{m=0}^{\infty}\frac{\langle f,P_m\rangle}{\|P_m\|^2}P_m(x)$$

It can be shown that

$$\|P_m\|^2=\frac{2}{2m+1}$$

Example

$$f=\sin(\pi x)\Rightarrow a_m=\frac{2m+1}{2}\int_{-1}^{1}\sin(\pi x)P_m(x)dx$$

$f=0.95493P_1-1.15824P_3+0.21929P_5-0.01664P_7+0.00068P_9-0.00002P_{11}+\ldots$

Generalized Fourier series

Fourier-Bessel series

Bessel's J_n functions are solutions of the ODE

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

That is

$$\tilde{x}^2 \frac{d^2 J_n}{d\tilde{x}^2}(\tilde{x}) + \tilde{x}\frac{dJ_n}{d\tilde{x}}(\tilde{x}) + (\tilde{x}^2 - n^2)J_n(\tilde{x}) = 0$$

Where for convenience we have used the variable $\tilde{x}$ instead of x. We now perform the change of variable

$$\tilde{x} = kx \Rightarrow x = \frac{\tilde{x}}{k}$$

$$\frac{dJ_n}{d\tilde{x}} = \frac{dJ_n}{dx}\frac{dx}{d\tilde{x}} = \frac{dJ_n}{dx}\frac{1}{k}$$

$$\frac{d^2 J_n}{d\tilde{x}^2} = \frac{d}{d\tilde{x}}\left(\frac{dJ_n}{d\tilde{x}}\right) = \frac{d}{dx}\left(\frac{dJ_n}{dx}\frac{1}{k}\right)\frac{dx}{d\tilde{x}} = \frac{d^2 J_n}{dx^2}\frac{1}{k^2}$$

Fourier-Bessel series

So Bessel's equation becomes

$$(kx)^2\frac{d^2J_n}{dx^2}(kx)\frac{1}{k^2}+(kx)\frac{dJ_n}{dx}(kx)\frac{1}{k}+((kx)^2-n^2)J_n(kx)=0$$

$$x^2J_n''(kx)+xJ_n'(kx)+(k^2x^2-n^2)J_n(kx)=0$$

Dividing by x

$$xJ_n''(kx)+J_n'(kx)+(k^2x-\frac{n^2}{x})J_n(kx)=0$$

$$(xJ_n'(kx))'+(-\frac{n^2}{x}+k^2x)J_n(kx)=0$$

This is a Sturm-Liouville problem with $p=x$, $q=-\frac{n^2}{x}$, $\lambda=k^2$, and $r=x$. Let us choose $a=0$, $p(a)=0$, so the problem is singular. For the boundary conditions fix a value $b=R$ and find the values k such that

$$J_n(kR)=0$$

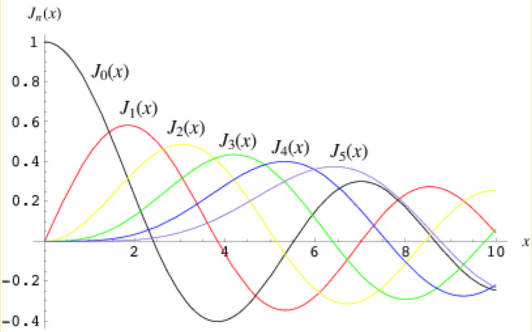
Generalized Fourier series

Fourier-Bessel series

$$J_n(kR) = 0$$

For every n, we find that this equation has infinite solutions that we may index with m

$$kR = \alpha_{n,m} \Rightarrow k_{n,m} = \frac{\alpha_{n,m}}{R}$$



Generalized Fourier series

Fourier-Bessel series

The set of functions $\{J_n(k_{n,1}x), J_n(k_{n,2}x), ...\}$ with $k_{n,m} = \frac{\alpha_{n,m}}{R}$ is orthogonal on the interval $[0, R]$ with respect to the weight function $r(x) = x$ since they are eigenfunctions associated to the eigenvalue $\lambda = k_{n,m}^2$. Additionally,

$$\|J_n(k_{n,m}x)\|_x^2 = \int_0^R xJ_n^2(k_{n,m}x)dx = \frac{R^2}{2}J_{n+1}^2(k_{n,m}R)$$

so the Fourier coefficients of the Fourier-Bessel series

$$\boxed{f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x)}$$

are

$$\boxed{a_m = \frac{\langle f(x), J_n(k_{n,m}x)\rangle_x}{\|J_n(k_{n,m}x)\|_x^2}}$$

Generalized Fourier series

Example

$$f(x) = \frac{1}{1-x^2}$$

Let us consider $n = 0$ and $R = 1$, then

$$k_{0,m} = 2.405, 5.520, 8.654, 11.792, ...$$

The Fourier coefficients are

$$a_m = \frac{4J_2(k_{0,m})}{k_{0,m}^2 J_1^2(k_{0,m})} = 1.1081, -0.1398, 0.0455, -0.0210, ...$$

And the function is approximated as

$$\frac{1}{1-x^2} = 1.1081 J_0(2.405x) - 0.1398 J_0(5.520x) + 0.0455 J_0(8.654x) - ...$$

Mean square convergence. Completeness

Mean square convergence

Let us define the functions

$$s_k = \sum_{m=1}^{k} a_m y_m$$

This sequence of functions tend to f in a mean-square sense if

$$\lim_{k \to \infty} \|s_k - f\|_r^2 = 0$$

where r is a weighting function.

Completeness

An orthonormal set of functions $y_0, y_1, ...$ in the interval $[a, b]$ is complete in a set of functions S defined on $[a, b]$ if

$$\forall f \in S, \forall \epsilon > 0 \Rightarrow \exists a_0, a_1, ... | \left\| f - \sum_{m=1}^{k} a_m y_m \right\| < \epsilon$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 11, Section 6:

- 11.6.2

Exercises

14. TEAM PROJECT. Orthogonality on the Entire Real Axis. Hermite Polynomials.[8] These orthogonal polynomials are defined by $He_0(1) = 1$ and

$$He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}), \qquad n = 1, 2, \cdots. \tag{19}$$

REMARK. As is true for many special functions, the literature contains more than one notation, and one sometimes defines as Hermite polynomials the functions

$$H_0^* = 1, \qquad H_n^*(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

This differs from our definition, which is preferred in applications.

(a) Small Values of *n*. Show that

$$He_1(x) = x, \qquad He_2(x) = x^2 - 1,$$
$$He_3(x) = x^3 - 3x, \qquad He_4(x) = x^4 - 6x^2 + 3.$$

(b) Generating Function. A generating function of the Hermite polynomials is

$$e^{tx - t^2/2} = \sum_{n=0}^{\infty} a_n(x) t^n \tag{20}$$

because $He_n(x) = n!\, a_n(x)$. Prove this. *Hint*: Use the formula for the coefficients of a Maclaurin series and note that $tx - \frac{1}{2}t^2 = \frac{1}{2}x^2 - \frac{1}{2}(x - t)^2$.

(c) Derivative. Differentiating the generating function with respect to x, show that

$$He_n'(x) = nHe_{n-1}(x). \tag{21}$$

(d) Orthogonality on the *x*-Axis needs a weight function that goes to zero sufficiently fast as $x \to \pm\infty$, (Why?)

Show that the Hermite polynomials are orthogonal on $-\infty < x < \infty$ with respect to the weight function $r(x) = e^{-x^2/2}$. *Hint.* Use integration by parts and (21).

(e) ODEs. Show that

$$He_n'(x) = xHe_n(x) - He_{n+1}(x). \tag{22}$$

Using this with $n - 1$ instead of n and (21), show that $y = He_n(x)$ satisfies the ODE

$$y'' = xy' + ny = 0. \tag{23}$$

Show that $w = e^{-x^2/4}y$ is a solution of **Weber's equation**

$$w'' + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)w = 0 \qquad (n = 0, 1, \cdots). \tag{24}$$

15. CAS EXPERIMENT. Fourier–Bessel Series. Use Example 2 and $R = 1$, so that you get the series

$$f(x) = a_1 J_0(\alpha_{0,1}x) + a_2 J_0(\alpha_{0,2}x) + a_3 J_0(\alpha_{0,3}x) + \cdots \tag{25}$$

With the zeros $\alpha_{0,1}\alpha_{0,2}, \cdots$ from your CAS (see also Table A1 in App. 5).

(a) Graph the terms $J_0(\alpha_{0,1}x), \cdots, J_0(\alpha_{0,10}x)$ for $0 \leqq x \leqq 1$ on common axes.

(b) Write a program for calculating partial sums of (25). Find out for what $f(x)$ your CAS can evaluate the integrals. Take two such $f(x)$ and comment empirically on the speed of convergence by observing the decrease of the coefficients.

(c) Take $f(x) = 1$ in (25) and evaluate the integrals for the coefficients analytically by (21a), Sec. 5.4, with $v = 1$. Graph the first few partial sums on common axes.

Outline

1. Fourier Analysis
   - Fourier series
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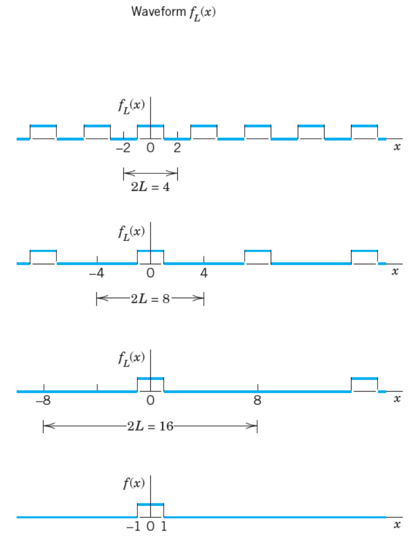
   - Orthogonal series. Generalized Fourier series
   - Fourier integral
   - Fourier transform

Fourier integral

Example



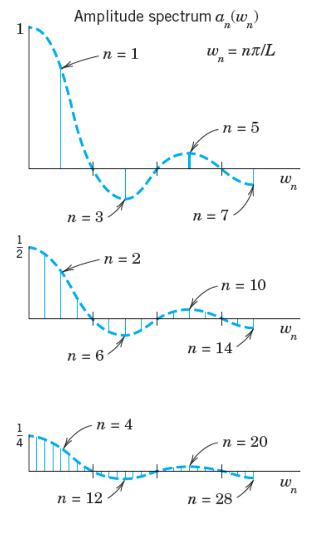
$$f_L(x) = \begin{cases} 0 & -L < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < L \end{cases}$$

$$f_L(x) = f_L(x + 2L)$$

$$\begin{aligned} f(x) &= \lim_{L\to\infty} f_L(x) \\ &= \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Fourier integral

Example (continued)



$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}$$

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-1}^{1} \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\frac{\sin(n\pi/L)}{n\pi/L} \\ &= \frac{2}{L}\mathrm{sinc}\left(\frac{n}{L}\right) \qquad \left[\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}\right] \\ &= \frac{2}{L}\mathrm{sinc}\left(\frac{\omega_n}{\pi}\right) \qquad \left[\omega_n = \frac{\pi}{L}n\right] \end{aligned}$$

Fourier integral

From Fourier series to Fourier integral

$$\begin{aligned}
f_L(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \\
&= a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(\omega_n x) + b_n \sin(\omega_n x)\right] \quad \left[\omega_n = \frac{\pi}{L}n\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f_L(v)\cos(\omega_n v)dv\right)\cos(\omega_n x) + \\
&\quad \sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f_L(v)\sin(\omega_n v)dv\right)\sin(\omega_n x) \quad \left[\Delta\omega_n = \frac{\pi}{L}\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f_L(v)\cos(\omega_n v)dv\right)\cos(\omega_n x)\Delta\omega + \\
&\quad \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f_L(v)\sin(\omega_n v)dv\right)\sin(\omega_n x)\Delta\omega
\end{aligned}$$

Fourier integral

From Fourier series to Fourier integral

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f_L(v)\cos(\omega_n v)dv\right)\cos(\omega_n x)\Delta\omega + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f_L(v)\sin(\omega_n v)dv\right)\sin(\omega_n x)\Delta\omega$$

We now take the limit when L goes to ∞

$$\lim_{L\to\infty} f_L(x) = 0 + \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv\right)\cos(\omega x)d\omega + \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv\right)\sin(\omega x)d\omega$$

$$\boxed{f(x) = \int_{0}^{\infty}(A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))d\omega}$$

Fourier integral

From Fourier series to Fourier integral

If f is piecewise continuous in every finite interval and has a left- and right-hand derivative at every point, and it is absolutely integrable ($\int_{-\infty}^{\infty} |f(x)| dx$), then f can be represented by a Fourier integral.

$$f(x) = \int_0^{\infty} (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)) d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v)\cos(\omega v) dv$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v)\sin(\omega v) dv$$

Where f is discontinuous, the value of the Fourier integral equals the average of the left- and right-hand limits of f at that point.

Outline

1. Fourier Analysis
   - Fourier series
   - Arbitrary period. Even and odd functions. Half-range expansions
   - Forced oscillations
   - Approximation by trigonometric polynomials
   - Sturm-Liouville problems. Orthogonal functions
   - Orthogonal series. Generalized Fourier series
   - Fourier integral
   - Fourier transform

Fourier transform

Complex form of the Fourier integral

$$f(x) = \int_0^\infty (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv$$

Let's substitute A and B into the Fourier integral

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\left(\cos(\omega v)\cos(\omega x) + \sin(\omega v)\sin(\omega x)\right)dv\right]d\omega \\ &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(\omega(x - v))dv\right]d\omega = \frac{1}{\pi}\int_0^\infty F(\omega, x)d\omega \end{aligned}$$

Fourier transform

Complex form of the Fourier integral

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v))dv\right]d\omega = \frac{1}{\pi}\int_{0}^{\infty} F(\omega,x)d\omega$$

Note that $F(\omega, x)$ is an even function in ω, that is $F(\omega, x) = F(-\omega, x)$, so we may symmetrize the integration limit and divide by two:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega,x)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v))dv\right]d\omega$$

The function

$$\int_{-\infty}^{\infty} f(v)\sin(\omega(x-v))dv$$

is odd and its integral over all ω must be 0.

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(\omega(x-v))dv\right]d\omega$$

Complex form of the Fourier integral

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v))dv\right]d\omega \\
0 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(\omega(x-v))dv\right]d\omega
\end{aligned}
$$

Now we calculate

$$
\begin{aligned}
f(x)+i0 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)(\cos(\omega(x-v))+i\sin(\omega(x-v)))dv\right]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}dv\right]d\omega \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right]e^{i\omega x}d\omega
\end{aligned}
$$

Fourier transform

Complex form of the Fourier integral

$$f(x) \;\; = \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right] e^{i\omega x} d\omega$$

Let us define the **Fourier transform** of f as

$$\boxed{\mathcal{F}\{f\} = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx}$$

From the Fourier transform we can recover the original function as

$$\boxed{\mathcal{F}^{-1}\{\hat{f}\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega}$$

If f is absolutely integrable and piecewise continuous, then its Fourier transform exists.

Fourier transform

Example

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned}
\mathcal{F}\{f\} &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{-i\omega x}}{i\omega} \right)_{-1}^{1} = -\frac{1}{i\omega\sqrt{2\pi}} (e^{-i\omega} - e^{i\omega}) \\
&= \frac{1}{i\omega\sqrt{2\pi}} (e^{i\omega} - e^{-i\omega}) = \frac{2}{\omega\sqrt{2\pi}} \frac{e^{i\omega} - e^{-i\omega}}{2i} = \frac{2}{\omega\sqrt{2\pi}} \sin(\omega) \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega} = \sqrt{\frac{2}{\pi}} \frac{\sin\left(\pi \frac{\omega}{\pi}\right)}{\pi \frac{\omega}{\pi}} \\
&= \sqrt{\frac{2}{\pi}} \mathrm{sinc}\left(\frac{\omega}{\pi}\right)
\end{aligned}$$

Fourier transform

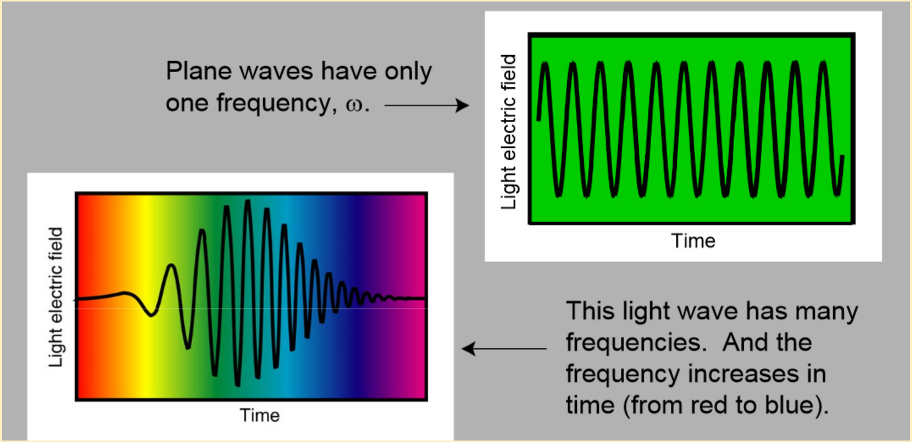
Example

$$f(x) = e^{-ax}u(x) \quad a > 0$$

Solution:

$$\mathcal{F}\{f\} \;=\; \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{-(a+i\omega)x}}{a+i\omega} \right)_0^{\infty} = \frac{1}{\sqrt{2\pi}(a+i\omega)}$$

Power spectral density



Power spectral density

Power spectral density

Let us consider the spring-mass system

$$my'' + ky = 0$$

Multiplying by y'

$$my'y'' + kyy' = 0$$

and integrating

$$m\frac{1}{2}(y')^2 + k\frac{1}{2}y^2 = E_0$$

$$\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = E_0$$

The first term is the kinetic energy of the system and the second term its (spring) potential energy, E_0 is the total energy.

Power spectral density

Power spectral density

The general solution of the ODE is

$$y = a_1 \cos(\omega_0 x) + b_1 \sin(\omega_0 x)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the system. We can rewrite it as

$$\begin{aligned} y &= a_1 \tfrac{e^{i\omega_0 x} + e^{-i\omega_0 x}}{2} + b_1 \tfrac{e^{i\omega_0 x} - e^{-i\omega_0 x}}{2i} \\ &= \tfrac{a_1 - ib_1}{2} e^{i\omega_0 x} + \tfrac{a_1 + ib_1}{2} e^{-i\omega_0 x} \\ &= c_1 e^{i\omega_0 x} + c_{-1} e^{-i\omega_0 x} \\ y' &= i\omega_0 (c_1 e^{i\omega_0 x} - c_{-1} e^{-i\omega_0 x}) \end{aligned}$$

Power spectral density

Power spectral density

Substituting in the energy equation

$$\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = E_0$$

$$\frac{1}{2}m\left(i\omega_0(c_1e^{i\omega_0 x} - c_{-1}e^{-i\omega_0 x})\right)^2 + \frac{1}{2}k\left(c_1e^{i\omega_0 x} + c_{-1}e^{-i\omega_0 x}\right)^2 = E_0$$

$$\frac{1}{2}m(i\omega_0)^2(A - A^*)^2 + \frac{1}{2}k(A + A^*)^2 = E_0$$

$$\frac{1}{2}m\left(-\frac{k}{m}\right)(A - A^*)^2 + \frac{1}{2}k(A + A^*)^2 = E_0$$

$$-\frac{1}{2}k(A - A^*)^2 + \frac{1}{2}k(A + A^*)^2 = E_0$$

$$\frac{1}{2}k\left[-(A - A^*)^2 + (A + A^*)^2\right] = E_0$$

$$\frac{1}{2}k\left[-A^2 - (A^*)^2 + 2AA^* + A^2 + (A^*)^2 + 2AA^*\right] = E_0$$

$$2k|A|^2 = 2k|c_1|^2 = E_0$$

Power spectral density

Power spectral density

So if y is a sum of two complex exponentials, then the energy is proportional to their amplitude

$$y = c_1 e^{i\omega_0 x} + c_{-1} e^{-i\omega_0 x} \Rightarrow E_0 \propto |c_1|^2$$

If we had a discrete sum of complex exponentials we would have

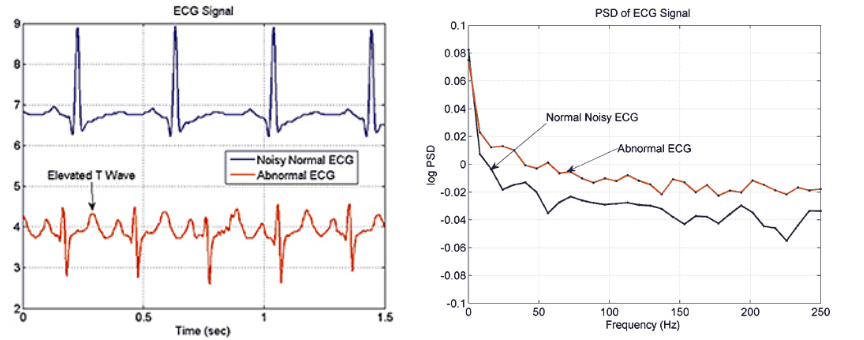
$$y = \sum_n c_n e^{i\omega_n x} + c_{-n} e^{-i\omega_n x} \Rightarrow E_0 \propto \sum |c_n|^2$$

and for a "continous" sum

$$y = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \Rightarrow \boxed{E_0 \propto \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega}$$

Power spectral density

Example



Linearity of the Fourier transform

Linearity of the Fourier transform

$$\mathcal{F}\{af+bg\} = a\mathcal{F}\{f\} + b\mathcal{F}\{g\}$$

Proof

$$\begin{aligned}
\mathcal{F}\{af+bg\} &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} (af(x)+bg(x))e^{-i\omega x}dx \\
&= \frac{a}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{-i\omega x}dx + \frac{b}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} g(x)e^{-i\omega x}dx \\
&= a\mathcal{F}\{f\} + b\mathcal{F}\{g\}
\end{aligned}$$

Fourier transform of the derivative

Fourier transform of the derivative

$$\mathcal{F}\{f'\} = i\omega\mathcal{F}\{f\}$$

Proof

$$\begin{aligned}
f &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \\
\frac{df}{dx} &= \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \frac{d}{dx}\left(e^{i\omega x}\right) d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)(i\omega) e^{i\omega x} d\omega \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(i\omega\hat{f}(\omega)\right) e^{i\omega x} d\omega
\end{aligned}$$

Fourier transform of the derivative

Fourier transform of the derivative

$$\mathcal{F}\{f''\} = (i\omega)^2 \mathcal{F}\{f\}$$

$$\mathcal{F}\{f^{(n)}\} = (i\omega)^n \mathcal{F}\{f\}$$

$$\mathcal{F}\{f^{(\alpha)}\} = (i\omega)^\alpha \mathcal{F}\{f\}$$

Fourier transform of the integral

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau) d\tau\right\} = \frac{\mathcal{F}\{f\}}{i\omega} + c\delta(f)$$

where c is a value such that

$$\int_{-\infty}^{t} (f(\tau) - c) d\tau = 0$$

It is normally referred to as the DC or average value.

Fourier transform of the convolution

Fourier transform of the convolution

$$f(x) \star g(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp$$

$$\mathcal{F}\{f \star g\} = \sqrt{2\pi}\mathcal{F}\{f\}\mathcal{F}\{g\}$$

Proof

$$\begin{aligned}
\mathcal{F}\{f \star g\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(p)g(x-p)dp \right) e^{-i\omega x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-i\omega x} dp dx \quad \text{[swap variables]} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-i\omega x} dx dp \quad [q = x - p] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(q)e^{-i\omega(q+p)} dq dp
\end{aligned}$$

Fourier transform of the convolution

$$\begin{aligned}
\mathcal{F}\{f \star g\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p) g(q) e^{-i\omega(q+p)} dq dp \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} f(p) e^{-i\omega p} dp \right) \left(\int_{-\infty}^{\infty} g(q) e^{-i\omega q} dq \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\sqrt{2\pi} \mathcal{F}\{f\} \right) \left(\sqrt{2\pi} \mathcal{F}\{g\} \right) \\
&= \sqrt{2\pi} \mathcal{F}\{f\} \mathcal{F}\{g\}
\end{aligned}$$

Fourier transform of the convolution

Calculation of the convolution

$$(f \star g)(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{i\omega x}d\omega$$

Proof

$$\begin{aligned}(f \star g)(x) &= \mathcal{F}^{-1}\left\{\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)\right\} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)\right)e^{i\omega x}d\omega \\ &= \int_{-\infty}^{\infty}\hat{f}(\omega)\hat{g}(\omega)e^{i\omega x}d\omega\end{aligned}$$

Table of Fourier transforms

	$f(x)$	$\hat{f}(w) = \mathscr{F}(f)$
1	$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin bw}{w}$
2	$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$
3	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|w\|}}{a}$
4	$\begin{cases} x & \text{if } 0 < x < b \\ 2x - b & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\frac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi} w^2}$
5	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (a > 0)$	$\frac{1}{\sqrt{2\pi}(a + iw)}$

Table of Fourier transforms

6	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
7	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin b(w - a)}{w - a}$
8	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}}\,\dfrac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
9	$e^{-ax^2} \quad (a > 0)$	$\dfrac{1}{\sqrt{2a}}\,e^{-w^2/4a}$
10	$\dfrac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}$ if $\lvert w\rvert < a$; $\;0$ if $\lvert w\rvert > a$

Outline

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   - Orthogonal series. Generalized Fourier series
   - Fourier integral
   - Fourier transform