Chapter 8. Partial Differential Equations

C.O.S. Sorzano

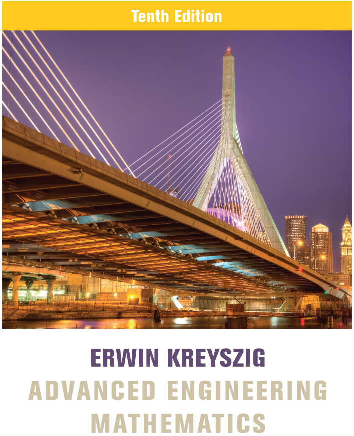
Biomedical Engineering

September 24, 2014

Outline

1. Partial Differential Equations
    - Basic concepts
    - Vibrating string. Wave equation
    - D'Alembert's solution of the wave equation. Characteristics
    - Heat flow from a body in space. Heat equation
    - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
    - 1D Heat equation: Solution by Fourier integrals and transforms
    - Membrane, 2D Wave equation
    - Rectangular membrane, double Fourier series
    - Circular membrane, Fourier-Bessel series
    - Laplace's equation in cylindrical and spherical coordinates. Potential

References



E. Kreyszig. Advanced Engineering Mathematics. John Wiley & sons. Chapter 12.

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential

Basic concepts

- PDE: unknown u and $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, ...
- Order: The highest order of derivation, e.g., order of $\frac{\partial^2 u}{\partial t \partial x}$ is 2.
- Linear PDE: it involves only first-order derivatives
- Homogeneous: each term contaisn u or one of its derivatives

Important Second-Order PDEs

(1)	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	*One-dimensional wave equation*
(2)	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	*One-dimensional heat equation*
(3)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	*Two-dimensional Laplace equation*
(4)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$	*Two-dimensional Poisson equation*
(5)	$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$	*Two-dimensional wave equation*
(6)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	*Three-dimensional Laplace equation*

Basic concepts

The set of solutions can be very large and one needs some constraints (boundary conditions of initial conditions) to restrict the solution to have physical meaning. For instance,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is satisfied by

$$u = x^2 - y^2$$
$$u = e^x \cos(y)$$
$$u = \sin(x) \cosh(y)$$
$$u = \log(x^2 + y^2)$$

Principle of superposition

If u_1 and u_2 are solutions of a homogeneous PDE, then $u = c_1 u_1 + c_2 u_2$ is also a solution.

Basic concepts

Example

Find solutions depending on x and y of

$$u_{xx} - u = 0$$

Solution: Since y does not appear, it is like solving

$$u'' - u = 0$$

whose general solution is

$$u = Ae^{x} + Be^{-x}$$

Here A and B may be functions of y

$$\boxed{u = A(y)e^{x} + B(y)e^{-x}}$$

Basic concepts

Example

Find solutions depending on x and y of

$$u_{xy} = -u_x$$

Solution: Setting $v = u_x$, we have the equation

$$v_y = -v \;\Rightarrow \frac{dv}{v} = -dy \Rightarrow \log|v| = -y + c_1(x) \Rightarrow v = c_2(x)e^{-y}$$

Integrating with respect to x

$$u = \int c_2(x)e^{-y}dx = c_3(x)e^{-y} + c_4(y)$$

That is

$$\boxed{u = f(x)e^{-y} + g(y)}$$

Exercises

Exercises

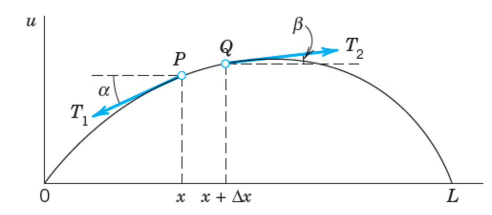
From Kreyszig (10th ed.), Chapter 12, Section 1:

- 12.1.2
- 12.1.5
- 12.1.19

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential

Vibrating string



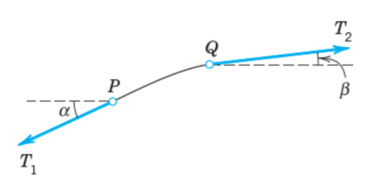
Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

https://www.youtube.com/watch?v=ttgLyWFINJI

Vibrating string. Wave equation

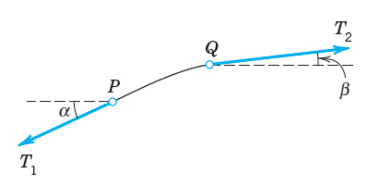
Vibrating string. Wave equation



Since the string offers no resistance to bending the tension is tangential to the curve at each point. Let T_1 and T_2 be the tension at the points P and Q. Since the points move vertically (not horizontally) the horizontal tension must cancel at every point

$$\text{Horizontally: } T_2 \cos(\beta) - T_1 \cos(\alpha) = 0 \Rightarrow T_2 \cos(\beta) = T_1 \cos(\alpha) = T(\text{const})$$

Vibrating string. Wave equation



Vertically, the difference of the forces translates into an acceleration

$$\text{Vertically:}\, T_2 \sin(\beta) - T_1 \sin(\alpha) = (\rho \Delta x) u_{tt}$$

where ρ is the mass density of the string and Δx is the distance between $P = x$ and $Q = x + \Delta x$.
Dividing by T we have

$$\frac{T_2 \sin(\beta)}{T_2 \cos(\beta)} - \frac{T_1 \sin(\alpha)}{T_1 \cos(\alpha)} = \frac{\rho \Delta x}{T} u_{tt}$$

Vibrating string. Wave equation

Vibrating string. Wave equation

$$\frac{T_2 \sin(\beta)}{T_2 \cos(\beta)} - \frac{T_1 \sin(\alpha)}{T_1 \cos(\alpha)} = \frac{\rho \Delta x}{T} u_{tt}$$

$$\frac{\tan(\beta) - \tan(\alpha)}{\Delta x} = \frac{\rho}{T} u_{tt}$$

$$\frac{u_x(x) - u_x(x + \Delta x)}{\Delta x} = \frac{\rho}{T} u_{tt}$$

Taking the limit when Δx goes to 0

$$u_{xx} = \frac{\rho}{T} u_{tt}$$

$$u_{tt} = \frac{T}{\rho} u_{xx}$$

$$\boxed{u_{tt} = c^2 u_{xx}}$$

This is the **1D wave equation** and c is the propagation speed.

Vibrating string. Wave equation

Vibrating string. Wave equation

The model of the vibrating string consists of the 1D wave equation

$$u_{tt} = c^2 u_{xx}$$

plus some boundary conditions

$$u(0,t) = 0, u(L,t) = 0$$

plus some initial conditions on the initial shape and velocity of the string

$$u(x,0) = f(x), u_t(x,0) = g(x)$$

The solution has three steps:

1. Separating variables
2. Satisfying the boundary conditions
3. Staisfying the initial conditions

Vibrating string. Wave equation

Separating variables

Let us look for a solution of the form

$$u(x,t) = F(x)G(t)$$

$$u_{tt} = FG_{tt}, u_{xx} = F_{xx}G$$

So the PDE becomes

$$FG_{tt} = c^2 F_{xx} G$$

$$\frac{1}{c^2}\frac{G_{tt}}{G} = \frac{F_{xx}}{F}$$

The left-hand side depends only of t, while the right-hand side depends only on x. The only way this is feasible is

$$\frac{1}{c^2}\frac{G_{tt}}{G} = \frac{F_{xx}}{F} = k \Rightarrow \begin{cases} F_{xx} - kF = 0 \\ G_{tt} - c^2 kG = 0 \end{cases}$$

Vibrating string. Wave equation

Satisfying the boundary conditions

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0, u(L,t) = F(L)G(t) = 0$$

$G(t)$ cannot be 0 because, it would be a solutio $u = 0$ of no interest. So it must be

$$F(0) = F(L) = 0$$

Consider the ODE for F

$$F_{xx} - kF = 0$$

If $k = 0$, then general solution is

$$F = ax + b$$

and the two boundary conditions would make $a = 0 = b$, which is again of no interest.

Vibrating string. Wave equation

Satisfying the boundary conditions

$$F_{xx} - kF = 0$$

If $k = \mu^2 > 0$, then the general solution is

$$F = ae^{\mu x} + be^{-\mu x}$$

and the two boundary conditions would make $a = 0 = b$, which is of no interest.
If $k = -\mu^2 < 0$, then the general solution is

$$F = a\cos(\mu x) + b\sin(\mu x)$$

and the two boundary conditions would make

$$0 = a, 0 = b\sin(\mu L) \Rightarrow \mu L = n\pi \Rightarrow \mu = \frac{\pi}{L}n$$

That is, there are infinitely many solutions of the form

$$\boxed{F(x) = F_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$$

Vibrating string. Wave equation

Satisfying the boundary conditions

We now solve

$$G_{tt} - c^2 kG = 0$$

where $k = -(frac\pi Ln)^2$. Let us define

$$\lambda_n = c\mu = \frac{c\pi}{L}n$$

Then

$$G_{tt} + \lambda_n^2 G = 0$$

The general solution is

$$G(t) = a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)$$

And an eigenfunction of the vibration problem with boundary conditions is

$$\boxed{u_n(x,t) = FG = \left(\sin\left(\frac{n\pi}{L}x\right)\right)(a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t))}$$

associated to the eigenvalue λ_n.

Vibrating string. Wave equation

Satisfying the boundary conditions

$$u_n(x,t) = \left(\sin\left(\frac{n\pi}{L}x\right)\right)\left(a_n\cos\left(c\frac{n\pi}{L}t\right) + b_n\sin\left(c\frac{n\pi}{L}t\right)\right)$$

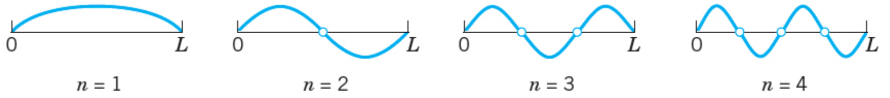


Fig. 287. Normal modes of the vibrating string

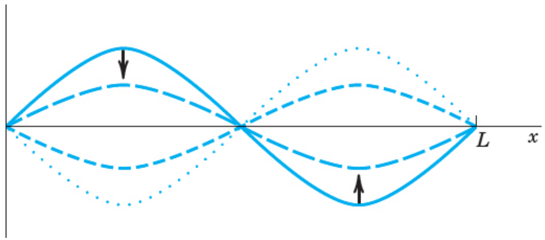


Fig. 288. Second normal mode for various values of t

Satisfying the boundary conditions

$$u_n(x,t) = \left(\sin\left(\frac{n\pi}{L}x\right)\right)\left(a_n\cos\left(c\frac{n\pi}{L}t\right) + b_n\sin\left(c\frac{n\pi}{L}t\right)\right)$$

Remind that $c = \sqrt{\frac{T}{\rho}}$ so that tuning an instrument amounts to changing T and, ultimately, c. The other two variables to control are ρ and L

Vibrating string. Wave equation

Satisfying the initial conditions: initial shape

The general solution of the vibrating string problem is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)\right)$$

For the initial shape condition we have

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

If we do the Fourier series expansion of $f(x)$ assuming we make an odd extension of it and make it of period $2L$, then f can be expressed as

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(v)\sin\left(\frac{n\pi}{L}v\right)dv\right)\sin\left(\frac{n\pi}{L}x\right) \Rightarrow \boxed{a_n = \frac{2}{L}\int_0^L f(v)\sin\left(\frac{n\pi}{L}v\right)dv}$$

Satisfying the initial conditions: initial speed

The derivative of the general solution is

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(-a_n\lambda_n \sin(\lambda_n t) + b_n\lambda_n \cos(\lambda_n t)\right)$$

For the initial shape condition we have

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n\lambda_n \sin\left(\frac{n\pi}{L}x\right) = g(x)$$

If we do the Fourier series expansion of $g(x)$ assuming we make an odd extension of it and make it of period $2L$, then g can be expressed as

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L g(v)\sin\left(\frac{n\pi}{L}v\right)dv\right)\sin\left(\frac{n\pi}{L}x\right) \Rightarrow \boxed{b_n = \frac{2}{\lambda_n L}\int_0^L g(v)\sin\left(\frac{n\pi}{L}v\right)dv}$$

Vibrating string. Wave equation

Particular solution

Finally the particular solution is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(a_n\cos(\lambda_n t) + b_n\sin(\lambda_n t)\right) \\
& a_n = \frac{2}{L}\int_0^L f(v)\sin\left(\frac{n\pi}{L}v\right)dv \\
& b_n = \frac{2}{\lambda_n L}\int_0^L g(v)\sin\left(\frac{n\pi}{L}v\right)dv \\
& \lambda_n = c\mu = \frac{c\pi}{L}n
\end{aligned}$$

Vibrating string. Wave equation

Particular solution

We may reformulate this solution as

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} a_n \sin\left(\tfrac{n\pi}{L}x\right)\cos\left(\tfrac{c\pi}{L}nt\right) + \\
&\quad \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{n\pi}{L}x\right)\sin\left(\tfrac{c\pi}{L}nt\right) \\
&= \sum_{n=1}^{\infty} a_n \tfrac{1}{2}\left[\sin\left(\tfrac{n\pi}{L}(x-ct)\right) + \sin\left(\tfrac{\pi}{L}n(x+ct)\right)\right] + \\
&\quad \sum_{n=1}^{\infty} b_n \tfrac{1}{2}\left[\cos\left(\tfrac{n\pi}{L}(x-ct)\right) + \cos\left(\tfrac{\pi}{L}n(x+ct)\right)\right] \\
&= \sum_{n=1}^{\infty} \tfrac{a_n}{2}\sin\left(\tfrac{n\pi}{L}(x-ct)\right) + \tfrac{b_n}{2}\cos\left(\tfrac{n\pi}{L}(x-ct)\right) + \\
&\quad \sum_{n=1}^{\infty} \tfrac{a_n}{2}\sin\left(\tfrac{n\pi}{L}(x+ct)\right) + \tfrac{b_n}{2}\cos\left(\tfrac{n\pi}{L}(x+ct)\right)
\end{aligned}
$$

Particular solution

We may reformulate this solution as

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \tfrac{a_n}{2}\sin\left(\tfrac{n\pi}{L}(x-ct)\right) + \tfrac{b_n}{2}\cos\left(\tfrac{n\pi}{L}(x-ct)\right) + \\ &\quad \sum_{n=1}^{\infty} \tfrac{a_n}{2}\sin\left(\tfrac{n\pi}{L}(x+ct)\right) + \tfrac{b_n}{2}\cos\left(\tfrac{n\pi}{L}(x+ct)\right) \end{aligned}$$

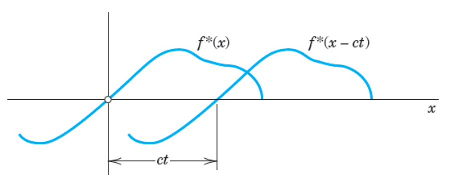
If we define

$$f^*(\xi) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}\xi\right) + b_n \cos\left(\frac{n\pi}{L}\xi\right)$$

Then

$$\boxed{u(x,t) = \frac{1}{2}\left(f^*(x-ct) + f^*(x+ct)\right)}$$

That is u is the sum of two **travelling waves**.



http://www.animations.physics.unsw.edu.au/jw/travelling_sine_wave.htm

Vibrating string. Wave equation

Example

$$f(x) = \begin{cases} \frac{2k}{L}x & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \frac{L}{2} < x < L \end{cases}$$

$$g(x) = 0$$

<u>Solution:</u>

$$g(x) = 0 \Rightarrow b_n = 0$$

For $f(x)$ see Example in Chapter 7 (Half-range expansion)

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{c\pi}{L}t\right) - \frac{1}{3^2}\sin\left(\frac{3\pi}{L}x\right)\cos\left(\frac{3c\pi}{L}t\right) + \dots\right)$$

Example

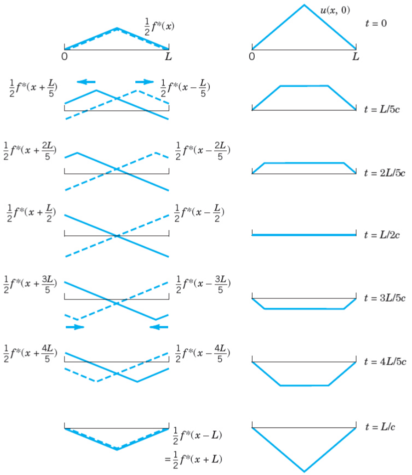


Fig. 291. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential

D'Alembert's solution of the wave equation

D'Alembert's solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

Let us introduce the variables

$$v = x + ct \quad w = x - ct$$

Then the derivaritives of u can be calculated as

$$u_x = u_v v_x + u_w w_x = u_v + u_w$$

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{wv} + u_{ww}$$

$$u_t = u_v v_t + u_w w_t = cu_v - cu_w$$

$$u_{tt} = c(u_v - u_w)_t = c[(u_v - u_w)_v v_t + (u_v - u_w)_w w_t] = c^2(u_{vv} - 2u_{wv} + u_{ww})$$

The PDE becomes

$$c^2(u_{vv} - 2u_{wv} + u_{ww}) = c^2(u_{vv} + 2u_{wv} + u_{ww})$$

D'Alembert's solution of the wave equation

D'Alembert's solution of the wave equation

$$c^2(u_{vv} - 2u_{wv} + u_{ww}) = c^2(u_{vv} + 2u_{wv} + u_{ww})$$

$$-u_{wv} = u_{wv} \Rightarrow u_{wv} = 0$$

Integrating in v

$$u_w = f_1(w)$$

And now in w

$$u = \int f_1(w)dw = \psi(w) + \phi(v)$$

That is

$$\boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)}$$

where ϕ and ψ are two, maybe different, travelling waves.

D'Alembert's solution of the wave equation

Initial conditions

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

Now we impose the initial conditions

$$u(x,0) = f(x) \quad u_t(x,0) = g(x)$$

Let us calculate u_t

$$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct)$$

Now the two initial conditions impose

$$u(x,0) = \phi(x) + \psi(x) = f(x)$$

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$$

D'Alembert's solution of the wave equation

Initial conditions (continued)

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$$

Dividing by c and integrating with respect to x, we get

$$\int_{x_0}^{x} \phi'(x)dx - \int_{x_0}^{x} \psi'(x)dx = \frac{1}{c}\int_{x_0}^{x} g(s)ds$$

$$\phi(x) - \phi(x_0) - \psi(x) + \psi(x_0) = \frac{1}{c}\int_{x_0}^{x} g(s)ds$$

$$\phi(x) - \psi(x) = \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)ds$$

D'Alembert's solution of the wave equation

Initial conditions (continued)

$$\phi(x) - \psi(x) = \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)ds$$

Adding this equation to

$$\phi(x) + \psi(x) = f(x)$$

We get

$$2\phi(x) = k(x_0) + f(x) + \frac{1}{c}\int_{x_0}^{x} g(s)ds \Rightarrow \phi(x) = \frac{1}{2}\left(k(x_0) + f(x) + \frac{1}{c}\int_{x_0}^{x} g(s)ds\right)$$

Similarly subtracting the first equation from the second

$$2\psi(x) = -k(x_0) + f(x) - \frac{1}{c}\int_{x_0}^{x} g(s)ds \Rightarrow \psi(x) = \frac{1}{2}\left(-k(x_0) + f(x) - \frac{1}{c}\int_{x_0}^{x} g(s)ds\right)$$

D'Alembert's solution of the wave equation

Initial conditions (continued)

The wave equation solution was

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

Substituting ϕ and ψ as calculated above

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}k(x_0) + \tfrac{1}{2}f(x+ct) + \tfrac{1}{2c}\int_{x_0}^{x+ct} g(s)ds \\ &\quad -\tfrac{1}{2}k(x_0) + \tfrac{1}{2}f(x-ct) - \tfrac{1}{2c}\int_{x_0}^{x-ct} g(s)ds \\ &= \tfrac{1}{2}(f(x+ct)+f(x-ct)) + \tfrac{1}{2c}\int_{x_0}^{x+ct} g(s)ds + \tfrac{1}{2c}\int_{x-ct}^{x_0} g(s)ds \\ &= \boxed{\frac{1}{2}(f(x+ct)+f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds} \end{aligned}$$

D'Alembert's solution of the wave equation

Initial conditions (continued)

$$u(x,t) \quad = \quad \tfrac{1}{2}(f(x+ct)+f(x-ct)) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

If the initial velocity is 0, then

$$u(x,t) = \boxed{\frac{1}{2}(f(x+ct)+f(x-ct))}$$

Method of characteristics

Method of characteristics

D'Alembert's solution is a special case of the method of characteristics that deals with the problem

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

This problem is classified as

Type	Defining Condition	Example in Sec. 12.1
Hyperbolic	$AC - B^2 < 0$	Wave equation (1)
Parabolic	$AC - B^2 = 0$	Heat equation (2)
Elliptic	$AC - B^2 > 0$	Laplace equation (3)

A, B and C may be functions of x and y, so the problem is of a mixed type, that is different type in different regions of space.

Method of characteristics

Example

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

Consider the 1D wave equation

$$u_{tt} = c^2 u_{xx}$$

Make the change of variable

$$y = ct \Rightarrow u_{tt} = c^2 u_{yy}$$

Then the PDE becomes

$$u_{xx} - u_{yy} = 0 \Rightarrow AC - B^2 = 1(-1) - 0^2 = -1 < 0 \Rightarrow \text{Hyperbolic}$$

Method of characteristics

Example

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

Consider the 1D heat equation

$$u_t = c^2 u_{xx}$$

Make the change of variable

$$y = c^2 t \Rightarrow u_t = c^2 u_y$$

Then the PDE becomes

$$u_{xx} = u_y \Rightarrow AC - B^2 = 1(0) - 0^2 = 0 \Rightarrow \text{Parabolic}$$

Method of characteristics

Transformation to normal form

The normal forms of the PDE

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

depend on the solutions of the **characteristic equation**

$$A(y')^2 - 2By' + C = 0$$

that are called **characteristics** of the PDE and are written in the form

$$\Psi(x, y) = C_1 \quad \Phi(x, y) = C_2$$

Method of characteristics

Transformation to normal form

The transformations giving the new variables v and w as well as the normal forms are

Type	New Variables		Normal Form
Hyperbolic	$v = \Phi$	$w = \Psi$	$u_{vw} = F_1$
Parabolic	$v = x$	$w = \Phi = \Psi$	$u_{ww} = F_2$
Elliptic	$v = \frac{1}{2}(\Phi + \Psi)$	$w = \frac{1}{2i}(\Phi - \Psi)$	$u_{vv} + u_{ww} = F_3$

Method of characteristics

Example: D'Alembert solution

$$u_{tt} - c^2 u_{xx} = 0$$

We do the change of variable $y = ct$, and transform the PDE into

$$u_{yy} - u_{xx} = 0 \Rightarrow u_{xx} - u_{yy} = 0$$

The characteristic equation is

$$(y')^2 - 1 = 0$$

$$(y'+1)(y'-1) = 0 \Rightarrow \begin{cases} y' + 1 = 0 \Rightarrow y = -x + C_1 \Rightarrow \Phi(x,y) = x + y = C_1 \\ y' - 1 = 0 \Rightarrow y = x + C_2 \Rightarrow \Psi(x,y) = x - y = C_2 \end{cases}$$

Since the equation is hyperbolic, the change of variables is

$$v = \Phi(x,y) = x + y = x + ct$$
$$w = \Psi(x,y) = x - y = x - ct$$

And the associated normal form

$$u_{vw} = 0$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 12, Section 4:

- 12.4.11
- 12.4.19

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential

Heat equation

Heat equation

Physical Assumptions

1. The *specific heat* σ and the *density* ρ of the material of the body are constant. No heat is produced or disappears in the body.

2. Experiments show that, in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient (cf. Sec. 9.7) of the temperature; that is, the velocity $\mathbf{v}$ of the heat flow in the body is of the form

$$\mathbf{v} = -K \operatorname{grad} u \tag{1}$$

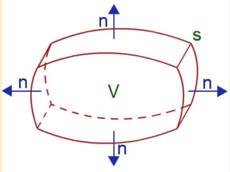
where $u(x, y, z, t)$ is the temperature at a point (x, y, z) and time t.

3. The *thermal conductivity* K is constant, as is the case for homogeneous material and nonextreme temperatures.

https://www.youtube.com/watch?v=TvlIfSlLB0c

Heat equation

Heat equation



Let V be a region in space bounded by a surface S, with outer unit normal vector $\mathbf{n}$. Then

$$\mathbf{v} \cdot \mathbf{n}$$

is the component of $\mathbf{v}$ (the velocity of heat flow) in the direction of $\mathbf{n}$, and

$$(\mathbf{v} \cdot \mathbf{n})dS$$

is the amount of heat leaving (if $\mathbf{v} \cdot \mathbf{n} > 0$) or entering V (if $\mathbf{v} \cdot \mathbf{n} < 0$) per unit time in a small portion of the surface of area dS. So the total amount of heat that flows through the whole surface is

$$\iint_S (\mathbf{v} \cdot \mathbf{n})dS = \iint_S ((-K\nabla u) \cdot \mathbf{n})dS$$

being K the thermal conductivity inside the body.

Heat equation

Heat equation

Now we use Gauss theorem

$$\boxed{\iint\limits_S \mathbf{v} \cdot \mathbf{n} dS = \iiint\limits_V \operatorname{div}(\mathbf{v}) dV = \iiint\limits_V \nabla \cdot \mathbf{v} dV}$$

to convert the total heat flow into

$$\iint\limits_S (\mathbf{v} \cdot \mathbf{n}) dS = \iiint\limits_V \nabla \cdot (-K \nabla u) dV = -K \iiint\limits_V \nabla^2 u dx dy dz$$

where ∇^2 is the Laplacian operator

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

Heat equation

The total amount of heat is

$$H = \iiint\limits_V \rho\sigma u dxdydz$$

where σ is the specific heat of the material and ρ its density. So, the time rate of decrease of heat is

$$-H_t = -\iiint\limits_V \rho\sigma u_t dxdydz$$

This must be equal to the amount of heat leaving the body since the body does not create heat or makes it disappear

$$-\iiint\limits_V \rho\sigma u_t dxdydz = -K\iiint\limits_V \nabla^2 u dxdydz$$

Heat equation

$$-\iiint\limits_V \rho\sigma u_t dxdydz = -K \iiint\limits_V \nabla^2 u dxdydz$$

$$\iiint\limits_V \left(\rho\sigma u_t - K\nabla^2 u\right) dxdydz = 0$$

$$\iiint\limits_V \left(u_t - \frac{K}{\rho\sigma}\nabla^2 u\right) dxdydz = 0$$

$$\iiint\limits_V \left(u_t - c^2\nabla^2 u\right) dxdydz = 0 \quad c^2 = \frac{K}{\rho\sigma}$$

Since this holds for every region in the body, the integrand must be 0 everywhere

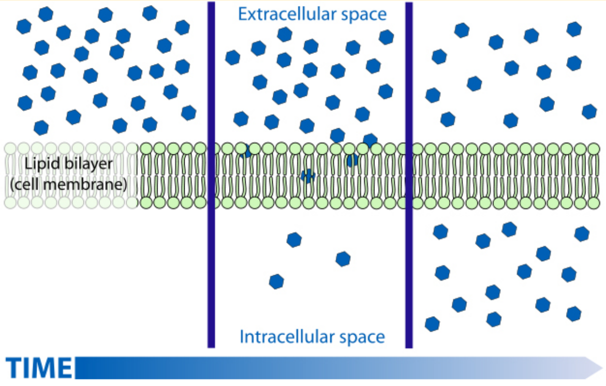
$$\boxed{u_t - c^2\nabla^2 u = 0} \Rightarrow u_t = c^2(u_{xx} + u_{yy} + u_{zz})$$

Diffusion equation

Diffusion equation

Heat equation is also the diffusion equation

$$u_t - c^2\nabla^2 u = 0$$



https://www.youtube.com/watch?v=RBFkmRapqts

Outline

1 Partial Differential Equations
- Basic concepts
- Vibrating string. Wave equation
- D'Alembert's solution of the wave equation. Characteristics
- Heat flow from a body in space. Heat equation
- 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
- 1D Heat equation: Solution by Fourier integrals and transforms
- Membrane, 2D Wave equation
- Rectangular membrane, double Fourier series
- Circular membrane, Fourier-Bessel series
- Laplace's equation in cylindrical and spherical coordinates. Potential

1D Heat equation

1D Heat equation

Fig. 294. Bar under consideration

$$u_t = c^2 u_{xx}$$

plus the boundary conditions

$$u(0,t) = 0 \quad u(L,t) = 0$$

plus the initial condition

$$u(x,0) = f(x)$$

The solution has three steps:

1. Separating variables
2. Satisfying the boundary conditions
3. Satisfying the initial condition

1D Heat equation

Separating variables

$$u_t = c^2 u_{xx}$$

Let us try a solution of the form

$$u(x,t) = F(x)G(t)$$

Substituting in the PDE we have

$$FG_t = c^2 F_{xx} G$$

$$\frac{G_t}{c^2 G} = \frac{F_{xx}}{F}$$

The left side depends only on t and the right side only on x, so it must be

$$\frac{G_t}{c^2 G} = \frac{F_{xx}}{F} = -p^2$$

[If their ratio is not negative, then the only solution is $u = 0$.]

1D Heat equation

Separating variables

$$\frac{G_t}{c^2 G} = \frac{F_{xx}}{F} = -p^2$$

This gives us the two equations

$$\boxed{\begin{array}{c} F_{xx} + p^2 F = 0 \\ G_t + c^2 p^2 G = 0 \end{array}}$$

Let us find the general solutions of both equations

$$F_{xx} + p^2 F = 0 \Rightarrow F = A\cos(px) + B\sin(px)$$

$$G_t + c^2 p^2 G = 0 \Rightarrow G = Ce^{-c^2 p^2 t}$$

$$\boxed{u(x,t) = (A\cos(px) + B\sin(px))e^{-c^2 p^2 t}}$$

1D Heat equation

Satisfying the boundary conditions

$$u(x,t) = (A\cos(px) + B\sin(px))e^{-c^2p^2t}$$

Let us impose the boundary conditions

$$u(0,t) = 0 = A$$

$$u(L,t) = 0 = B\sin(pL) \Rightarrow pL = n\pi$$

Let us define

$$\lambda_n = c\frac{n\pi}{L}$$

So the eigenfunctions of the problem are the functions

$$\boxed{u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}}$$

1D Heat equation

Satisfying the initial condition

The solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \boxed{\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}}$$

Let us impose the initial condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

So that B_n must be the coefficients of the sine Fourier series

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)} \quad n = 1, 2, \ldots$$

Example

Sinusoidal Initial Temperature

Find the temperature $u(x, t)$ in a laterally insulated copper bar 80 cm long if the initial temperature is $100 \sin(\pi x/80)$ °C and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C? First guess, then calculate. *Physical data for copper:* density 8.92 g/cm^3, specific heat 0.092 cal/(g °C), thermal conductivity 0.95 cal/(cm sec °C).

<u>Solution</u>

As stated in the problem

$$f(x) = 100 \sin\left(\frac{\pi}{80}x\right) \Rightarrow B_1 = 100, B_n = 0 \quad (n = 2, 3, \ldots)$$

Let us calculate $\lambda_1^2 = c^2\pi^2/L^2$, for that we need

$$c^2 = \frac{K}{\sigma\rho} = \frac{0.95\left[\frac{cal}{cm\cdot s\cdot{}^\circ C}\right]}{0.092\left[\frac{cal}{g\cdot{}^\circ C}\right] 8.92\left[\frac{g}{cm^3}\right]} = 1.158\left[\frac{cm^2}{s}\right]$$

Example (continued)

$$c^2 = 1.158 \left[\frac{cm^2}{s}\right]$$

$$\lambda_1 = c^2 \frac{\pi^2}{L^2} = 1.158 \left[\frac{cm^2}{s}\right] \frac{\pi^2}{(80)^2[cm^2]} = 1.785 \cdot 10^{-3}[s^{-1}]$$

So the solution is

$$\boxed{u(x,t) = 100 \sin\left(\frac{\pi}{80}x\right) e^{-1.785 \cdot 10^{-3} t}}$$

To calculate the time for the maximum temperature to drop to 50°C

$$100e^{-1.785 \cdot 10^{-3} t} = 50 \Rightarrow t = \frac{\log(0.5)}{-1.785 \cdot 10^{-3}} = 388[s] \approx 6.5[min]$$

1D Heat equation

Example

Let's solve the same problem with $n = 3$

$$f(x) = 100 \sin\left(3\frac{\pi}{80}x\right)$$

Solution

$$B_3 = 100, B_n = 0 \quad (n = 1, 2, 4, 5, \ldots)$$

$$\lambda_3 = 3^2 \lambda_1^2 = 1.607 \cdot 10^{-2}$$

$$\boxed{u(x,t) = 100 \sin\left(3\frac{\pi}{80}x\right) e^{-1.607 \cdot 10^{-2} t}}$$

$$100 e^{-1.607 \cdot 10^{-2} t} = 50 \Rightarrow t = \frac{\log(0.5)}{-1.607 \cdot 10^{-2}} = 43[s]$$

1D Heat equation

Example

Let's solve the same problem with insulated ends Solution
The equation and initial conditions remain the same

$$u_t = c^2 u_{xx}$$

$$u(x,0) = f(x)$$

But the boundary conditions change to

$$u_x(0,t) = 0 \quad u_x(L,t) = 0$$

Since the equation has not changed the solution is still of the form

$$u(x,t) = (A\cos(px) + B\sin(px))e^{-c^2p^2t}$$

Let us calculate $u_x(x,t)$

$$u_x(x,t) = F_x(x)G(t) = (-Ap\sin(px) + Bp\cos(px))e^{-c^2p^2t}$$

Example (continued)

$$u_x(x,t) = (-Ap\sin(px) + Bp\cos(px))e^{-c^2p^2t}$$

The the two boundary conditions imply

$$u_x(0,t) = 0 = Bp$$

Let us choose $B = 0$, otherwise, the number of solutions is rather limited.

$$u_x(L,t) = 0 = -Ap\sin(pL) \Rightarrow pL = n\pi \Rightarrow p_n = \frac{n\pi}{L}$$

Then, we have the eigenfunctions

$$u_n(x,t) = A_n\cos(p_n x)\, e^{-c^2p_n^2t} = A_n\cos\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t} \quad \lambda_n = \frac{cn\pi}{L}$$

Note that now $n = 0, 1, 2, \ldots$ instead of $n = 1, 2, \ldots$, that is, we can have the solution

$$u_0 = A_0$$

1D Heat equation

Example (continued)

$$u_n(x,t) = A_n \cos(p_n x)\, e^{-c^2 p_n^2 t} = A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} \quad \lambda_n = \frac{cn\pi}{L}$$

The particular solution must be of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}$$

Imposing the initial condition

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

That is the A_n coefficients are the coefficients of the Fourier cosine series of $f(x)$.

1D Heat equation

Example (continued)

$$u_n(x,t) = A_n \cos(p_n x)\, e^{-c^2 p_n^2 t} = A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t} \quad \lambda_n = \frac{cn\pi}{L}$$

The particular solution must be of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}$$

Imposing the initial condition

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

That is the A_n coefficients are the coefficients of the Fourier cosine series of $f(x)$.

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

2D Heat equation

Steady 2D heat problems. Laplace's equation

The 2D heat problem

$$u_t = c^2 \nabla^2 u = c^2 (u_{xx} + u_{yy})$$

In **steady state**, there is no variation with time

$$0 = u_{xx} + u_{yy}$$

The boundary value problem is

- A **Dirichlet problem** if u is known on the boundary of a region R.
- A **Neumann problem** if the normal derivative of u, $u_n = \frac{\partial u}{\partial n}$, is known on the boundary of a region R.
- A **Robin problem** if u is known on a part of the boundary and u_n on the rest.

Dirichlet's problem

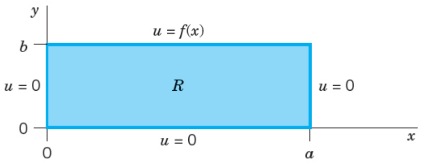


Fig. 296. Rectangle R and given boundary values

We solve the problem by separating variables

$$u(x, y) = F(x)G(y)$$

$$F_{xx}G + FG_{yy} = 0$$

Dividing by FG

$$\frac{F_{xx}}{F} = -\frac{G_{yy}}{G}$$

2D Heat equation

Dirichlet's problem

$$\frac{F_{xx}}{F} = -\frac{G_{yy}}{G}$$

The left part depends on x and the right part on y, so it must be

$$\frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = -k$$

$$\frac{F_{xx}}{F} = -k \Rightarrow F_{xx} + kF = 0 \Rightarrow F = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$$

The boundary conditions imply

$$F(0) = 0 = A$$

$$F(a) = 0 = B\sin(\sqrt{k}a) \Rightarrow \sqrt{k}a = n\pi \Rightarrow k = \left(\frac{n\pi}{a}\right)^2$$

The non-zero solutions are $\boxed{F_n(x) = \sin\left(\frac{n\pi}{a}x\right)}$

2D Heat equation

Dirichlet's problem

$$\frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = -k$$

$$-\frac{G_{yy}}{G} = -k \Rightarrow G_{yy} - kG = 0$$

$$G_n = A_n e^{-\sqrt{k}y} + B_n e^{\sqrt{k}y} = \boxed{A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}}$$

The boundary conditions

$$G_n(0) = 0 = A_n + B_n \Rightarrow B_n = -A_n$$

This gives

$$G_n = A_n e^{\frac{n\pi}{a}y} - A_n e^{-\frac{n\pi}{a}y} = 2A_n \sinh\left(\frac{n\pi}{a}y\right)$$

The eigenfunctions are thus

$$\boxed{u_n(x,y) = F_n(x)G_n(y) = A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)}$$

2D Heat equation

Dirichlet's problem

$$u_n(x,y) = F_n(x)G_n(y) = A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

and the particular solution

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)}$$

Finally we impose the boundary condition

$$u(x,b) = f(x) = \sum_{n=1}^{\infty}\left[A_n \sinh\left(\frac{n\pi}{a}b\right)\right]\sin\left(\frac{n\pi}{a}x\right)$$

That is $A_n \sinh\left(\frac{n\pi}{a}b\right)$ is the coefficient of $f(x)$ of the sine series

$$\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx = A_n \sinh\left(\frac{n\pi}{a}b\right)$$

Dirichlet's problem

$$\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx = A_n \sinh\left(\frac{n\pi}{a}b\right)$$

$$\boxed{A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx}$$

2D Heat equation

Dirichlet's problem

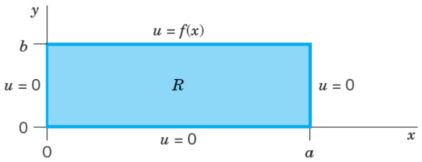


Fig. 296. Rectangle R and given boundary values

The solution found is the solution of ...

- ... the steady 2D heat problem.
- ... the electrostatic potential in the region R with the constraints shown.
- ... the displacement of a rubber band fixed on three sides and with the fourth side with a displacement $f(x)$.

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 12, Section 6:

- 12.6.11

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential

1D Heat equation

$$u_t = c^2 u_{xx}$$

Let us assume that the bar is very long (like a wire), it goes to infinity (from $-\infty$ to ∞). We do not have boundary conditions, but only the initial condition

$$u(x,0) = f(x) \quad (-\infty < x < \infty)$$

We use separation of variables $u(x,t) = F(x)G(t)$

$$FG_t = c^2 F_{xx} G$$

$$\frac{G_t}{G} = c^2 \frac{F_{xx}}{F} = -p^2$$

$$F_{xx} + p^2 F = 0 \Rightarrow F = A\cos(px) + B\sin(px)$$

$$G_t + c^2 p^2 G = 0 \Rightarrow G = e^{-c^2 p^2 t}$$

The solution is $\boxed{u(x,t) = (A\cos(px) + B\sin(px))e^{-c^2 p^2 t}}$

1D Heat equation

$$u_t = c^2 u_{xx}$$

Let us assume that the bar is very long (like a wire), it goes to infinity (from $-\infty$ to ∞). We do not have boundary conditions, but only the initial condition

$$u(x,0) = f(x) \quad (-\infty < x < \infty)$$

We use separation of variables $u(x,t) = F(x)G(t)$

$$FG_t = c^2 F_{xx} G$$

$$\frac{G_t}{G} = c^2 \frac{F_{xx}}{F} = -p^2$$

$$F_{xx} + p^2 F = 0 \Rightarrow F = A\cos(px) + B\sin(px)$$

$$G_t + c^2 p^2 G = 0 \Rightarrow G = e^{-c^2 p^2 t}$$

The eigenfunctions are $\boxed{u_p(x,t) = (A_p \cos(px) + B_p \sin(px))e^{-c^2 p^2 t}}$

1D Heat equation

1D Heat equation

The eigenfunctions are

$$u_p(x,t) = (A_p \cos(px) + B_p \sin(px))e^{-c^2p^2t}$$

and the solution

$$\boxed{u(x,t) = \int_0^\infty u_p(x,t)dp = \int_0^\infty (A_p \cos(px) + B_p \sin(px))e^{-c^2p^2t}dp}$$

1D Heat equation

$$u(x,t) = \int_0^\infty u_p(x,t)dp = \int_0^\infty (A_p \cos(px) + B_p \sin(px))e^{-c^2p^2t}dp$$

The initial condition implies

$$u(x,0) = f(x) = \int_0^\infty (A_p \cos(px) + B_p \sin(px))dp$$

But this is the Fourier integral (see Chapter 7) and the A_p and B_p coefficients are given by

$$A_p = \frac{1}{\pi}\int_0^\infty f(v)\cos(pv)dv \quad B_p = \frac{1}{\pi}\int_0^\infty f(v)\sin(pv)dv$$

1D Heat equation

As we saw in the case of the Fourier transform, the Fourier integral can be rewritten as

$$u(x,0) = \int_0^\infty (A_p \cos(px) + B_p \sin(px)) dp = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) \cos(px - pv) dv \right] dp$$

In the same way

$$\begin{aligned} u(x,t) &= \int_0^\infty (A_p \cos(px) + B_p \sin(px)) e^{-c^2p^2t} dp \\ &= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) \cos(px - pv) dv \right] e^{-c^2p^2t} dp \\ &= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) \cos(px - pv) e^{-c^2p^2t} dv \right] dp \\ &= \frac{1}{\pi} \int_{-\infty}^\infty f(v) \left[\int_0^\infty \cos(px - pv) e^{-c^2p^2t} dp \right] dv \end{aligned}$$

1D Heat equation

$$u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\int_{0}^{\infty} \cos(px - pv) e^{-c^2 p^2 t} dp \right] dv$$

Now, we can evaluate the inner integral using

$$\int_{0}^{\infty} \cos(2bs) e^{-s^2} ds = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

If we make the change of variable

$$s = cp\sqrt{t} \Rightarrow p = \frac{s}{c\sqrt{t}}, dp = \frac{ds}{c\sqrt{t}}$$

we obtain

$$\int_{0}^{\infty} \cos(px - pv) e^{-c^2 p^2 t} dp = \int_{0}^{\infty} \cos\left(\frac{s}{c\sqrt{t}}(x - v)\right) e^{-s^2} \frac{ds}{c\sqrt{t}}$$

1D Heat equation

$$\int_0^\infty \cos\left(\tfrac{s}{c\sqrt{t}}(x-v)\right) e^{-s^2} \tfrac{ds}{c\sqrt{t}} = \tfrac{1}{c\sqrt{t}} \int_0^\infty \cos\left(\tfrac{x-v}{c\sqrt{t}} s\right) e^{-s^2} ds \quad \left[b = \tfrac{1}{2}\tfrac{x-v}{c\sqrt{t}}\right]$$

$$= \tfrac{1}{c\sqrt{t}} \int_0^\infty \cos(2bs)\, e^{-s^2} ds = \tfrac{1}{c\sqrt{t}} \tfrac{\sqrt{\pi}}{2} e^{-b^2} = \tfrac{\sqrt{\pi}}{2c\sqrt{t}} e^{-\frac{(x-v)^2}{4c^2 t}}$$

Substituting in the solution

$$u(x,t) = \tfrac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\int_0^\infty \cos(px - pv) e^{-c^2 p^2 t} dp\right] dv = \tfrac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\tfrac{\sqrt{\pi}}{2c\sqrt{t}} e^{-\frac{(x-v)^2}{4c^2 t}}\right] dv$$

$$= \boxed{\frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) e^{-\frac{(x-v)^2}{4c^2 t}} dv} \quad \left[z = \tfrac{v-x}{2c\sqrt{t}}\right]$$

$$= \boxed{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2cz\sqrt{t}) e^{-z^2} dz}$$

1D Heat equation

Example

Find the temperature in the infinite bar if

$$f(x) = \begin{cases} T_0 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

<u>Solution:</u>

$$u(x,t) = \frac{T_0}{2c\sqrt{\pi t}} \int_{1}^{1} e^{-\frac{(x-v)^2}{4c^2 t}} dv$$

Example

$$u(x,t) = \frac{T_0}{2c\sqrt{\pi t}} \int_{1}^{1} e^{-\frac{(x-v)^2}{4c^2 t}}\, dv$$

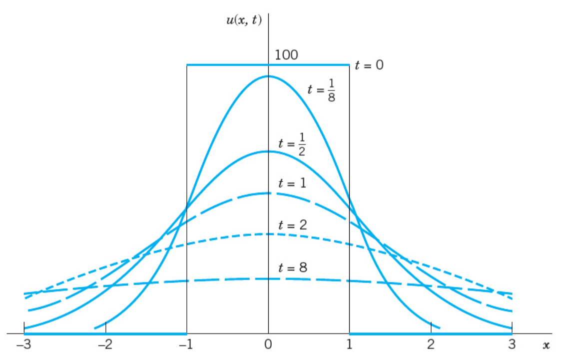


Fig. 299. Solution $u(x, t)$ in Example 1 for $U_0 = 100°C$, $c^2 = 1\,cm^2/sec$, and several values of t

1D Heat equation

Example: with Fourier transforms

Let us solve the same problem using Fourier transforms (that are useful for problems that extend from $-\infty$ to ∞)

Solution:

$$u_t = c^2 u_{xx}$$

Let's take the Fourier transform with respect to x of both sides

$$\mathcal{F}_x\{u_t\} = c^2 \mathcal{F}_x\{u_{xx}\}$$

If we now consider u as only a function of x (and not (x, t)), then

$$\begin{aligned} \mathcal{F}_x\{u_t\} &= \tfrac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u_t e^{-i\omega x} dx = \tfrac{1}{\sqrt{2\pi}} \tfrac{\partial}{\partial t} \int\limits_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx \\ &= \tfrac{\partial \mathcal{F}_x\{u\}}{\partial t} = \tfrac{\partial \hat{u}(\omega,t)}{\partial t} = \hat{u}_t \end{aligned}$$

1D Heat equation

Example: with Fourier transforms

The PDE becomes

$$u_t = c^2 u_{xx} \Rightarrow \hat{u}_t = -c^2\omega^2\hat{u}$$

$$\frac{d\hat{u}}{\hat{u}} = -c^2\omega^2 dt$$

$$\log \hat{u} = -c^2\omega^2 t + C(\omega)$$

$$\hat{u}(\omega, t) = C(\omega)e^{-c^2\omega^2 t}$$

The initial condition makes

$$\hat{u}(\omega, 0) = \mathcal{F}_x\{f(x)\} = \hat{f}(\omega) = C(\omega)$$

Finally we calculate the inverse Fourier transform

$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-c^2\omega^2 t}e^{i\omega x}dx}$$

1D Heat equation

Example: with convolutions

We can further elaborate the previous answer

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-c^2\omega^2 t} e^{i\omega x} dx$$

by realizing that it can be written as the product of two functions in Fourier space

$$u(x,t) = \int_{-\infty}^{\infty} \left(\hat{f}(\omega)\right) \left(\frac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t}\right) e^{i\omega x} dx$$

then,

$$u(x,t) = f(x) \star_x g(x,t)$$

where $g(x,t)$ is the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t}$

1D Heat equation

Example: with convolutions

We know the Fourier transform

$$\mathcal{F}\{e^{-ax^2}\} = \frac{1}{\sqrt{2a}} e^{-\frac{1}{4a}\omega^2}$$

Consequently our function $\frac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t}$ has an inverse Fourier transform given by

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\tfrac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t}\right\} &= \tfrac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{e^{-c^2\omega^2 t}\right\} = \tfrac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{e^{-\frac{1}{4\frac{1}{4c^2 t}}\omega^2}\right\} \\
&= \tfrac{1}{\sqrt{2\pi}} \sqrt{2\tfrac{1}{4c^2 t}}\, e^{-\frac{1}{4c^2 t}x^2} = \tfrac{1}{\sqrt{4\pi c^2 t}} e^{-\frac{x^2}{4c^2 t}}
\end{aligned}$$

$$u(x,t) = f(x) \star_x g(x,t) = \boxed{\frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} f(p) e^{-\frac{(x-p)^2}{4c^2 t}}\, dp}$$

Exercises

Exercises

From Kreyszig (10th ed.), Chapter 12, Section 7:

- 12.7.3

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential

2D Wave equation

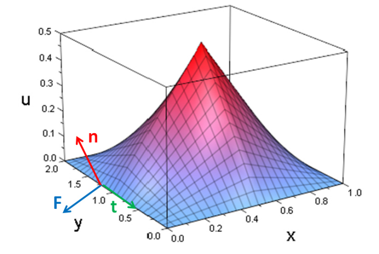
https://www.youtube.com/watch?v=34C6oKF4tag

Physical Assumptions

1. The mass of the membrane per unit area is constant ("homogeneous membrane"). The membrane is perfectly flexible and offers no resistance to bending.
2. The membrane is stretched and then fixed along its entire boundary in the xy-plane. The tension per unit length T caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.
3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

2D Wave equation

2D Wave equation



n is the unit normal vector at each point of the edge of the membrane.
t is the unit tangent vector at each point of the edge of the membrane.
The tensile force acting at each point of the edge of the membrane is

$$\mathbf{F} = T_0(\mathbf{t} \times \mathbf{n})$$

2D Wave equation

2D Wave equation

Since movement is vertical, we concentrate in this direction

$$F_z = T_0(\mathbf{t} \times \mathbf{n}) \cdot \mathbf{e}_3$$

and this force translates into a local acceleration of the membrane

$$\boxed{\iint_R \rho u_{tt} dA = \int_{\partial R} T_0(\mathbf{t} \times \mathbf{n}) \cdot \mathbf{e}_3 dl}$$

where R is the whole membrane, dA a differential area of it, ρ its mass density so that ρdA is the mass of the differential area, ∂R is the boundary of the membrane, and dl a differential arc length of it.

2D Wave equation

We now make use of the triple vector product

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$$

to transform

$$\iint_R \rho u_{tt} dA = \int_{\partial R} T_0 (\mathbf{t} \times \mathbf{n}) \cdot \mathbf{e}_3 dl$$

into

$$\iint_R \rho u_{tt} dA = \int_{\partial R} T_0 (\mathbf{n} \times \mathbf{e}_3) \cdot \mathbf{t} dl$$

Now we use Stokes' theorem that transforms an integral of a force on the boundary of a region into the integral of the curl of the force in the region

$$\int_{\partial R} \mathbf{F} \cdot \mathbf{t} dl = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA$$

That is

$$\boxed{\iint_R \rho u_{tt} dA = \iint_R T_0 \left[\nabla \times (\mathbf{n} \times \mathbf{e}_3)\right] \cdot \mathbf{n} dA}$$

2D Wave equation

$$\iint_R \rho u_{tt} dA = \iint_R T_0 \left[\nabla \times (\mathbf{n} \times \mathbf{e}_3)\right] \cdot \mathbf{n} dA$$

since the identity holds for any region R, we must have

$$\boxed{\rho u_{tt} = T_0 \left[\nabla \times (\mathbf{n} \times \mathbf{e}_3)\right] \cdot \mathbf{n}}$$

The surface of the membrane is given by

$$z = u(x, y)$$

and its normal is given by

$$\mathbf{n} = \frac{-u_x \mathbf{e}_1 - u_y \mathbf{e}_2 + \mathbf{e}_3}{\sqrt{(u_x)^2 + (u_y)^2 + 1}}$$

If we have small displacements, then $\mathbf{n} \approx -u_x \mathbf{e}_1 - u_y \mathbf{e}_2 + \mathbf{e}_3$

2D Wave equation

$$\mathbf{n} \approx -u_x\mathbf{e}_1 - u_y\mathbf{e}_2 + \mathbf{e}_3$$

Now

$$\mathbf{n} \times \mathbf{e}_3 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -u_x & -u_y & 1 \\ 0 & 0 & 1 \end{vmatrix} = -u_y\mathbf{e}_1 + u_x\mathbf{e}_2$$

Now, let's calculate the curl of this force

$$\nabla \times (\mathbf{n} \times \mathbf{e}_3) = \nabla \times (-u_y\mathbf{e}_1 + u_x\mathbf{e}_2) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial_x & \partial_y & \partial_z \\ -u_y & u_x & 0 \end{vmatrix} = (u_{xx} + u_{yy})\mathbf{e}_3$$

Finally

$$\nabla \times (\mathbf{n} \times \mathbf{e}_3) \cdot \mathbf{e}_3 = ((u_{xx} + u_{yy})\mathbf{e}_3)(-u_x\mathbf{e}_1 - u_y\mathbf{e}_2 + \mathbf{e}_3) = u_{xx} + u_{yy}$$

2D Wave equation

2D Wave equation

$$\nabla \times (\mathbf{n} \times \mathbf{e}_3) \cdot \mathbf{e}_3 = ((u_{xx} + u_{yy})\mathbf{e}_3)(-u_x\mathbf{e}_1 - u_y\mathbf{e}_2 + \mathbf{e}_3) = u_{xx} + u_{yy}$$

The PDE

$$\rho u_{tt} = T_0 \left[\nabla \times (\mathbf{n} \times \mathbf{e}_3)\right] \cdot \mathbf{n}$$

becomes

$$\rho u_{tt} = T_0(u_{xx} + u_{yy})$$

$$u_{tt} = \frac{T_0}{\rho}(u_{xx} + u_{yy})$$

$$\boxed{u_{tt} = c^2(u_{xx} + u_{yy})}$$

Laplace's equation: Steady-state $u_{xx} + u_{yy} = 0$

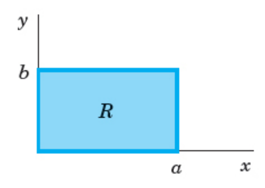
Poisson's equation: Steady-state with external force $u_{xx} + u_{yy} = f(x, y)$

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential

2D Wave equation on a rectangular membrane

2D Wave equation on a rectangular membrane



Let us solve the problem

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

$$u = 0 \quad \text{on the boundary}$$

$$u(x, y, 0) = f(x, y)$$

$$u_t(x, y, 0) = g(x, y)$$

The solution will have 3 steps:

1. Separating variables
2. Finding eigenfunctions satisfying the boundary conditions
3. Finding solution satisfying initial conditions

2D Wave equation on a rectangular membrane

Separating variables

Let's find a solution of the form

$$u(x, y, t) = F(x, y)G(t)$$

The PDE

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

translates

$$FG_{tt} = c^2(F_{xx} + F_{yy})G$$

$$\frac{G_{tt}}{c^2 G} = \frac{F_{xx} + F_{yy}}{F}$$

The left side depends on t while the second on x and y, so actually both must be constant. In fact, a negative constant (otherwise, the only solution is $u = 0$)

$$\frac{G_{tt}}{c^2 G} = \frac{F_{xx} + F_{yy}}{F} = -\nu^2$$

2D Wave equation on a rectangular membrane

Separating variables

$$\frac{G_{tt}}{c^2 G} = \frac{F_{xx} + F_{yy}}{F} = -\nu^2 \Rightarrow \begin{cases} G_{tt} + c^2\nu^2 G = 0 \\ F_{xx} + F_{yy} + \nu^2 F = 0 \end{cases}$$

Let's analyze the equation (Helmholtz's equation)

$$F_{xx} + F_{yy} + \nu^2 F = 0$$

and solve it by separating variables

$$F(x, y) = H(x)Q(y)$$

$$H_{xx}Q + HQ_{yy} + \nu^2 HQ = 0$$

$$\frac{H_{xx}}{H} + \frac{Q_{yy}}{Q} + \nu^2 = 0$$

2D Wave equation on a rectangular membrane

Separating variables

$$\frac{H_{xx}}{H} + \frac{Q_{yy}}{Q} + \nu^2 = 0$$

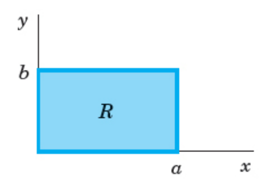
$$\frac{H_{xx}}{H} = -\left(\frac{Q_{yy}}{Q} + \nu^2\right) = -k^2$$

$$\begin{cases} H_{xx} + k^2 H = 0 & & \Rightarrow H = A\cos(kx) + B\sin(kx) \\ Q_{yy} + p^2 Q = 0 & [p^2 = \nu^2 - k^2] & \Rightarrow Q = C\cos(py) + D\sin(py) \end{cases}$$

2D Wave equation on a rectangular membrane

Satisfying boundary conditions

$$u(x,y,t) = F(x,y)G(t) = H(x)Q(y)G(t)$$



$$u = 0 \quad \text{on the boundary} \Rightarrow$$

$$\begin{cases} u(0,y,t) = 0 = H(0)Q(y)G(t) \Rightarrow H(0) = 0 \\ u(a,y,t) = 0 = H(a)Q(y)G(t) \Rightarrow H(a) = 0 \\ u(x,0,t) = 0 = H(x)Q(0)G(t) \Rightarrow Q(0) = 0 \\ u(x,b,t) = 0 = H(x)Q(b)G(t) \Rightarrow Q(b) = 0 \end{cases}$$

$$H = A\cos(kx) + B\sin(kx) \Rightarrow \begin{cases} H(0) = 0 \Rightarrow A = 0 \\ H(a) = 0 \Rightarrow B\sin(ka) = 0 \Rightarrow ka = m\pi \end{cases}$$

$$Q = C\cos(py) + D\sin(py) \Rightarrow \begin{cases} Q(0) = 0 \Rightarrow C = 0 \\ Q(b) = 0 \Rightarrow D\sin(pb) = 0 \Rightarrow pb = n\pi \end{cases}$$

The eigenfunctions are $\boxed{F_{mn}(x,y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}$ $\quad m,n = 1,2,\ldots$

Satisfying boundary conditions

Let us solve now for the time dependence

$$G_{tt} + c^2\nu^2 G = 0 \Rightarrow G = A_g \cos(c\nu t) + B_g \sin(c\nu t)$$

Remind that

$$k = \frac{m\pi}{a} \quad p = \frac{n\pi}{b} \quad p^2 = \nu^2 - k^2$$

from where

$$\nu_{mn} = \sqrt{p^2 + k^2} = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

The eigenvalue is

$$\boxed{\lambda_{mn} = c\nu_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad m, n = 1, 2, \ldots}$$

$$G_{mn} = B_{mn} \cos(\lambda_{mn} t) + B^*_{mn} \sin(\lambda_{mn} t)$$

2D Wave equation on a rectangular membrane

Satisfying boundary conditions

The eigenfunctions are

$$\begin{aligned} u_{mn}(x,y,t) &= F_{mn}(x,y)G_{mn}(t) \\ &= \boxed{\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\left(B_{mn}\cos(\lambda_{mn}t)+B^*_{mn}\sin(\lambda_{mn}t)\right)} \end{aligned}$$

The frequency of each one of these modes is

$$f_{mn} = \frac{\lambda_{mn}}{2\pi}$$

Note that there can be several modes associated to the same frequency (as shown in the following example)

2D Wave equation on a rectangular membrane

Vibration modes

Consider $a = b = 1$, the eigenvalues are

$$\lambda_{mn} = c\pi\sqrt{m^2 + n^2} \Rightarrow \lambda_{mn} = \lambda_{nm}$$

but its eigenfunctions are different

$$u_{mn} = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right)$$

$$u_{nm} = \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\left(B_{nm}\cos(\lambda_{nm}t) + B^*_{nm}\sin(\lambda_{nm}t)\right)$$

Vibration modes (continued)

$$u_{mn} = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\left(B_{mn}\cos(\lambda_{mn}t) + B_{mn}^{*}\sin(\lambda_{mn}t)\right)$$

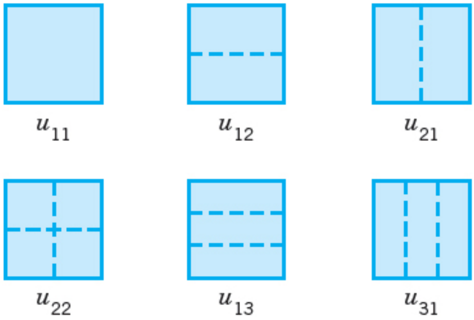


Fig. 303. Nodal lines of the solutions u_{11}, u_{12}, u_{21}, u_{22}, u_{13}, u_{31} in the case of the square membrane

2D Wave equation on a rectangular membrane

Satisfying initial conditions: Double Fourier series

The solution of the PDE is of the form

$$\begin{aligned} u(x,y,t) &= \sum_{m,n=1}^{\infty} u_{mn}(x,y,t) \\ &= \sum_{m,n=1}^{\infty} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right) \end{aligned}$$

The initial condition $u(x,y,0) = f(x,y)$ imposes

$$\begin{aligned} u(x,y,0) = f(x,y) &= \sum_{m,n=1}^{\infty} B_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \\ &= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} B_{mn}\sin\left(\frac{n\pi}{b}y\right)\right)\sin\left(\frac{m\pi}{a}x\right) \\ &= \sum_{m=1}^{\infty} K_m(y)\sin\left(\frac{m\pi}{a}x\right) \end{aligned}$$

Satisfying initial conditions: Double Fourier series

$$f(x,y) = \sum_{m=1}^{\infty} K_m(y)\sin\left(\frac{m\pi}{a}x\right)$$

That is, if we consider a fixed value of y, then $K_m(y)$ are the Fourier coefficients of the sine Fourier series of $f(x,y)$

$$K_m(y) = \frac{2}{a}\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right)dx$$

On its turn $K_m(y) = \sum_{n=1}^{\infty} B_{mn}\sin\left(\frac{n\pi}{b}y\right)$ That is B_{mn} are the Fourier coefficients of the sine Fourier series of $K_m(y)$

$$B_{mn} = \frac{2}{b}\int_0^b K_m(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

2D Wave equation on a rectangular membrane

Satisfying initial conditions: Double Fourier series

$$
\begin{aligned}
B_{mn} &= \frac{2}{b}\int_0^b K_m(y)\sin\left(\frac{n\pi}{b}y\right)dy \\
&= \frac{2}{b}\int_0^b \left(\frac{2}{a}\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right)dx\right)\sin\left(\frac{n\pi}{b}y\right)dy \\
&= \boxed{\frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dxdy}
\end{aligned}
$$

This is called the **double Fourier series**. It exists as long as f, f_x, f_y, f_{xy} are continuous functions in R.

2D Wave equation on a rectangular membrane

Satisfying initial conditions: Double Fourier series

The other initial condition is $u_t(x, y, 0) = g(x, y)$

$$\begin{aligned}
u_t(x, y, t) &= \frac{\partial}{\partial t}\left[\sum_{m,n=1}^{\infty} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right)\right] \\
&= \sum_{m,n=1}^{\infty} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\lambda_{mn}\left(-B_{mn}\sin(\lambda_{mn}t) + B^*_{mn}\cos(\lambda_{mn}t)\right) \\
u_t(x, y, 0) &= \sum_{m,n=1}^{\infty} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\lambda_{mn}B^*_{mn}\cos(\lambda_{mn}t)
\end{aligned}$$

and with a development analogous to the previous one

$$B^*_{mn} = \boxed{\frac{4}{ab}\int_0^b\int_0^a g(x, y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dxdy}$$

2D Wave equation on a rectangular membrane

Solution

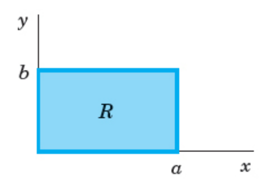
Summarizing, the solution is

$$u(x,y,t) = \sum_{m,n=1}^{\infty} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\left(B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right)$$

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \quad \left[c = \frac{T_0}{\rho}\right]$$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dxdy$$

$$B^*_{mn} = \frac{4}{ab}\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dxdy$$



2D Wave equation on a rectangular membrane

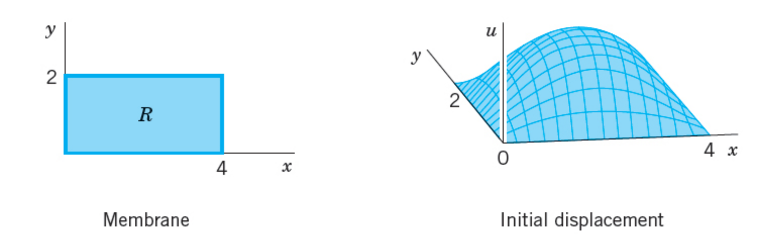
Example

$$f(x,y) = 0.1(4x - x^2)(2y - y^2)[ft]$$

$$g(x,y) = 0$$

$$\rho = 2.5[slugs/ft^2]$$

$$T_0 = 12.5[lb/ft]$$



2D Wave equation on a rectangular membrane

Example (continued)

Solution:

$$c = \frac{T_0}{\rho} = \frac{12.5[lb/ft]}{2.5[slugs/ft^2]} = 5[ft^2/s^2]$$

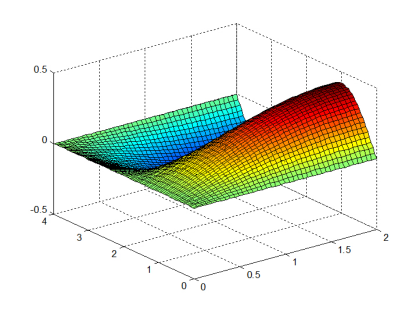
$$g = 0 \Rightarrow B^*_{mn} = 0$$

$$\begin{aligned} B_{mn} &= \frac{4}{2\cdot 8}\int_0^2\int_0^4 0.1(4x - x^2)(2y - y^2)\sin\left(\frac{m\pi}{4}x\right)\sin\left(\frac{n\pi}{2}y\right)dxdy \\ &= \begin{cases} \frac{256\cdot 32}{20\pi^6 m^3 n^3} \approx \frac{0.426050}{m^3 n^3} & m,n \neq \dot{2} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} u = 0.42605 \quad & \left(\sin\left(\frac{\pi x}{4}\right)\sin\left(\frac{\pi y}{2}\right)\cos\left(\frac{\sqrt{5}\pi\sqrt{5}}{4}t\right) + \quad [m=1, n=1] \right. \\ & \frac{1}{27}\sin\left(\frac{\pi x}{4}\right)\sin\left(\frac{3\pi y}{2}\right)\cos\left(\frac{\sqrt{5}\pi\sqrt{37}}{4}t\right) + \quad [m=1, n=3] \\ & \frac{1}{27}\sin\left(\frac{3\pi x}{4}\right)\sin\left(\frac{\pi y}{2}\right)\cos\left(\frac{\sqrt{5}\pi\sqrt{13}}{4}t\right) + \quad [m=3, n=1] \\ & \frac{1}{729}\sin\left(\frac{3\pi x}{4}\right)\sin\left(\frac{3\pi y}{2}\right)\cos\left(\frac{\sqrt{5}\pi\sqrt{45}}{4}t\right) + \quad [m=3, n=3] \\ & \left. \ldots \right) \end{aligned}$$

Example (continued)

```
[X,Y] = meshgrid(0:.05:2, 0:0.05:4);
for t=0:0.01:pi/2
 u=0.42605*cos(5*pi/4*t).*sin(pi/4*X).*sin(pi/2*Y);
 surf(X,Y,u)
 axis([0 2 0 4 -0.5 0.5])
 pause(0.05)
end
```



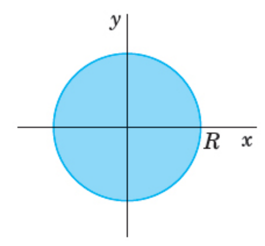
Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential

2D Wave equation on a circular membrane

http://www.youtube.com/watch?v=v4ELxKKT5Rw

$$u_{tt} = c^2(u_{xx} + u_{yy})$$



We make the change of variables

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \operatorname{atan}\frac{y}{x} \end{aligned}$$

Whose derivatives are

$$\begin{aligned}
r_x &= \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} & r_y &= \frac{y}{r} \\
r_{xx} &= \frac{r - xr_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3} & r_{yy} &= \frac{x^2}{r^3} \\
\theta_x &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2} & \theta_y &= \frac{x}{r^2} \\
\theta_{xx} &= -y\left(-\frac{2}{r^3}\right) r_x = \frac{2xy}{r^4} & \theta_{yy} &= -\frac{2xy}{r^4}
\end{aligned}$$

2D Wave equation on a circular membrane

Let us now calculate

$$
\begin{aligned}
u_x &= u_r r_x + u_\theta \theta_x \\
u_y &= u_r r_y + u_\theta \theta_y \\
u_{xx} &= (u_r r_x)_x + (u_\theta \theta_x)_x \\
&= (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx} \\
&= (u_{rr} r_x + u_{r\theta} \theta_x) r_x + u_r r_{xx} + (u_{\theta r} r_x + u_{\theta\theta} \theta_x) \theta_x + u_\theta \theta_{xx} \\
u_{yy} &= (u_{rr} r_y + u_{r\theta} \theta_y) r_y + u_r r_{yy} + (u_{\theta r} r_y + u_{\theta\theta} \theta_y) \theta_y + u_\theta \theta_{yy}
\end{aligned}
$$

Substituting the values above, we get

$$
\begin{aligned}
u_{xx} &= \tfrac{x^2}{r^2} u_{rr} - 2\tfrac{xy}{r^3} u_{r\theta} + \tfrac{y^2}{r^4} u_{\theta\theta} + \tfrac{y^2}{r^3} u_r + 2\tfrac{xy}{r^4} u_\theta \\
u_{yy} &= \tfrac{y^2}{r^2} u_{rr} + 2\tfrac{xy}{r^3} u_{r\theta} + \tfrac{x^2}{r^4} u_{\theta\theta} + \tfrac{x^2}{r^3} u_r - 2\tfrac{xy}{r^4} u_\theta
\end{aligned}
$$

Summing

$$
\boxed{\nabla^2 u} = u_{xx} + u_{yy} = \frac{x^2+y^2}{r^2} u_{rr} + \frac{y^2+x^2}{r^4} u_{\theta\theta} + \frac{y^2+x^2}{r^3} u_r = \boxed{u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}}
$$

2D Wave equation on a circular membrane

The wave equation becomes

$$u_{tt} = c^2(u_{xx} + u_{yy})$$
$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \quad c^2 = \frac{T}{\rho}$$

For the moment, we will study radially symmetric solutions so $u_{\theta\theta} = 0$ and the 2D wave equation with boundary and initial conditions becomes

$$\boxed{\begin{array}{l} u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right) \quad c^2 = \frac{T}{\rho} \\ u(R,t) = 0 \\ u(r,0) = f(r) \\ u_t(r,0) = g(r) \end{array}}$$

The solution involves:

1. Separating variables
2. Satisfying the boundary conditions
3. Satisfying the initial conditions

2D Wave equation on a circular membrane

Separating variables. Bessel's equation

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r \right)$$

Let us find a solution of the form $u(r,t) = W(r)G(t)$

$$WG_{tt} = c^2 \left(W_{rr} G + \frac{1}{r} W_r G \right)$$

$$\frac{G_{tt}}{c^2 G} = \left(\frac{W_{rr}}{W} + \frac{1}{r} \frac{W_r}{W} \right) = -k^2 \Rightarrow \begin{cases} G_{tt} + c^2 k^2 G = 0 \\ W_{rr} + r^{-1} W_r + k^2 W = 0 \end{cases}$$

2D Wave equation on a circular membrane

Separating variables. Bessel's equation

Let us analyze

$$W_{rr} + r^{-1}W_r + k^2W = 0$$

Let us make the change of variable $s = kr$, then

$$W_r = W_s s_r = W_s k$$

$$W_{rr} = (W_s k)_s s_r = k^2 W_{ss}$$

and the ODE is

$$k^2W_{ss} + \frac{k}{s}kW_s + k^2W = 0$$

$$W_{ss} + s^{-1}kW_s + W = 0$$

This is Bessel's equation with $\nu = 0$

$$y'' + x^{-1}y' + \frac{x^2 - \nu^2}{x^2}y = 0$$

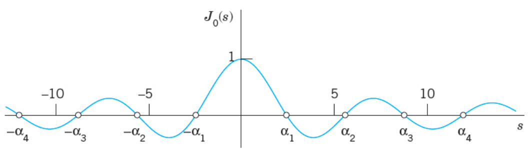
2D Wave equation on a circular membrane

Boundary conditions

$$W_{ss} + s^{-1} k W_s + W = 0$$

The solution of Bessel's equation is

$$W(r) = J_0(s) = J_0(kr)$$



On the boundary we have

$$W(R) = 0 = J_0(kR) \Rightarrow k_m = \frac{\alpha_m}{R} \quad m = 1, 2, \ldots$$

$$\alpha_1 = 2.4048 \quad \alpha_2 = 5.5201 \quad \alpha_3 = 8.6537 \quad \ldots$$

2D Wave equation on a circular membrane

Eigenvalues and eigenfunctions

So the solutions

$$\boxed{W_m(r) = J_0(k_m r)}$$

are solutions that vanish at the boundary. We now solve for the time equation

$$G_{tt} + c^2 k_m^2 G = 0 \Rightarrow \boxed{G_m = A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t)} \quad \left[\lambda_m = c k_m = c\frac{\alpha_m}{R}\right]$$

So the eigenfunction associated to the eigenvalue λ_m is

$$\boxed{u_m(r,t) = W_m(r) G_m(t) = J_0(k_m r)(A_m \cos(\lambda_m t) + B_m \sin(\lambda_m t))}$$

These are called vibration **normal modes** and their frequency is $\frac{\lambda_m}{2\pi}$.

2D Wave equation on a circular membrane

Eigenvalues and eigenfunctions

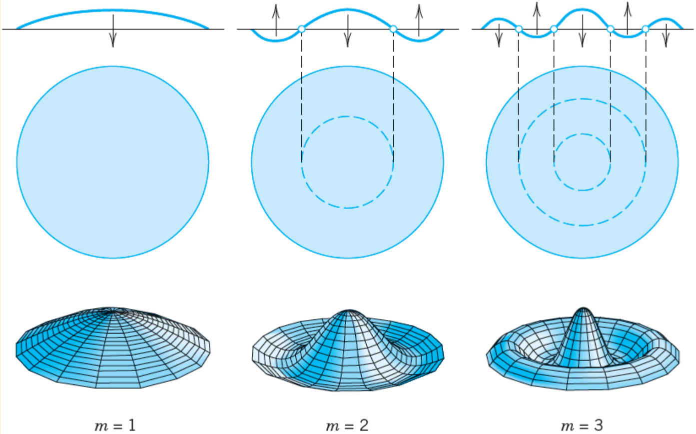


Fig. 309. Normal modes of the circular membrane in the case of vibrations independent of the angle

2D Wave equation on a circular membrane

Satisfying the initial conditions

The solution of the PDE is of the form

$$\boxed{u(r,t)=\sum_{m=1}^{\infty}u_m(r,t)=\sum_{m=1}^{\infty}J_0\left(\frac{\alpha_m}{R}r\right)\left(A_m\cos(\lambda_m t)+B_m\sin(\lambda_m t)\right)}$$

The initial condition $u(r,0)=f(r)$ implies

$$u(r,0)=f(r)=\sum_{m=1}^{\infty}A_m J_0\left(\frac{\alpha_m}{R}r\right)$$

This is the Fourier-Bessel series (see Chapter 7) whose coefficients are

$$\boxed{A_m=\frac{2}{R^2J_1^2(\alpha_m)}\int_0^R rf(r)J_0\left(\frac{\alpha_m}{R}r\right)}$$

2D Wave equation on a circular membrane

Satisfying the initial conditions

The initial condition $u_t(r,0) = g(r)$ implies

$$u_t(r,t) = \sum_{m=1}^{\infty} J_0\left(\frac{\alpha_m}{R}r\right)\left(-A_m\lambda_m \sin(\lambda_m t) + B_m\lambda_m \cos(\lambda_m t)\right)$$

$$u_t(r,0) = g(r) = \sum_{m=1}^{\infty} B_m\lambda_m J_0\left(\frac{\alpha_m}{R}r\right)$$

This is the Fourier-Bessel series (see Chapter 7) whose coefficients are

$$\lambda_m B_m = \frac{2}{R^2 J_1^2(\alpha_m)}\int_0^R r g(r) J_0\left(\frac{\alpha_m}{R}r\right) \Rightarrow \boxed{B_m = \frac{2}{\lambda_m R^2 J_1^2(\alpha_m)}\int_0^R r g(r) J_0\left(\frac{\alpha_m}{R}r\right)}$$

2D Wave equation on a circular membrane

Example

$R = 1[ft]$, $\rho = 2[slugs/ft^2]$, $T_0 = 8[lb/ft]$, $f(r) = 1 - r^2[ft]$, $g(r) = 0[ft/s]$

Solution

$$c^2 = \frac{T_0}{\rho} = \frac{8}{2} = 4[ft^2/s^2]$$

$$g = 0 \Rightarrow B_m = 0$$

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r(1-r)^2 J_0(\alpha_m) dr = \frac{8}{\alpha_m^3 J_1(\alpha_m)}$$

$$\begin{aligned} u(r,t) \quad = \quad & 1.108 J_0(2.4048r)\cos(4.8097t) \\ & -0.140 J_0(5.5201r)\cos(11.0402t) \\ & +0.045 J_0(8.6537r)\cos(17.3075t) \\ & -\ldots \end{aligned}$$

Exercises

4. TEAM PROJECT. Series for Dirichlet and Neumann Problems

(a) Show that $u_n = r^n \cos n\theta$, $u_n = r^n \sin n\theta$, $n = 0, 1, \cdots$, are solutions of Laplace's equation $\nabla^2 u = 0$ with $\nabla^2 u$ given by (5). (What would u_n be in Cartesian coordinates? Experiment with small n.)

(b) Dirichlet problem (See Sec. 12.6) Assuming that termwise differentiation is permissible, show that a solution of the Laplace equation in the disk $r < R$ satisfying the boundary condition $u(R, \theta) = f(\theta)$ (R and f given) is

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{r}{R}\right)^n \cos n\theta + b_n \left(\frac{r}{R}\right)^n \sin n\theta \right] \tag{20}$$

where a_n, b_n are the Fourier coefficients of f (see Sec. 11.1).

(c) Dirichlet problem. Solve the Dirichlet problem using (20) if $R = 1$ and the boundary values are $u(\theta) = -100$ volts if $-\pi < \theta < 0$, $u(\theta) = 100$ volts if $0 < \theta < \pi$. (Sketch this disk, indicate the boundary values.)

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - **Laplace's equation in cylindrical and spherical coordinates. Potential**

Laplace's equation

Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

It appears in

- Gravitation
- Electrostatics
- Steady-state heat flow
- Fluid flow

The theory of solutions is called **Potential theory** and its solutions with continuous second derivatives are called **harmonic functions**.
We normally use a coordinate system in which the boundary surface has a simple representation.

Laplace's equation in cylindrical coordinates

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

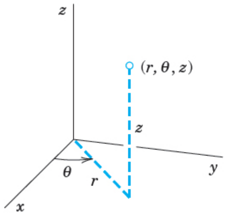


Fig. 311. Cylindrical coordinates ($r \geqq 0, 0 \leqq \theta \leqq 2\pi$)

$$\boxed{\begin{array}{l} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{array} \quad \Leftrightarrow \quad \begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \mathrm{atan}\frac{y}{x} \\ z = z \end{array}}$$

In the case of the circular 2D membrane we obtained

$$u_{xx} + u_{yy} = 0$$

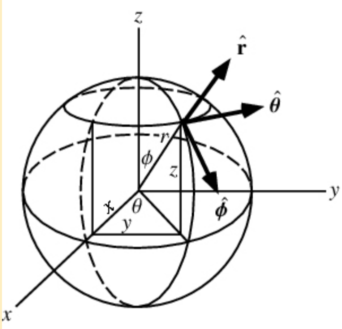
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

In cylindrical coordinates, we simply need to add u_{zz}

$$\boxed{u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0}$$

Laplace's equation in spherical coordinates

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$



$$\begin{aligned} x &= r\cos(\theta)\sin(\phi) \\ y &= r\sin(\theta)\sin(\phi) \\ z &= r\cos(\phi) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \operatorname{atan}\frac{y}{x} \\ \phi &= \operatorname{acos}\frac{z}{\sqrt{x^2+y^2+z^2}} \end{aligned}$$

Using a similar approach we get

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{1}{r^2\tan(\phi)}u_\phi + \frac{1}{r^2\sin^2(\phi)}u_{\theta\theta} = 0$$

Laplace's equation

Spherical harmonics

By separating variables, let us try a solution of the form

$$u(r,\theta,\phi) = R(r)Y(\theta,\phi)$$

The PDE becomes

$$R_{rr}Y + \frac{2}{r}R_rY + \frac{1}{r^2}RY_{\phi\phi} + \frac{1}{r^2\tan(\phi)}RY_\phi + \frac{1}{r^2\sin^2(\phi)}RY_{\theta\theta} = 0$$

Multiplying by $\frac{r^2}{RY}$

$$\left(r^2\frac{R_{rr}}{R} + 2r\frac{R_r}{R}\right) + \left(\frac{Y_{\phi\phi}}{Y} + \frac{1}{\tan(\phi)}\frac{Y_\phi}{Y} + \frac{1}{\sin^2(\phi)}\frac{Y_{\theta\theta}}{Y}\right) = 0$$

Laplace's equation

Spherical harmonics

$$\left(r^2 \frac{R_{rr}}{R} + 2r \frac{R_r}{R} \right) + \left(\frac{Y_{\phi\phi}}{Y} + \frac{1}{\tan(\phi)} \frac{Y_\phi}{Y} + \frac{1}{\sin^2(\phi)} \frac{Y_{\theta\theta}}{Y} \right) = 0$$

which gives the two equations

$$r^2 \frac{R_{rr}}{R} + 2r \frac{R_r}{R} = \lambda$$

$$\frac{Y_{\phi\phi}}{Y} + \frac{1}{\tan(\phi)} \frac{Y_\phi}{Y} + \frac{1}{\sin^2(\phi)} \frac{Y_{\theta\theta}}{Y} = -\lambda$$

Laplace's equation

Spherical harmonics

Let us solve the second equation

$$\frac{Y_{\phi\phi}}{Y} + \frac{1}{\tan(\phi)}\frac{Y_{\phi}}{Y} + \frac{1}{\sin^2(\phi)}\frac{Y_{\theta\theta}}{Y} = -\lambda$$

We also look for solutions with separated variables $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$

$$\frac{\Theta\Phi_{\phi\phi}}{\Theta\Phi} + \frac{1}{\tan(\phi)}\frac{\Theta\Phi_{\phi}}{\Theta\Phi} + \frac{1}{\sin^2(\phi)}\frac{\Theta_{\theta\theta}\Phi}{\Theta\Phi} = -\lambda$$

$$\frac{\Phi_{\phi\phi}}{\Phi} + \frac{1}{\tan(\phi)}\frac{\Phi_{\phi}}{\Phi} + \frac{1}{\sin^2(\phi)}\frac{\Theta_{\theta\theta}}{\Theta} = -\lambda$$

Multiplying by $\sin^2(\phi)$

$$\left(\sin^2(\phi)\frac{\Phi_{\phi\phi}}{\Phi} + \cos(\phi)\sin(\phi)\frac{\Phi_{\phi}}{\Phi}\right) + \left(\frac{\Theta_{\theta\theta}}{\Theta}\right) = -\lambda\sin^2(\phi)$$

Laplace's equation

Spherical harmonics

$$\left(\sin^2(\phi)\frac{\Phi_{\phi\phi}}{\Phi} + \cos(\phi)\sin(\phi)\frac{\Phi_\phi}{\Phi}\right) + \left(\frac{\Theta_{\theta\theta}}{\Theta}\right) = -\lambda\sin^2(\phi)$$

This gives the two equations

$$\frac{\Theta_{\theta\theta}}{\Theta} = -m^2$$

$$\sin^2(\phi)\frac{\Phi_{\phi\phi}}{\Phi} + \cos(\phi)\sin(\phi)\frac{\Phi_\phi}{\Phi} = m^2 - \lambda\sin^2(\phi)$$

The solution to the first one is

$$\boxed{\Theta_m(\theta) = C_m\cos(m\theta) + S_m\sin(m\theta)} \quad (m = 0, 1, 2, ...)$$

or

$$\boxed{\Theta_m(\theta) = A_m e^{jm\theta}} \quad (m = -\infty, ..., \infty)$$

Spherical harmonics

$$\sin^2(\phi)\frac{\Phi_{\phi\phi}}{\Phi} + \cos(\phi)\sin(\phi)\frac{\Phi_\phi}{\Phi} = m^2 - \lambda\sin^2(\phi)$$

$$\sin^2(\phi)\Phi_{\phi\phi} + \cos(\phi)\sin(\phi)\Phi_\phi = (m^2 - \lambda\sin^2(\phi))\Phi$$

$$\Phi_{\phi\phi} + \frac{\cos(\phi)}{\sin(\phi)}\Phi_\phi + \frac{\lambda\sin^2(\phi) - m^2}{\sin^2(\phi)}\Phi = 0$$

Now we do the change of variables

$$\begin{aligned}
x &= \cos(\phi) \Rightarrow \sin(\phi) = \sqrt{1-x^2} \\
\Phi_\phi &= \Phi_x x_\phi = \Phi_x(-\sin(\phi)) = -\sqrt{1-x^2}\Phi_x \\
\Phi_{\phi\phi} &= (\Phi_\phi)_x x_\phi = (-\sqrt{1-x^2}\Phi_x)_x(-\sqrt{1-x^2}) \\
&= -\left(-\frac{x}{\sqrt{1-x^2}}\Phi_x + \sqrt{1-x^2}\Phi_{xx}\right)(-\sqrt{1-x^2}) \\
&= -x\Phi_x + (1-x^2)\Phi_{xx}
\end{aligned}$$

Laplace's equation

Spherical harmonics

$$\Phi_{\phi\phi} + \frac{\cos(\phi)}{\sin(\phi)}\Phi_\phi + \frac{\lambda\sin^2(\phi) - m^2}{\sin^2(\phi)}\Phi = 0$$

$$-x\Phi_x + (1-x^2)\Phi_{xx} + \frac{x}{\sqrt{1-x^2}}(-\sqrt{1-x^2}\Phi_x) + \frac{\lambda(1-x^2) - m^2}{1-x^2}\Phi = 0$$

$$(1-x^2)\Phi_{xx} - 2x\Phi_x + \left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = 0$$

If $\underline{m = 0}$, then

$$(1-x^2)\Phi_{xx} - 2x\Phi_x + \lambda\Phi = 0$$

This is Legendre's equation with $\lambda = l(l+1)$ and its solution is

$$\Phi(x) = P_l(x) = P_l(\cos(\phi))$$

being $P_l(x)$ Legendre's polynomial of order l.

Spherical harmonics

$$(1-x^2)\Phi_{xx} - 2x\Phi_x + \left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = 0$$

If $m \neq 0$, then this is the associated Legendre's equation with $\lambda = l(l+1)$ and its solution is

$$\Phi_{ml}(x) = P_l^m(x) = P_l^m(\cos(\phi)) \quad m = -l, -l+1, ..., l-1, l$$

being

$$P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}} \frac{d^m P_l(x)}{dx^m}$$

Spherical harmonics

So the eigenfunctions of

$$\frac{Y_{\phi\phi}}{Y} + \frac{1}{\tan(\phi)}\frac{Y_{\phi}}{Y} + \frac{1}{\sin^2(\phi)}\frac{Y_{\theta\theta}}{Y} = -\lambda$$

is

$$\boxed{Y_{ml}(\theta,\phi) = \Theta_m(\theta)\Phi_{ml}(\phi) = e^{im\theta}P_l^m(\cos(\phi))} \quad m = -l, -l+1, ..., l-1, l$$

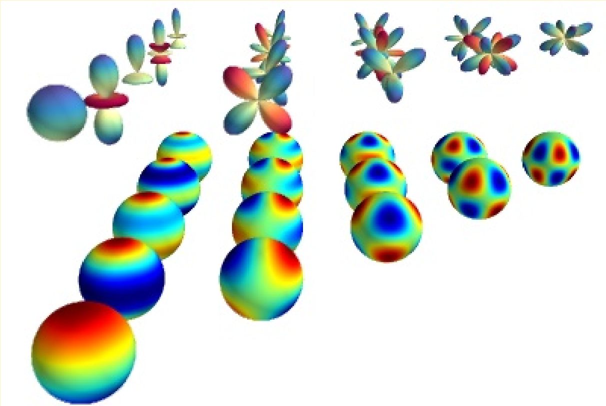
These functions are called **Spherical harmonics**, and $\lambda = l(l+1)$. The general solution of the ODE can be expressed as

$$\begin{aligned} Y(\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{ml}e^{im\theta}P_l^m(\cos(\phi)) \\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{l} (C_{ml}\cos(m\theta) + S_{ml}\sin(m\theta))P_l^m(\cos(\phi)) \end{aligned}$$

Spherical harmonics

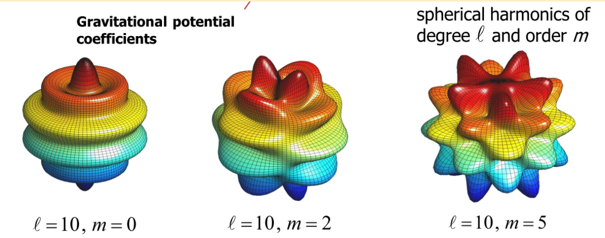
$$Y_{ml}(\theta,\phi)^{re} = \cos(m\theta)P_l^m(\cos(\phi))$$

$$Y_{ml}(\theta,\phi)^{im} = \sin(m\theta)P_l^m(\cos(\phi))$$



Laplace's equation

Spherical harmonics



Laplace's equation

Spherical harmonics

The radial component can be calculated from

$$r^2 \frac{R_{rr}}{R} + 2r \frac{R_r}{R} = \lambda = l(l+1)$$

This is an Euler-Cauchy equation whose solution is

$$\boxed{R_l(r) = A_l r^l + B_l r^{-l-1}}$$

Laplace's equation

Spherical harmonics

Finally, the general solution of the Laplacian problem is

$$\begin{aligned} u(r,\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_l(r)Y_{ml}(\theta,\phi) \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (A_l r^l + B_l r^{-l-1})A_{ml}e^{im\theta}P_l^m(\cos(\phi)) \end{aligned}$$

Outline

1. Partial Differential Equations
   - Basic concepts
   - Vibrating string. Wave equation
   - D'Alembert's solution of the wave equation. Characteristics
   - Heat flow from a body in space. Heat equation
   - 1D Heat equation: Solution by Fourier series. Steady 2D heat problems. Dirichlet problem
   - 1D Heat equation: Solution by Fourier integrals and transforms
   - Membrane, 2D Wave equation
   - Rectangular membrane, double Fourier series
   - Circular membrane, Fourier-Bessel series
   - Laplace's equation in cylindrical and spherical coordinates. Potential