A Primer on Complex Numbers

1. Imaginary and complex numbers.

One of the fundamental properties of the real numbers is that the *square* of a real number is always *nonnegative*. I.e., if x is a real number, then $x^2 \geq 0$. This implies, among other things, that certain *quadratic* equations don't have real solutions. In particular, the equation $x^2 = -1$, has *no solution* in real numbers. This doesn't mean however, that the equation cannot have a solution.

By the 16th century, it became apparent to various mathematicians that $\sqrt{-1}$ would be very useful and this new number, and its multiples, entered slowly into common use. As with all important constants, a special symbol was eventually designated to represent $\sqrt{-1}$.

> **Definition 1.**
> *An* ***imaginary number*** *is a number of the form $b\boldsymbol{i}$, where b is real and*
> $$\boldsymbol{i} = \sqrt{-1}. \tag{1.1}$$

With the set of imaginary numbers in hand, we can find a square root for *every* real number, positive or negative. Indeed, following the usual algebraic rules, we have

$$(b\boldsymbol{i})^2 = b^2 \cdot \boldsymbol{i}^2 = b^2 \cdot (-1) = -b^2,$$

for any real number b. Now, if $\alpha < 0$ and $b = \sqrt{|\alpha|}$ (remember, $|\alpha| > 0$ so it has a real square root), then $-b^2 = \alpha$, so

$$b\boldsymbol{i} = \sqrt{\alpha}.$$

Thus, for example, $\sqrt{-9} = 3\boldsymbol{i}$, $\sqrt{-100} = 10\boldsymbol{i}$ and $\sqrt{-2} = \sqrt{2} \cdot \boldsymbol{i}$.

More is true. By combining real and imaginary numbers, we can solve any quadratic equation.

Example 1.1. *Solve the equation* $x^2 + 2x + 2 = 0$. Using the quadratic formula, we find the two solutions

$$z_1 = \frac{-2 + \sqrt{4-8}}{2} = -1 + 2\boldsymbol{i} \quad \text{and} \quad z_2 = \frac{-2 - \sqrt{4-8}}{2} = -1 - 2\boldsymbol{i}.$$

Note that the two solutions are neither real numbers, nor are they purely imaginary.

Definition 2.

*A **complex number** is a number of the form* $z = a + b\boldsymbol{i}$, *, where* a *and* b *are real numbers, and* $\boldsymbol{i} = \sqrt{-1}$*. The (real) numbers* a *and* b *are called the **real** and **imaginary** parts of* z*, respectively, and we often use the notation*

$$a = \mathbf{Re}(z) \quad \text{and} \quad b = \mathbf{Im}(z).$$

If $\mathbf{Re}(z) = 0$, then z is an imaginary number and if $\mathbf{Im}(z) = 0$, then z is a real number. In other words, the real numbers and the imaginary numbers are subsets of the complex numbers. The boldfaced letter $\mathbb{R}$ is used to denote the set of real numbers and the boldfaced letter $\mathbb{C}$ is used to denote the set of all complex numbers.

The two solutions of the quadratic equation in Example 1.1 have the same real part and their imaginary parts are opposite. I.e., using the notation above, we have $\mathbf{Re}(z_1) = \mathbf{Re}(z_2)$ and $\mathbf{Im}(z_1) = -\mathbf{Im}(z_2)$. This is not a coincidence, and there is also a name for this.

Definition 3.

*The **complex conjugate** of* $a + b\boldsymbol{i}$ *is the number* $a - b\boldsymbol{i}$*. We use a bar over the number to denote the conjugate, i.e.,* $\overline{a + b\boldsymbol{i}} = a - b\boldsymbol{i}$*.*

If a, b and c are *real* numbers, and $b^2 - 4ac < 0$, then the quadratic equation

$$ax^2 + bx + c = 0 \tag{1.2}$$

has *no* real solutions. But, as in Example 1.1, there are always *complex* solutions and these solutions come in *conjugate* pairs. Namely, the solutions to equation (1.2) are given by $\alpha + \beta\boldsymbol{i}$ and $\alpha - \beta\boldsymbol{i}$, where (from the quadratic formula)

$$\alpha = -\frac{b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{|b^2 - 4ac|}}{2a}.$$

Observe that if $z = a = a + 0 \cdot \boldsymbol{i}$ is a *real* number, then $\bar{z} = a - 0 \cdot \boldsymbol{i} = a = z$. In other words, real numbers are their own complex conjugates. Another thing to note is that $\bar{\bar{z}} = z$ for any complex number z (check!), that is, the conjugate of the conjugate is the original number. We'll see a geometric interpretation of these two observations a little later.

Comments:

a. Complex numbers first appeared explicitly in the work of the 16th century Italian mathematician Cardano. The term '*imaginary number*' was introduced later by Descartes who did not think highly of the concept. His opinion notwithstanding, complex numbers have important *real* applications, as we will see.

b. Introducing a 'new' number as the solution of an equation that didn't already have a solution among the accepted set of numbers was not new, even in the 16th century. The *irrational* number $\sqrt{2}$ was introduced around 2500 years ago, when it became apparent that the equation $x^2 = 2$ had no *rational* solutions.

Exercises

1.1. Find the solutions of the equation $2x^2 + 4x + 5 = 0$.

1.2. Find the real and imaginary parts of the solutions of the equation $x^2+3x+5 = 0$.

1.3. Show that $\overline{\overline{z}} = z$ for any complex number z.

2. *Complex arithmetic.*

Complex numbers may be added and multiplied, just like real numbers, and the usual properties (commutativity, associativity and distributivity) continue to hold.

2.1 *Addition and subtraction.*

To add two complex numbers, we add their real and imaginary components separately and use the distributive rule $b\boldsymbol{i} + d\boldsymbol{i} = (b + d)\boldsymbol{i}$ for the imaginary parts. In other words,

$$(a + b\boldsymbol{i}) + (c + d\boldsymbol{i}) = (a + c) + (b + d)\boldsymbol{i}.$$

Subtraction is defined similarly:

$$(a + b\boldsymbol{i}) - (c + d\boldsymbol{i}) = (a - c) + (b - d)\boldsymbol{i}.$$

Example 2.1. If $z_1 = 3 + 2\boldsymbol{i}$, $z_2 = 4 - 3\boldsymbol{i}$ and $z_3 = -3 + 5\boldsymbol{i}$, then

$$z_1 + z_2 = (3 + 2\boldsymbol{i}) + (4 - 3\boldsymbol{i}) = (3 + 4) + (2 - 3)\boldsymbol{i} = 7 - \boldsymbol{i}$$

and

$$z_2 - z_3 = (4 - 3\boldsymbol{i}) - (-3 + 5\boldsymbol{i}) = (4 + 3) + (-3 - 5)\boldsymbol{i} = 7 - 8\boldsymbol{i}.$$

Addition of complex numbers satisfies the usual rules of *commutativity* and *associativity*. I.e., if u, v and w are complex numbers then

$$u + v = v + u \quad \text{and} \quad (u + v) + w = u + (v + w).$$

To see this, we write $u = a + b\boldsymbol{i}$, $v = c + d\boldsymbol{i}$ and $w = e + f\boldsymbol{i}$, and check, using the fact that these properties hold for addition of *real* numbers:

$$(a + b\boldsymbol{i}) + (c + d\boldsymbol{i}) = (a + c) + (b + d)\boldsymbol{i} = (c + a) + (d + b)\boldsymbol{i} = (c + d\boldsymbol{i}) + (a + b\boldsymbol{i})$$

and

$$\begin{aligned}[(a + b\boldsymbol{i}) + (c + d\boldsymbol{i})] + (e + f\boldsymbol{i}) &= [(a + c) + e] + [(b + d) + f]\boldsymbol{i} \\ &= [a + (c + e)] + [b + (d + f)]\boldsymbol{i} \\ &= (a + b\boldsymbol{i}) + [(c + d\boldsymbol{i}) + (e + f\boldsymbol{i})].\end{aligned}$$

The number 0 continues to be the *additive identity*, i.e., $z + 0 = z$ for every complex number z, as you can check for yourself. Also, every complex number has an *opposite* with respect to addition. Specifically, $-(a+b\boldsymbol{i}) = -a - b\boldsymbol{i}$, since

$$a + b\boldsymbol{i} + (-a - b\boldsymbol{i}) = (a-a) + (b-b)\boldsymbol{i} = 0.$$

Proposition 2.1. *For any complex number z,*

$$z + \overline{z} = 2 \cdot \mathbf{Re}(z) \quad \text{and} \quad z - \overline{z} = 2 \cdot \mathbf{Im}(z)\boldsymbol{i}.$$

Proof: Suppose that $z = a + b\boldsymbol{i}$, then

$$z + \overline{z} = (a + b\boldsymbol{i}) + (a - b\boldsymbol{i}) = (a + a) + (b - b)\boldsymbol{i} = 2a = 2\mathbf{Re}(z).$$

Likewise,

$$z - \overline{z} = (a + b\boldsymbol{i}) - (a - b\boldsymbol{i}) = 2b\boldsymbol{i} = 2\mathbf{Im}(z)\boldsymbol{i}.$$

$\square$

2.2 Multiplication.

To multiply the numbers $a + b\boldsymbol{i}$ and $c + d\boldsymbol{i}$, we use the familiar '*foil*' rule from elementary algebra while keeping in mind that $\boldsymbol{i}^2 = -1$, and collecting real and imaginary terms:

$$(a + b\boldsymbol{i}) \cdot (c + d\boldsymbol{i}) = ac + ad\boldsymbol{i} + cb\boldsymbol{i} + bd\boldsymbol{i}^2 = (ac - bd) + (ad + bc)\boldsymbol{i}. \tag{2.1}$$

Example 2.2. Using the numbers from Example 2.1, we have

$$\begin{aligned} z_1 \cdot z_2 &= (3 + 2\boldsymbol{i})(4 - 3\boldsymbol{i}) \\ &= 12 - 9\boldsymbol{i} + 8\boldsymbol{i} - 6\boldsymbol{i}^2 \\ &= (12 - (-6)) + (-9 + 8)\boldsymbol{i} = 18 - \boldsymbol{i} \end{aligned}$$

and

$$\begin{aligned} z_3 \cdot z_1 &= (-3 + 5\boldsymbol{i})(3 + 2\boldsymbol{i}) \\ &= -9 - 6\boldsymbol{i} + 15\boldsymbol{i} + 10\boldsymbol{i}^2 \\ &= (-9 + (-10)) + (15 + 10)\boldsymbol{i} \\ &= -19 + 25\boldsymbol{i}. \end{aligned}$$

When one of the two factors is real, then multiplication is particularly simple, since real numbers have no imaginary parts. E.g., $7 \cdot z_2 = 7(4 - 3\boldsymbol{i}) = 28 - 21\boldsymbol{i}$.

As with addition, the usual properties of commutativity and associativity hold for multiplication of complex numbers, as you can verify for yourself (see exercises). And, multiplication distributes over addition, as usual:

$$\begin{aligned} (a + b\boldsymbol{i}) \cdot \big((c + d\boldsymbol{i}) + (e + f\boldsymbol{i})\big) &= (a + b\boldsymbol{i}) \cdot \big((c + e) + (d + f)\boldsymbol{i}\big) \\ &= \big(a(c + e) - b(d + f)\big) + \big(a(d + f)) + b(c + e)\big)\, \boldsymbol{i} \\ &= (ac + ae - bd - bf) + (ad + af + bc + be)\boldsymbol{i} \\ &= \big((ac - bd) + (ad + bc)\boldsymbol{i}\big) + \big((ae - bf) + (af + be)\boldsymbol{i}\big) \\ &= (a + b\boldsymbol{i})(c + d\boldsymbol{i}) + (a + b\boldsymbol{i})(e + f\boldsymbol{i}). \end{aligned}$$

2.3 Division.

Division is properly thought of as multiplication by the inverse, where the inverse of a complex number z is the complex number z^{-1} satisfying $z \cdot z^{-1} = 1$. As with the real numbers, a complex number has a multiplicative inverse if and only if it is not zero. Finding the multiplicative inverse is simply a matter of solving a pair of linear equations in two unknowns.

Specifically, given a (nonzero) complex number $a + b\boldsymbol{i}$, we want to find a complex number $x + y\boldsymbol{i}$ such that $(a + b\boldsymbol{i})(x + y\boldsymbol{i}) = 1$. Using the rule for multiplication of complex numbers this gives

$$(ax - by) + (ay + bx)\boldsymbol{i} = 1 = 1 + 0\boldsymbol{i},$$

which reduces to a pair of linear equations in the variables x and y:

$$ax - by = 1 \tag{2.2}$$

$$bx + ay = 0. \tag{2.3}$$

Multiplying equation (2.2) by a and equation (2.3) by b and adding the results together gives

$$(a^2x - aby) + (b^2x + aby) = a + 0 \implies (a^2 + b^2)x = a \implies \boxed{x = \frac{a}{a^2 + b^2}}.$$

Note that division by $(a^2 + b^2)$ is allowed since $a + b\boldsymbol{i} \neq 0$, which means that either $a \neq 0$, $b \neq 0$ or both, so $a^2 + b^2 > 0$.

In similar fashion, multiplying equation (2.2) by $-b$ and equation (2.3) by a and adding the results together gives

$$(-abx + b^2y) + (abx + a^2y) = -b + 0 \implies (a^2 + b^2)y = -b \implies \boxed{y = -\frac{b}{a^2 + b^2}}.$$

We can summarize these calculations as follows:

Proposition 2.2. *If $a + b\boldsymbol{i} \neq 0$, then*

$$(a + b\boldsymbol{i})^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\boldsymbol{i}.$$

Now, to divide $c + d\boldsymbol{i}$ by $a + b\boldsymbol{i}$, we multiply by $(a + b\boldsymbol{i})^{-1}$:

$$\frac{c + d\boldsymbol{i}}{a + b\boldsymbol{i}} = (c+d\boldsymbol{i}) \cdot (a+b\boldsymbol{i})^{-1} = (c+d\boldsymbol{i}) \cdot \left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\boldsymbol{i}\right) = \frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2}\boldsymbol{i}.$$

Example 2.3. *Compute $\frac{z_1}{z_2}$ and $\frac{z_1}{z_3}$, for the numbers z_1, z_2 and z_3 from Example 2.1.*

$$\frac{z_1}{z_2} = \frac{3 + 2\boldsymbol{i}}{4 - 3\boldsymbol{i}} = \frac{12 - 6}{25} + \frac{-9 - 8}{25}\boldsymbol{i} = \frac{6}{25} - \frac{17}{25}\boldsymbol{i},$$

and

$$\frac{z_1}{z_3} = \frac{3 + 2\boldsymbol{i}}{-3 + 5\boldsymbol{i}} = \frac{-9 + 10}{34} + \frac{-15 - 6}{34}\boldsymbol{i} = \frac{1}{34} - \frac{21}{34}\boldsymbol{i}.$$

2.4 Arithmetic and conjugation.

The complex conjugate plays nicely with addition, multiplication and division.

Proposition 2.3. *For complex numbers z and w,* $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$; *and if $w \neq 0$, then* $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$.

Proof: Let $z = a + b\boldsymbol{i}$ and $w = c + d\boldsymbol{i}$ (where a, b, c and d are real, as usual). Then for addition we have

$$\begin{aligned} \overline{(a+b\boldsymbol{i}) + (c+d\boldsymbol{i})} &= \overline{(a+c) + (b+d)\boldsymbol{i}} \\ &= (a+c) - (b+d)\boldsymbol{i} \\ &= (a-b\boldsymbol{i}) + (c-d\boldsymbol{i}) = \overline{(a+b\boldsymbol{i})} + \overline{(c+d\boldsymbol{i})}; \end{aligned}$$

and for multiplication we have

$$\begin{aligned} \overline{(a+b\boldsymbol{i}) \cdot (c+d\boldsymbol{i})} &= \overline{(ac-bd) + (ad+bc)\boldsymbol{i}} \\ &= (ac-bd) - (ad+bc)\boldsymbol{i} \\ &= (a-b\boldsymbol{i}) \cdot (c-d\boldsymbol{i}) = \overline{(a+b\boldsymbol{i})} \cdot \overline{(c+d\boldsymbol{i})} \end{aligned}$$

The proof for the case of division is left as an exercise. □

Exercises

2.1. Let $u = 3 + 2\boldsymbol{i}$ and $v = 5 - 2\boldsymbol{i}$. Compute $u + v$, $u \cdot v$, u/v and $v/\overline{u}$.

2.2. Compute the following products, quotients and powers:
(a) $(1 + 4\boldsymbol{i}) \cdot (2 - 3\boldsymbol{i}) =$
(b) $2/(3 + \boldsymbol{i}) =$
(c) $(1 + \boldsymbol{i})^3 =$
(d) $(2 - \boldsymbol{i})^{-2} =$

2.3. Compute $(1 + \boldsymbol{i})^{-1}$ and $(3 - 4\boldsymbol{i})^{-1}$.

2.4. Let $\rho = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$. Compute ρ^2, ρ^3 and ρ^4. Can you guess what ρ^{25} is?

2.5. Solve the pair of equations

$$\begin{aligned} \boldsymbol{i}u + v &= 1 \\ 4u - 2\boldsymbol{i}v &= 2. \end{aligned}$$

The solution will be a pair of complex numbers, u and v.

2.6. Show that if $w \neq 0$, then $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$.

2.7. Show that if n is a positive integer and ζ is a complex number, then $\overline{\zeta^n} = \left(\overline{\zeta}\right)^n$.†

2.8. Show that $u \cdot v = v \cdot u$ for any two complex numbers u and v.

2.9. Show that $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ for any three complex numbers u, v and w.

† ζ is the greek letter *zeta*.

3. *The geometry of complex numbers.*

3.1 Complex numbers as points in the plane.

Every complex number has two real coordinates, namely its real and imaginary parts. It is natural therefore to represent complex numbers as points in what is called the ***complex plane***. In this representation, the convention is to plot the real part of the complex number on the horizontal axis and the imaginary part on the vertical axis. For this reason, the horizontal axis of the complex plane is called the ***real axis*** and the vertical axis is called the ***imaginary axis***. In this representation, the real numbers correspond to the points on the real axis and the (purely) imaginary numbers correspond to the points on the imaginary axis. The real and imaginary parts of a complex number are also called the *rectangular coordinates* of that number.

To each complex number we can also associate a *vector*: the directed line segment (arrow) whose tail is at the origin and whose head is at the point in the plane corresponding to the complex number. Addition of complex numbers can be described geometrically, using vectors. Given complex numbers u and v, thought of as vectors in the complex plane, *translate* the vector v so that it's tail is at the head of u (or *vice versa*, translate the tail of u to the head of v), then the head of the translated vector is the point corresponding to the sum $u+v$, see Figure 1.[†] We'll learn a useful geometric description of multiplication once we have studied the representation of complex numbers using *polar-coordinates*.

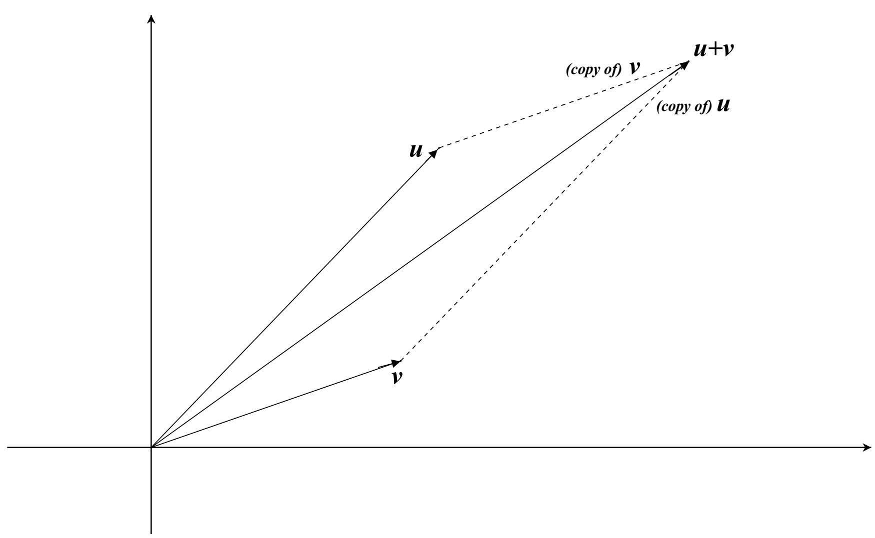

FIGURE 1. Addition, viewed geometrically.

[†]There is a 'coordinate-free' description of addition as well. The sum $u+v$ is the complex number such that the triangle with vertices $0, u$ and v in the complex plane is *congruent* to the triangle with vertices u, v and $u+v$.

3.2 The modulus of a complex number.

The *magnitude* of a real number a is given by its *absolute value*, $|a|$, which may be described as the *distance* of the number from 0. This idea is easily extended to all complex numbers.

Definition 4.

The ***modulus****, or* ***absolute value****, of a complex number $z = a + b\boldsymbol{i}$, is denoted by $|z|$ and defined to be the* ***distance*** *in the complex plane between the point z and the point 0, (see Figure 2). Equivalently, $|z|$ is the* ***length*** *of the vector corresponding to z.*

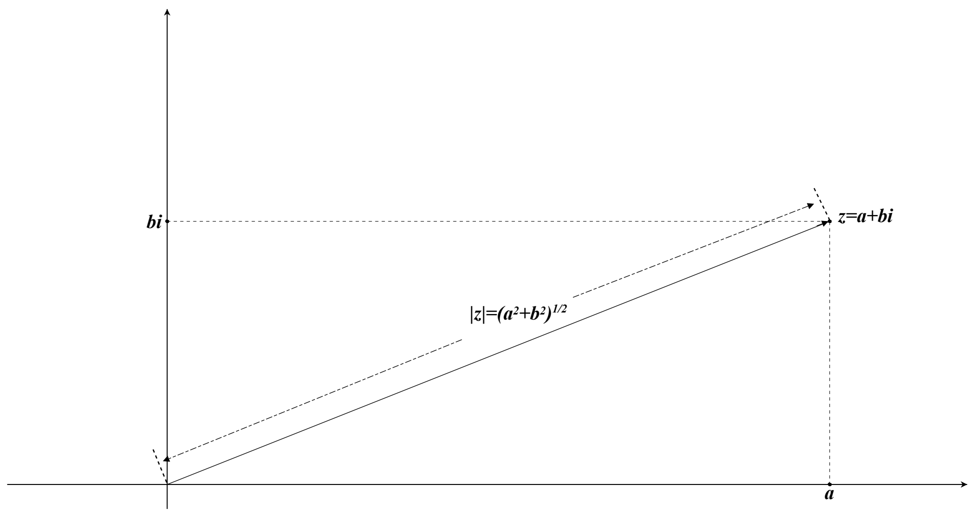

FIGURE 2. The modulus (absolute value) of z.

Comments:

a. The modulus of a complex number may be computed from its rectangular coordinates (the real and imaginary parts) using the Pythagorean theorem. I.e.,

$$|a + b\boldsymbol{i}| = \sqrt{a^2 + b^2}.$$

b. It follows directly from the definition that $|\overline{z}| = |z|$, since

$$|a - b\boldsymbol{i}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |a + b\boldsymbol{i}|.$$

Proposition 3.1. *For any complex number z*

$$|z|^2 = z \cdot \overline{z}.$$

Proof: Let $z = a + b\boldsymbol{i}$, then $|z|^2 = a^2 + b^2$ and

$$z \cdot \overline{z} = (a + b\boldsymbol{i})(a - b\boldsymbol{i}) = a^2 - ab\boldsymbol{i} + ab\boldsymbol{i} - b^2\boldsymbol{i}^2 = a^2 + b^2,$$

since $\boldsymbol{i}^2 = -1$. $\square$

Using the modulus and the conjugate, we can also shorten the expression for z^{-1}. Namely, if $z \neq 0$, then

$$z^{-1} = \frac{\overline{z}}{|z|^2}, \tag{3.1}$$

as you can check by comparing to Proposition 2.2. It is also important to note that the modulus is a *multiplicative* function. I.e., the modulus of a product is equal to the product of the moduli.

Proposition 3.2. *For any complex numbers z and w, $|z \cdot w| = |z| \cdot |w|$.*

Proof: It follows from Propositions 3.1 and 2.3 and the commutative property of multiplication that

$$|z \cdot w|^2 = (z \cdot w) \cdot \overline{(z \cdot w)} = z \cdot w \cdot \overline{z} \cdot \overline{w} = (z \cdot \overline{z}) \cdot (w \cdot \overline{w}) = |z|^2 \cdot |w|^2.$$

Taking square roots of both sides then gives

$$|z \cdot w| = \sqrt{|z \cdot w|^2} = \sqrt{|z|^2 \cdot |w|^2} = \sqrt{|z|^2} \cdot \sqrt{|w|^2} = |z| \cdot |w|,$$

as claimed. □

Proposition 3.3. *If $z \neq 0$ then $\left|z^{-1}\right| = (|z|)^{-1}$.*

Proof: See exercises.

3.3 The argument of a complex number.

The modulus of a complex number defines its magnitude. We can also associate an angle to each nonzero complex number.

> ***Definition 5.***
>
> *The **argument** (or **phase**) of $z = a + b\boldsymbol{i}$ ($z \neq 0$) is 'the' angle, ϕ, between the positive real axis and the line segment connecting z to 0, measured in the counterclockwise direction, (see Figure 3). The argument of z is denoted by* $\arg(z)$.

Comments:

a. Unless specified otherwise, we'll measure angles in **radians**. Recall that there are 2π radians in a circle.

b. In the definition of the argument, the second definite article ('the') appears in quotation marks because for any complex number z, $\arg(z)$ is only determined up to a multiple of 2π. This is also illustrated in Figure 3.

c. Among the (infinitely many) possible choices for $\arg(z)$, we'll usually choose the one that lies between 0 and 2π. This value of $\arg(z)$ is called the ***principal value*** of the argument, e.g., the angle ϕ in Figure 3.

The principal value of the argument of $z = a+b\boldsymbol{i}$ can be found from the rectangular coordinates of z, $a = \mathbf{Re}(z)$ and $b = \mathbf{Im}(z)$, by using the *arctangent* function (or *inverse tangent* function) and some basic trigonometry. This is done as follows.

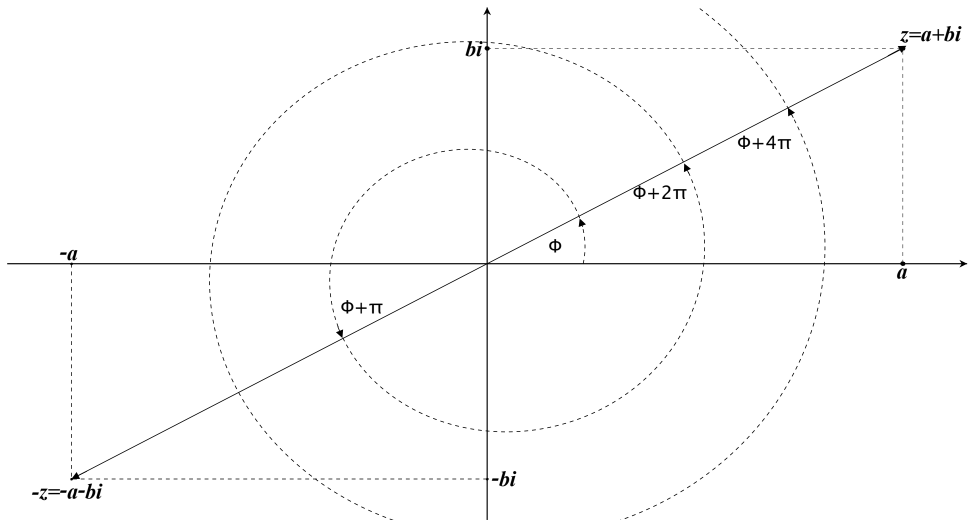

FIGURE 3. z, $-z$ and their arguments.

Recall that if $-\infty < x < \infty$, then $\arctan(x)$ is defined to be the angle ϕ, satisfying $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ and $\tan(\phi) = x$.[‡] Now, as you can see in Figure 3, if $\phi = \arg(a + b\boldsymbol{i})$, then $\tan(\phi) = b/a$ (*tangent* $=$ *opposite/adjacent*). This might lead you to the conclusion that $\arg(a + b\boldsymbol{i}) = \arctan(b/a)$, however this is not necessarily so.

For example, $(-b)/(-a) = b/a$, so that the arctangent function doesn't distinguish the arguments of z and $-z$, even though they are *different*, since $\arg(-z) = \arg(z) \pm \pi$, as illustrated in Figure 3. Moreover, $\arctan(x)$ produces values between $-\pi/2$ and $\pi/2$, and (for the principal value of the argument) we want values between 0 and π.

The correct approach is to adjust the value of $\arctan(b/a)$ according to the *quadrant* of the complex plane in which $a + b\boldsymbol{i}$ lies.

a. If $a+b\boldsymbol{i}$ is in the first quadrant, then the arctangent function produces the correct value, so no adjustment is necessary. This is the case of z_1 in Figure 4. I.e.,

A1. $\boxed{\textit{if } a > 0 \textit{ and } b > 0, \textit{ then } \arg(a + b\boldsymbol{i}) = \arctan(b/a).}$

b. If $a+b\boldsymbol{i}$ is in the second quadrant, then $-\pi/2 < \arctan(b/a) < 0$, and the correct value of the argument is $\pi + \arctan(b/a)$. In Figure 4, this corresponds to the case of z_2, in which case $\arctan(b/-a) = -\phi$ (in red), but $\arg(z_2) = \pi - \phi$. I.e.,

A2. $\boxed{\textit{if } a < 0 \textit{ and } b > 0, \textit{ then } \arg(a + b\boldsymbol{i}) = \pi + \arctan(b/a) = \pi - \arctan(|b/a|).}$

c. If $a + b\boldsymbol{i}$ is in the third quadrant, then $0 < \arctan(b/a) < \pi/2$, and the correct value of the argument is once again given by $\pi + \arctan(b/a)$. In Figure 4, this corresponds to the case of z_3, in which case $\arctan(-b/-a) = \phi$, but $\arg(z_3) = \pi + \phi$. I.e.,

[‡]By specifying that the angle fall in this range, we are technically defining what is called the *principal branch* of the arctangent function. These are the values produced by (most) calculators.

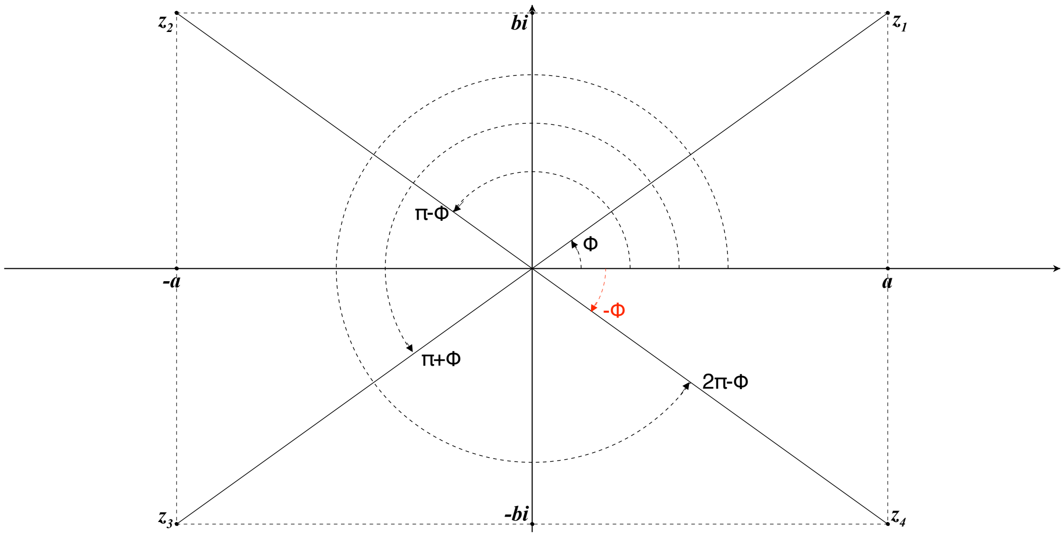

FIGURE 4. The argument, by quadrant.

A3. $\boxed{\textit{if } a < 0 \textit{ and } b < 0, \textit{ then } \arg(a + b\boldsymbol{i}) = \pi + \arctan(b/a).}$

d. If $a + b\boldsymbol{i}$ is in the fourth quadrant, then $-\pi/2 < \arctan(b/a) < 0$, and the correct value of the argument is given by $2\pi + \arctan(b/a)$. In Figure 4, this corresponds to the case of z_4, in which case $\arctan(-b/a) = -\phi$, but $\arg(z_4) = 2\pi - \phi$. I.e.,

A4. $\boxed{\textit{if } a > 0 \textit{ and } b < 0, \textit{ then } \arg(a + b\boldsymbol{i}) = 2\pi + \arctan(b/a) = 2\pi - \arctan(|b/a|).}$

e. Finally, if $a = 0$, then $\arctan(b/a)$ is not defined (why?), but the argument is easily determined without the arctangent function. Specifically, If $\mathbf{Re}(z) = 0$ and $\mathbf{Im}(z) > 0$ then $\arg(z) = \pi/2$ $(= 90°)$, and if $\mathbf{Re}(z) = 0$ and $\mathbf{Im}(z) < 0$ then $\arg(z) = 3\pi/2$ $(= 270°)$.

Example 3.1. *Find the moduli and the arguments of* $u = 3 + 4\boldsymbol{i}$ *and* $v = 12 - 5\boldsymbol{i}$.[§]

The moduli are easy to compute, even without a calculator:

$$|u| = \sqrt{9 + 16} = 5 \quad \text{and} \quad |v| = \sqrt{144 + 25} = 13.$$

Since u lies in the first quadrant (both $\mathbf{Re}(u) > 0$ and $\mathbf{Im}(u) > 0$), it follows from rule **A1** that

$$\arg(u) = \arctan(4/3) \approx 0.9273.$$

Likewise, since v lies in the fourth quadrant ($\mathbf{Re}(v) > 0$ and $\mathbf{Im}(v) < 0$), it follows from rule **A4** that

$$\arg(v) = 2\pi + \arctan(-5/12) \approx 5.8884.$$

[§]The plural of *modulus* is *moduli*.

(Remember — angles are measured here in *radians*. To convert to degrees, you multiply by $180/\pi$. For example, measuring the arguments above in degrees, we have $\arg(u) \approx 53.13°$ and $\arg(v) \approx 337.38°$.)

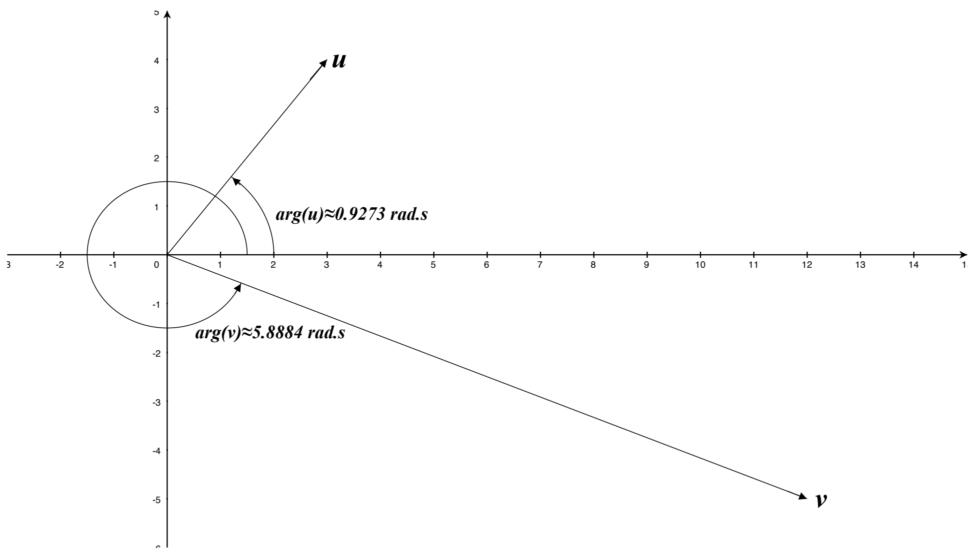

FIGURE 5. Arguments of the numbers in Example 3.1.

3.4 The polar-coordinate representation of complex numbers.

The modulus and argument of a complex number are called the ***polar coordinates*** of the number, and they determine that number completely. I.e., given the polar coordinates of z, we can easily find $\mathbf{Re}(z)$ and $\mathbf{Im}(z)$ using basic trigonometry.

Proposition 3.4. *If $|z| = r$ and $\arg(z) = \theta$, then $\mathbf{Re}(z) = r\cos\theta$ and $\mathbf{Im}(z) = r\sin\theta$. In other words, $z = r\cos\theta + r\sin\theta\boldsymbol{i}$, or more succinctly*

$$z = r(\cos\theta + \sin\theta\boldsymbol{i}). \tag{3.2}$$

Proof: See Figure 6, in which $r = |z|$ and $\theta = \arg(z)$. □

Comment: When expressing a complex number in polar coordinates, as in equation (3.2), it is common to write $... + \sin\theta\boldsymbol{i}$ instead of $... \sin\theta\boldsymbol{i}$.

The addition of complex numbers is easiest to express in terms of their rectangular coordinates. Multiplication however, is simplest to express in terms of polar coordinates. This follows from Proposition 3.2 and the following fact.

Proposition 3.5. *For complex numbers z_1 and z_2, we have*

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2).$$

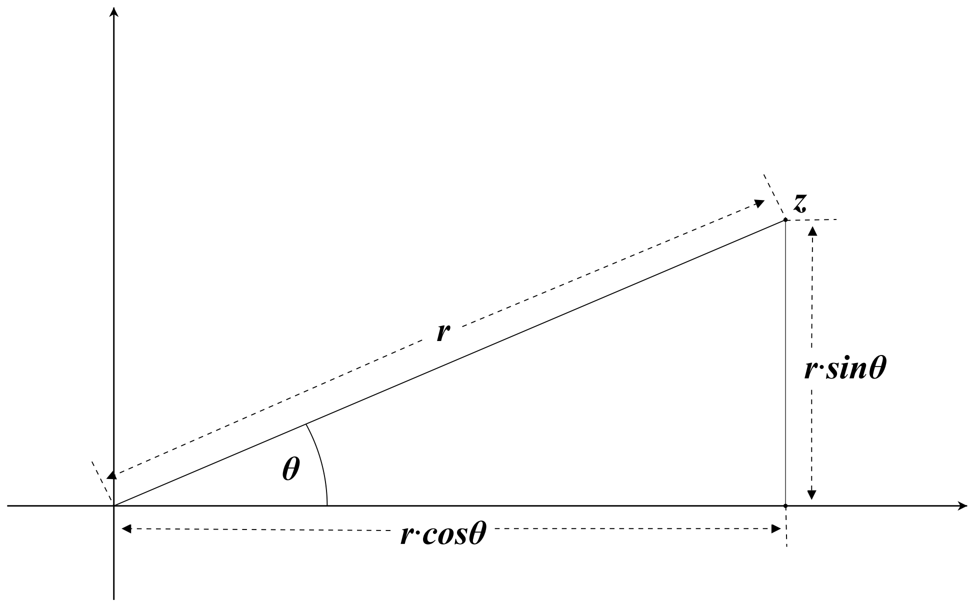

FIGURE 6. ‘Proof’ of Proposition 3.4.

Proof: Let $z_1 = r(\cos\theta + \sin\theta \boldsymbol{i})$ and $z_2 = \rho(\cos\phi + \sin\phi \boldsymbol{i})$, then it follows from equation (2.1) and the rules for sine and cosine of a sum of angles that

$$\begin{aligned} z_1 z_2 &= \big((r\cos\theta)(\rho\cos\phi) - (r\sin\theta)(\rho\sin\phi)\big) + \big((r\cos\theta)(\rho\sin\phi) + (r\sin\theta)(\rho\cos\phi)\big)\boldsymbol{i} \\ &= r\cdot\rho\,\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + (\cos\theta\sin\phi + \sin\theta\cos\phi)\boldsymbol{i}\big) \\ &= r\cdot\rho\,\big(\cos(\theta+\phi) + \sin(\theta+\phi)\boldsymbol{i}\big)\,, \end{aligned}$$

Which implies that $\theta + \phi$ is equal to (one of the possible values of) $\arg(z_1 z_2)$. □

Comments:

a. If $\arg(z_1)$ and $\arg(z_2)$ are the principal values of z_1 and z_2 and if their sum is greater than 2π, then we need to subtract 2π from $\arg(z_1) + \arg(z_2)$ to obtain the principal value of $\arg(z_1 z_2)$.

b. Some authors use the shorthand $\mathrm{cis}(\theta) = \cos\theta + \sin\theta \boldsymbol{i}$, but we shall not. Instead, we'll use *exponential notation.*

Combining Propositions 3.2 and 3.5, we obtain a simple geometric characterization of multiplication by a (nonzero) complex number.

Proposition 3.6. *If $z \neq 0$ and w is any complex number, then multiplying w by z rotates w by the angle* $\arg(z)$ *and scales (stretches or shrinks) the modulus of w by the factor $|z|$.*

Example 3.2. Returning to the numbers u and v from Example 3.1, and using formula (2.1), we have

$$uv = (36 + 20) + (-15 + 48)\boldsymbol{i} = 56 + 33\boldsymbol{i}.$$

It follows that

$$|uv| = \sqrt{56^2 + 33^2} = \sqrt{4225} = 65$$

and

$$\arg(uv) = \arctan(33/56) \approx 0.5325.$$

Now, $|u||v| = 5 \cdot 13 = 65$, as it should, but

$$\arg(u) + \arg(v) \approx 0.9273 + 5.8884 = 6.8157 \neq 0.5325.$$

This discrepancy is immediately cleared up, however, when we subtract 2π (≈ 6.2832) from the sum of the arguments, as you should verify.

3.5 Exponential notation.

The polar coordinate representation of a complex number, equation (3.2), can be simplified by using *Euler's formula*:[¶]

$$\cos\theta + \sin\theta \boldsymbol{i} = e^{\theta \boldsymbol{i}}, \tag{3.3}$$

where θ is real and e^z is the familiar exponential function defined by the *Taylor series*,[‖]

$$e^z = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!}. \tag{3.4}$$

In most calculus classes, the variable in the series is assumed to be a real number. But the operations of addition and multiplication make perfectly good sense for complex numbers too and it is not hard to show that the series above converges for every complex number z. Furthermore, the resulting function has all of its familiar properties, most notably

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} \tag{3.5}$$

and

$$e^{z_1 z_2} = (e^{z_1})^{z_2}$$

for all complex numbers z_1 and z_2.

Using Euler's formula (3.3), we can restate Proposition 3.4 as

Proposition 3.7. *If* $|z| = r > 0$ *and* $\theta = \arg(z)$*, then*

$$z = r \cdot e^{\theta \boldsymbol{i}}. \tag{3.6}$$

Proposition 3.5 (the fact that the argument of a product is equal to the sum of the arguments) can now be justified by using property (3.5) of the exponential function: if $z_1 = r_1 e^{\theta_1 \boldsymbol{i}}$ and $z_2 = r_2 e^{\theta_2 \boldsymbol{i}}$, then

$$z_1 \cdot z_2 = \left(r_1 e^{\theta \boldsymbol{i}_1}\right) \cdot \left(r_2 e^{\theta \boldsymbol{i}_2}\right) = r_1 r_2 e^{\theta_1 \boldsymbol{i} + \theta_2 \boldsymbol{i}} = r_1 r_2 e^{(\theta_1+\theta_2)\boldsymbol{i}}.$$

[¶] *Leonhard Euler* (pronounced 'oiler'), 1707 – 1783, is one of the most prolific and important mathematicians of all time. His work has influenced (almost) every field in mathematics and mathematical physics.

[‖] If Taylor series are unfamiliar or they fill you with a vague sense of dread, then you may skip the explanation that follows and accept formula (3.3) and Proposition(3.6) on faith.

Note that $\cos(2\pi) = \cos(0) = 1$ and $\sin(2\pi) = \sin(0) = 0$, and this together with Euler's formula (3.3) implies that

$$e^{2\pi \boldsymbol{i}} = 1.$$

Furthermore, since $e^{k\alpha} = (e^{\alpha})^k$ the previous identity generalizes to

$$e^{2k\pi \boldsymbol{i}} = \left(e^{2\pi \boldsymbol{i}}\right)^k = 1^k = 1 \tag{3.7}$$

for any *integer* k.

Combining the identities (3.7) and (3.4) proves the following useful fact.

Proposition 3.8. *For any number θ and any integer k,*

$$e^{(\theta+2k\pi)\boldsymbol{i}} = e^{\theta \boldsymbol{i}}.$$

The rest of this subsection explains Euler's formula. You may skip it without loss of continuity, but I recommend that you at least skim it.

To understand Euler's formula, we set $z = \theta \boldsymbol{i}$ in the series on the right-hand side of (3.4) (with a *real* number θ), and manipulate the series a little as follows.

To begin, it helps to recognize that the powers of $\boldsymbol{i}$ follow a simple pattern, namely $\boldsymbol{i}^0 = 1$, $\boldsymbol{i}^1 = \boldsymbol{i}$, $\boldsymbol{i}^2 = -1$, $\boldsymbol{i}^3 = -\boldsymbol{i}$, $\boldsymbol{i}^4 = 1$, $\boldsymbol{i}^5 = \boldsymbol{i}$, $\boldsymbol{i}^6 = -1$, $\boldsymbol{i}^7 = -\boldsymbol{i}$, etc. In other words, for any integer n,

$$\boldsymbol{i}^n = \begin{cases} 1 & : \text{ if } n \text{ leaves remainder 0 when divided by 4;} \\ \boldsymbol{i} & : \text{ if } n \text{ leaves remainder 1 when divided by 4;} \\ -1 & : \text{ if } n \text{ leaves remainder 2 when divided by 4;} \\ -\boldsymbol{i} & : \text{ if } n \text{ leaves remainder 3 when divided by 4.} \end{cases}$$

We can summarize this information even more usefully by distinguishing even and odd values of n. Specifically, note that if $n = 2k$ for some integer k (n is even), then

$$\boldsymbol{i}^n = \boldsymbol{i}^{2k} = (\boldsymbol{i}^2)^k = (-1)^k,$$

and if $n = 2k + 1$ for some integer k (n is odd), then

$$\boldsymbol{i}^n = \boldsymbol{i}^{2k+1} = \boldsymbol{i}^{2k} \cdot \boldsymbol{i} = (-1)^k \cdot \boldsymbol{i}.$$

Now, if θ is a real number, then when $n = 2k$ we have

$$(\theta \boldsymbol{i})^n = (\theta \boldsymbol{i})^{2k} = (-1)^k \theta^{2k}; \tag{3.8}$$

and if $n = 2k + 1$, we have

$$(\theta \boldsymbol{i})^n = (\theta \boldsymbol{i})^{2k+1} = (-1)^k \theta \boldsymbol{i}^{2k+1}. \tag{3.9}$$

Next, we substitute $z = \theta \boldsymbol{i}$ in the Taylor series for e^z in (3.4), split the series into two sub-series based on the *parity* (even*ness* or odd*ness*) of the power n, and use the formulas (3.8) and (3.9) to simplify.

$$
\begin{aligned}
e^{\theta \boldsymbol{i}} &= 1 + \sum_{n=1}^{\infty} \frac{(\theta \boldsymbol{i})^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(\theta \boldsymbol{i})^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(\theta \boldsymbol{i})^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + \left(\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \right) \boldsymbol{i}. \qquad (3.10)
\end{aligned}
$$

At this point, we are done because the series on the left-hand side of line (3.10) is the Taylor series for $\cos\theta$, and the series on the right-hand side is the Taylor series for $\sin\theta$, showing that

$$e^{\theta \boldsymbol{i}} = \cos\theta + \sin\theta \boldsymbol{i},$$

as claimed.

Exercises

3.1. Compute the moduli and arguments of the numbers

$$z_1 = -1 - i, \quad z_2 = 3 - 4\boldsymbol{i}, \quad z_3 = -8 + 6\boldsymbol{i}, \quad z_4 = \sqrt{3}/2 + 0.5\boldsymbol{i}.$$

3.2. Compute the arguments and moduli of $(z_1 \cdot z_2)$, $(z_1 \cdot z_4)$ and $(z_2 \cdot z_3)$, with z_1, z_2, z_3 and z_4 as above.

3.3. Express the four numbers in exercise 3.1 using exponential notation.

3.4. Show that multiplication by $\boldsymbol{i}$ corresponds to counterclockwise rotation by an angle of $\pi/2$.

3.5. Show that if $z \neq 0$, then $\left|z^{-1}\right| = |z|^{-1}$.

3.6. Show that $e^{\pi \boldsymbol{i}} + 1 = 0$.

3.7. Show that if $z \neq 0$, and $0 \leq \arg(z) < \pi$, then $\arg(-z) = \arg(z) + \pi$. How does this claim need to be adjusted if $\pi \leq \arg(z) < 2\pi$, assuming that we want the principal value of $\arg(-z)$?

3.8. Show that $\left|e^{\theta \boldsymbol{i}}\right| = 1$ for any *real number* θ.

4. Roots of polynomials.

4.1 The fundamental theorem of algebra.

Let $P(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$ be a polynomial with degree $n > 0$ and complex coefficients, $c_0, c_1, \ldots, c_n$. The ***Fundamental Theorem of Algebra*** states that $P(z)$ has a complex *root*, i.e., the equation

$$P(z) = 0, \qquad (4.1)$$

has a *solution* in the complex numbers.[†]

If ζ_1 is a solution of equation (4.1),[‡] then the polynomial $P(z)$ may be factored as

$$P(z) = (z - \zeta_1) \cdot P_1(z),$$

where $P_1(z)$ is a polynomial of degree $n-1$.[§] If $n - 1 \geq 1$, then we can apply the fundamental theorem of algebra again to conclude that the equation $P_1(z) = 0$ has a solution ζ_2, from which it follows that $P_1(z) = (z - \zeta_2) \cdot P_2(z)$ and so

$$P(z) = (z - \zeta_1) \cdot (z - \zeta_2) \cdot P_2(z),$$

where $P_2(z)$ is a polynomial of degree $n-2$. Continuing in this way, we can factor the polynomial completely, which gives the following proposition.

***Proposition* 4.1.** *If $P(z)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then P may be expressed as a product of linear factors,*

$$P(z) = c_n \cdot (z - \zeta_1) \cdot (z - \zeta_2) \cdots (z - \zeta_n), \tag{4.2}$$

where $\zeta_1, \zeta_2, \ldots, \zeta_n$ are solutions of the equation $P(z) = 0$.

Note: It is not necessarily the case that the roots $\zeta_1, \ldots, \zeta_n$ are all different from each other, so that some of the factors in (4.2) may be repeated. For example, the polynomial $Q(z) = z^4 - 2z^3 + 2z^2 - 2z + 1$ factors as

$$Q(z) = z^4 - 2z^3 + 2z^2 - 2z + 1 = (z-1)(z-1)(z - \boldsymbol{i})(z + \boldsymbol{i}),$$

as you should verify.

It is also important to note that while the roots $\zeta_1, \ldots, \zeta_n$ that appear in (4.2) may not all be different from each other, they do include *all* the roots of $P(z)$. This implies that a polynomial of degree n has no more than n roots. We can summarize all of this as follows.

***Proposition* 4.2.** *If $P(z)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then P has at least one root, but no more than n roots.*

While the fundamental theorem of algebra guarantees that every polynomial has roots, it says nothing about the important, practical problem of actually *finding* these roots.

For quadratic equations we have the quadratic formula which works just as well when the coefficients are complex numbers (as we will see below). There are analogous formulas for finding the solutions of cubic and quartic equations (degrees 3 and 4), but a deep theorem in algebra states that there are ***no*** general formulas for finding the roots of polynomials of degree 5 or more.

While there are *ad hoc* methods that can be used to find roots in special cases, it is generally very difficult to find the roots of a polynomial of degree greater than four. In most cases, we have to settle for *approximate* solutions using methods from calculus, e.g., *Newton's method* and the like.

[†]The proof of this theorem is beyond the scope of this course.

[‡]The symbol ζ is the greek letter *zeta*.

[§]This claim can be proved using long division of polynomials. See the exercises.

In the last few sections below, we'll make a couple of general observations about polynomials with *real* coefficients and then look at some simple types of polynomials whose roots are relatively easy to find.

4.2 Polynomials with real coefficients.

Since the real numbers are a subset of the complex numbers, it follows that any polynomial with *real* coefficients also has a root, and such polynomials may be factored as in equation (4.2). On the other hand, even though the coefficients of the polynomial may be real, there is no guarantee that its roots will be real, as in Example 1.1.

The complex roots of a real polynomial do have a special property however: they occur in *complex conjugate pairs*, as proved below.

Proposition 4.3. *If $a_0, a_1, \ldots, a_n$ are real numbers and ζ is a root of the polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$, then $\overline{\zeta}$ is also a root of $P(z)$.*

Proof: ζ is a root of $P(z)$ if and only if $a_n\zeta^n + \cdots + a_1\zeta + a_0 = 0$. Taking conjugates of both sides of this equation gives

$$\overline{a_n\zeta^n + \cdots + a_1\zeta + a_0} = \overline{0} = 0, \tag{4.3}$$

since 0 is real. Now, applying Proposition 2.3 (and exercise 2.7) to the left-hand side of equation (4.3) gives

$$\begin{aligned}\overline{a_n\zeta^n + \cdots + a_1\zeta + a_0} &= \overline{a_n\zeta^n} + \cdots + \overline{a_1\zeta} + \overline{a_0} \\ &= \overline{a_n}(\overline{\zeta})^n + \cdots + \overline{a_1}(\overline{\zeta}) + \overline{a_0} \\ &= a_n(\overline{\zeta})^n + \cdots + a_1(\overline{\zeta}) + a_0,\end{aligned} \tag{4.4}$$

since the coefficients $a_0, \ldots, a_n$ are real. Finally, replacing the left-hand side of (4.3) by the right-hand side of (4.4) shows that

$$P(\overline{\zeta}) = a_n(\overline{\zeta})^n + \cdots + a_1(\overline{\zeta}) + a_0 = 0,$$

so that $\overline{\zeta}$ is also a root of $P(z)$, as claimed. □

Example 4.1. Consider the polynomial $P(z) = z^4 - 2z^3 + z^2 + 2z - 2$. Suppose that a mystical experience reveals that $\zeta = 1 + \boldsymbol{i}$ is a root of P, then because the coefficients of P are all real, it follows that $\overline{\zeta} = 1 - \boldsymbol{i}$ is also a root of P.

This means that both $(z - (1 + \boldsymbol{i}))$ and $(z - (1 - \boldsymbol{i}))$ are factors of $P(z)$, so that

$$P(z) = (z - (1 + \boldsymbol{i})) \cdot (z - (1 - \boldsymbol{i})) \cdot Q(z) = (z^2 - 2z + 2) \cdot Q(z),$$

where $Q(z)$ is a polynomial of degree 2. Dividing $P(z)$ by $z^2 - 2z + 2$ gives

$$Q(z) = \frac{P(z)}{z^2 - 2z + 2} = z^2 - 1,$$

so that

$$P(z) = (z - (1 + \boldsymbol{i})) \cdot (z - (1 - \boldsymbol{i})) \cdot (z^2 - 1) = (z - (1 + \boldsymbol{i})) \cdot (z - (1 - \boldsymbol{i})) \cdot (z - 1)(z + 1),$$

which means that the roots of $P(z)$ are $\zeta_1 = 1 + \boldsymbol{i}$, $\zeta_2 = 1 - \boldsymbol{i}$, $\zeta_3 = 1$ and $\zeta_4 = -1$.

In the preceding example we saw that for $\zeta = 1 + \boldsymbol{i}$, the product $(z-\zeta)(z-\overline{\zeta})$ results in a (quadratic) polynomial with *real* coefficients. This phenomenon is true in general and in fact we can be more precise.

Proposition 4.4. *For any complex number ζ, we have*

$$(z-\zeta)(z-\overline{\zeta}) = z^2 - bz + c,$$

where $b = 2 \cdot \mathbf{Re}(\zeta)$ and $c = |\zeta|^2$ are both real numbers.

Proof: Exercise.

The representation of a polynomial with real coefficients as a product of *linear* factors, as in 4.2, may have complex coefficients, since the polynomial may have complex roots. Using Propositions 4.3 and 4.4 however, we can show that every real polynomial can be represented as a product of linear and quadratic factors, all of which have *real* coefficients.

Proposition 4.5. *If $a_0, a_1, \ldots, a_n$ are real numbers (with $a_n \neq 0$), then*

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 = a_n(z-\xi_1)\cdots(z-\xi_m)Q_1(z)Q_2(z)\cdots Q_k(z), \tag{4.5}$$

where

a. *$\xi_1, \ldots, \xi_m$ are all the real roots of $P(z)$, repeated as necessary;*
b. *and for $1 \leq j \leq k$,*

$$Q_j = z^2 - 2\mathbf{Re}(\zeta_j) + |\zeta_j|^2,$$

where $\zeta_1, \overline{\zeta}_1, \zeta_2, \overline{\zeta}_2, \ldots, \zeta_k, \overline{\zeta}_k$ are the complex roots of $P(z)$.

Proof: Exercise.

4.3 Square roots and quadratic equations.

Before we can solve a general quadratic equation, we need to be able to compute the square roots of a complex number. This is easiest to do using the exponential (or polar) representation.

Proposition 4.6. *If $u \neq 0$ then the solutions of the equation $z^2 = u$ are*

$$\zeta_1 = \sqrt{|u|}e^{(\theta/2)\boldsymbol{i}} \quad \textit{and} \quad \zeta_2 = \sqrt{|u|}e^{(\theta/2+\pi)\boldsymbol{i}}, \tag{4.6}$$

where $\theta = \arg(u)$ (the principal value of the argument) and $\sqrt{|u|}$ is the (positive) real square root of $|u|$. Moreover,

$$0 \leq \arg(\zeta_1) < \pi \quad \textit{and} \quad \arg(\zeta_2) = \arg(\zeta_1) + \pi.$$

Proof: First, we have

$$\zeta_2^2 = (-\zeta_1)^2 = \zeta_1^2 = \left(\sqrt{|u|}\right)^2 e^{\boldsymbol{i}(\theta/2+\theta/2)} = |u|e^{\theta\boldsymbol{i}} = u,$$

so ζ_1 and ζ_2 are certainly the two solutions of the equation $z^2 = u$.

Next, we have $0 \leq \theta < 2\pi$, so $0 \leq \arg(\zeta_1) < \pi$, because $\arg(\zeta_1) = \theta/2$. Finally, it follows from exercise 3.7 that

$$\arg(\zeta_2) = \arg(-\zeta_1) = \arg(\zeta_1) + \pi,$$

and therefore $\pi \leq \arg(\zeta_2) < 2\pi$. $\square$

Comment: By convention '*the*' square root of a nonzero complex number u is the solution ζ_1 above, i.e., it is the solution of $z^2 = u$ whose principal argument is half the principal argument of u. This generalizes with the convention that the square root of a positive real number α is the *positive* solution of the equation $z^2 = \alpha$.

Example 4.2. *Find the square roots of* $\boldsymbol{i}$, $-2\boldsymbol{i}$ *and* $-3+4\boldsymbol{i}$. *Express your answers in both exponential and rectangular coordinates.*

First, we have $|\boldsymbol{i}| = 1$ and $\arg(\boldsymbol{i}) = \pi/2$, so

$$\sqrt{\boldsymbol{i}} = e^{\pi \boldsymbol{i}/4} = \cos(\pi/4) + \sin(\pi/4)\boldsymbol{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}.$$

Next, we have $|-2\boldsymbol{i}| = 2$ and $\arg -2\boldsymbol{i} = 3\pi/2$, so

$$\sqrt{-2\boldsymbol{i}} = \sqrt{2}e^{3\pi\boldsymbol{i}/4} = \sqrt{2}\left(\cos(3\pi/4) + \sin(3\pi/4)\boldsymbol{i}\right) = -1 + \boldsymbol{i}.$$

Finally, $|-3+4\boldsymbol{i}| = \sqrt{9+16} = 5$ and $\arg(-3+4\boldsymbol{i}) = \pi - \arctan(4/3) \approx 2.2143$,[¶]

$$\sqrt{-3+4\boldsymbol{i}} = \sqrt{5}e^{1.10715\boldsymbol{i}} = \sqrt{5}\left(\cos(1.10715) + \sin(1.10715)\boldsymbol{i}\right) = 1 + 2\boldsymbol{i}.$$

To solve a quadratic equation with complex coefficients, we use the familiar quadratic formula.

Proposition 4.7. *The solutions of the equation* $az^2 + bz + c = 0$ *are given by*

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \textit{and} \quad z_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \tag{4.7}$$

where a, b *and* c *may be any complex numbers, as long as* $a \neq 0$.

The proof is identical to the case in which the coefficients are real (i.e., *completing the square*), because none of the steps make any use of the nature of the coefficients (real or complex), and I leave it to you as an exercise.

Example 4.3. *Find the solutions of the equation* $z^2 + (1-2\boldsymbol{i})z - 2\boldsymbol{i} = 0$.

Applying the quadratic formula (with $a = 1$, $b = 1 - 2\boldsymbol{i}$ and $c = -2\boldsymbol{i}$) we find that the two solutions are

$$\zeta_1 = \frac{-(1-2\boldsymbol{i}) + \sqrt{(1-2\boldsymbol{i})^2 + 8\boldsymbol{i}}}{2} \quad \text{and} \quad \zeta_2 = \frac{-(1-2\boldsymbol{i})^2 - \sqrt{(1-2\boldsymbol{i})^2 + 8\boldsymbol{i}}}{2}.$$

Next, note that $(1-2\boldsymbol{i})^2 + 8\boldsymbol{i} = -3 - 4\boldsymbol{i} + 8\boldsymbol{i} = -3 + 4\boldsymbol{i}$, so the two solutions simplify to

$$\zeta_1 = \frac{-1 + 2\boldsymbol{i} + 1 + 2\boldsymbol{i}}{2} = 2\boldsymbol{i} \quad \text{and} \quad \zeta_2 = \frac{-1 + 2\boldsymbol{i} - 1 - 2\boldsymbol{i}}{2} = -1,$$

as you can check.

4.4 The n^{th} roots of a complex number.

If n is a positive integer and $\alpha > 0$ is a *real* number then the n^{th} root of α is defined to be the positive real number β satisfying $\beta^n = \alpha$, which we denote by $\beta = \sqrt[n]{\alpha}$ or $\beta = \alpha^{1/n}$. In other words, β is a solution of the equation $z^n - \alpha = 0$. In this section, we'll generalize this to find the n^{th} root(s) of any complex number.

[¶] See rule A2 in subsection 3.3.

If $u \neq 0$ is a complex number and n is a positive integer, then Proposition 4.1 tells us that there are complex numbers $\zeta_1, \zeta_2, \ldots, \zeta_n$ such that

$$z^n - u = (z - \zeta_1)(z - \zeta_2)\cdots(z - \zeta_n). \tag{4.8}$$

Now, for a general polynomial of degree n, there is no guarantee that the ζ's that appear in the factorization (4.2) are all different from one another. But in this case, i.e., in (4.8), the roots *are* distinct.

Proposition 4.8. *If $u \neq 0$, $\theta = \arg(u)$ and n is a positive integer, then the solutions of the equation $z^n = u$ are given by*

$$\zeta_k = \sqrt[n]{|u|}e^{(\theta+2k\pi)\boldsymbol{i}/n}, \tag{4.9}$$

for $k = 0, 1, 2, \ldots, n-1$. Furthermore, these n numbers are all distinct from one another.

Proof: To see that $\zeta_k^n = u$ for each integer k, between 0 and $n-1$, we simply evaluate:

$$\begin{aligned}
\zeta_k^n &= \left(\sqrt[n]{|u|}e^{(\theta+2k\pi)\boldsymbol{i}/n}\right)^n \\
&= \left(\sqrt[n]{|u|}\right)^n \cdot e^{n\cdot(\theta+2k\pi)\boldsymbol{i}/n} \\
&= |u| \cdot e^{(\theta+2k\pi)\boldsymbol{i}} \\
&= |u|e^{\theta\boldsymbol{i}} \\
&= u,
\end{aligned}$$

where the transition from line 3 to line 4 is justified by Proposition 3.8.

Now suppose that $0 \leq j < k \leq n-1$, then

$$\frac{\zeta_k}{\zeta_j} = \frac{\sqrt[n]{|u|}e^{(\theta+2k\pi)\boldsymbol{i}/n}}{\sqrt[n]{|u|}e^{(\theta+2j\pi)\boldsymbol{i}/n}} = \frac{e^{(\theta+2k\pi)\boldsymbol{i}/n}}{e^{(\theta+2j\pi)\boldsymbol{i}/n}} = e^{\left(\frac{k-j}{n}\right)\cdot 2\pi\boldsymbol{i}},$$

as you should verify (using the fact that $e^\alpha/e^\beta = e^{\alpha-\beta}$). It follows that

$$\arg(\zeta_k/\zeta_j) = \frac{k-j}{n}\cdot 2\pi.$$

But $0 < k-j < n$, so $0 < \frac{k-j}{n} < 1$, which implies that $\zeta_k/\zeta_j \neq 1$, since the argument of 1 must be an *integer* multiple of 2π. Thus $\zeta_k \neq \zeta_j$, as claimed. $\square$

Example 4.4. The previous proposition applies to all complex numbers, including the real numbers. The n^{th} roots of the number 1 are called the n^{th} ***roots of unity***. Now, since $\arg(1) = 0$ and $\sqrt[n]{1} = 1$ (for any n), the n^{th} roots of unity are particularly simple to write down:

$$1,\ e^{2\pi\boldsymbol{i}/n},\ e^{4\pi\boldsymbol{i}/n},\ \ldots,\ e^{2k\pi\boldsymbol{i}/n},\ \ldots,\ e^{2(n-1)\pi\boldsymbol{i}/n}. \tag{4.10}$$

For example, the 4^{th} roots of unity are

$$\rho_0 = e^{0\cdot\pi\boldsymbol{i}/4} = 1, \quad \rho_1 = e^{2\pi\boldsymbol{i}/4} = \boldsymbol{i}, \quad \rho_2 = e^{4\pi\boldsymbol{i}/4} = -1, \quad \rho_3 = e^{6\pi\boldsymbol{i}/4} = -\boldsymbol{i};$$

and the 6^{th} roots of unity are

$$\rho_0 = e^{0\cdot\pi\boldsymbol{i}/6} = 1,\ \ \rho_1 = e^{2\pi\boldsymbol{i}/6} = \frac{1}{2}+\frac{\sqrt{3}}{2}\boldsymbol{i},\ \ \rho_2 = e^{4\pi\boldsymbol{i}/6} = -\frac{1}{2}+\frac{\sqrt{3}}{2}\boldsymbol{i},\ \ \rho_3 = e^{6\pi\boldsymbol{i}/6} = -1$$

$$\rho_4 = e^{8\pi\boldsymbol{i}/6} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i} \quad \text{and} \quad \rho_5 = e^{10\pi\boldsymbol{i}/6} = \frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}.$$

The arguments of the n^{th} roots of unity increase from 0 to $2(n-1)\pi/n$ in increments of $2\pi/n$, and the roots of unity themselves are equally spaced points on the *unit circle* in the complex plane. This is illustrated below, where the 10^{th} roots of unity are marked with blue dots on the unit circle.

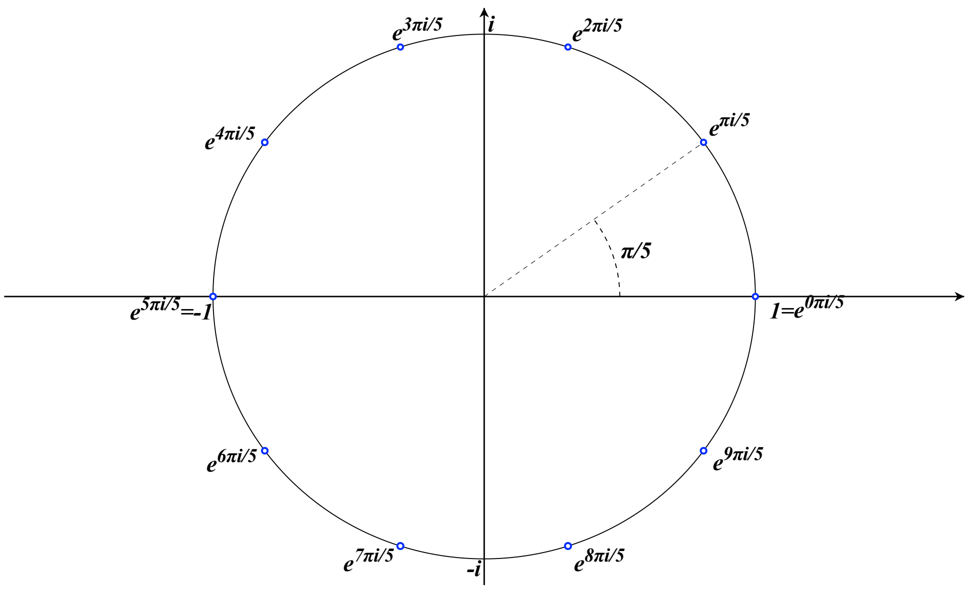

FIGURE 7. The tenth roots of unity.

If we set $\rho_n = e^{2\pi\boldsymbol{i}/n}$, then it follows from the basic properties of exponents that

$$e^{2k\pi\boldsymbol{i}/n} = \rho_n^k.$$

In other words, all of the n^{th} roots of unity may be expressed as powers of the root ρ_n. For this reason, ρ_n is sometimes called a *primitive* n^{th} root of unity.$^{\|}$

Example 4.5. If α is a *positive* real number, then $\arg(\alpha) = 0$ and the n^{th} roots of α may be expressed in terms of $\sqrt[n]{\alpha}$ and the n^{th} roots of unity. Indeed, according to formula (4.9), the n^{th} roots of α have the form

$$\zeta_k = \sqrt[n]{|\alpha|} \cdot e^{2k\pi\boldsymbol{i}/n} = \sqrt[n]{\alpha} \cdot \rho_n^k,$$

for $k = 0, \ldots, n-1$, since $|\alpha| = \alpha$ in this case.

$^{\|}\rho_n$ is called ***a*** primitive n^{th} root of unity and not ***the*** primitive n^{th} root of unity because for any n, there are other n^{th} roots of unity with the same property. E.g., it is possible to represent the 5^{th} roots of unity as powers of $e^{4\pi\boldsymbol{i}/5}$.

Example 4.6. If β is a *negative* real number, then $\arg(\beta) = \pi$ and the n^{th} roots of β may be expressed in terms of $\sqrt{|\beta|}$, the n^{th} roots of unity and an n^{th} root of $\boldsymbol{i}$. Specifically, the n^{th} roots of β are given by

$$\zeta_k = \sqrt[n]{|\beta|} \cdot e^{(\pi + 2k\pi)\boldsymbol{i}/n} = \sqrt[n]{|\beta|} \cdot e^{\pi \boldsymbol{i}/n} \cdot \rho_n^k,$$

for $k = 0, \ldots, n-1$, where you should note that $(e^{\pi \boldsymbol{i}/n})^n = \boldsymbol{i}$.

Exercises

4.1. Find the solutions of the quadratic equations below. Express your answers in rectangular coordinates (i.e., in the form $a + b\boldsymbol{i}$).
(a) $z^2 + \boldsymbol{i}z + 1 = 0$.
(b) $2z^2 - 3\boldsymbol{i}z + 2 = 0$.
(c) $\boldsymbol{i}z^2 + 2z - 1 = 0$.

4.2. Write down the 8^{th} roots of unity using rectangular coordinates. Do not round your answers, instead use $\sqrt{2}$ as needed. Hint: first express your answer in *polar* coordinates.

4.3. Write down the 12^{th} roots of unity using rectangular coordinates. Do not round your answers, instead use $\sqrt{3}$ as needed. Hint: first express your answer in *polar* coordinates.

4.4. Prove Proposition 4.4.

4.5. Prove Proposition 4.5.

4.6. Use Proposition 4.5 to show that a polynomial with real coefficients and *odd* degree always has at least one *real* root.

4.7. Show that if n is even then -1 is an n^{th} root of unity and if n is divisible by 4, then $\boldsymbol{i}$ is an n^{th} roots of unity.

4.8. Show that if $\boldsymbol{i}$ is an n^{th} root of unity, then n is divisible by 4.

4.9. Show that if η is an n^{th} root of unity, then so is η^k for any integer k.

4.10. Show that if ζ is a root of the polynomial $P(z) = c_n z^n + \cdots + c_1 z + c_0$, then

$$P(z) = (z - \zeta) \cdot P_1(z),$$

where $P_1(z)$ is a polynomial of degree $n - 1$.

Hints: If you use '*long division*' to divide $(z - \zeta)$ into $P(z)$ you obtain an identity of the form

$$P(z) = (z - \zeta) \cdot P_1(z) + R(z),$$

where $R(z)$ is a polynomial whose degree is ***less*** than the degree of $(z - \zeta)$. What is the degree of $(z - \zeta)$ and what is the nature of a 'polynomial' whose degree is less than this? Finally, use the fact that $P(\zeta) = 0$.