Universidad San Pablo - CEU

Mathematical Fundaments of Biomedical Engineering 1

Problems

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1 Chapter 1

Lay, 1.1.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Solve the following equation system applying row operations on the augmented matrix:

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$$

Solution: Let us construct the augmented matrix of the equation system

$$\left(\begin{array}{rr|r} 1 & 5 & 7 \\ -2 & -7 & -5 \end{array}\right)$$

Now we add twice row 1 to row 2

$$\left(\begin{array}{rr|r} 1 & 5 & 7 \\ 0 & 3 & 9 \end{array}\right)$$

Now we divide the second row by 3

$$\left(\begin{array}{rr|r} 1 & 5 & 7 \\ 0 & 1 & 3 \end{array}\right)$$

Finally, we subtract 5 times row 2 from row 1

$$\left(\begin{array}{rr|r} 1 & 0 & -8 \\ 0 & 1 & 3 \end{array}\right)$$

This equation system is compatible determinate and its solution is $x_1 = -8$ and $x_2 = 3$.

Lay, 1.1.4

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the point of intersection of the lines $x_1 + 2x_2 = -13$ and $3x_1 - 2x_2 = 1$

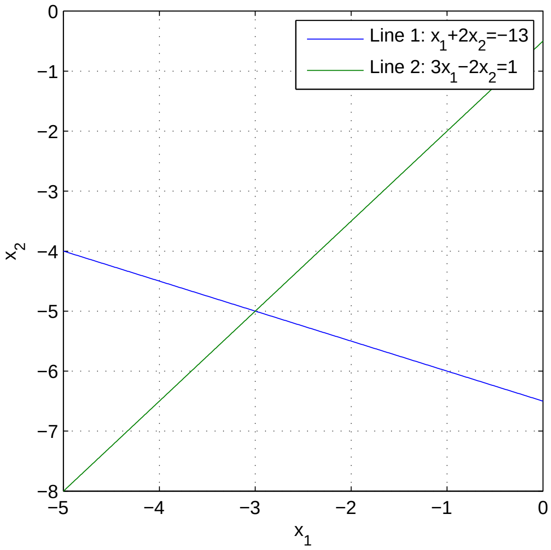
Solution: Let us construct the augmented system matrix

$$\left(\begin{array}{rr|r} 1 & 2 & -13 \\ 3 & -2 & 1 \end{array}\right)$$

Now, we apply row operations to solve it

$$\begin{array}{cc} \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1 & \left(\begin{array}{rr|r} 1 & -5 & 1 \\ 0 & -8 & 40 \end{array}\right) \\ \mathbf{r}_2 \leftarrow -\frac{1}{8}\mathbf{r}_2 & \left(\begin{array}{rr|r} 1 & -5 & 1 \\ 0 & 1 & -5 \end{array}\right) \\ \mathbf{r}_1 \leftarrow \mathbf{r}_1 + 5\mathbf{r}_2 & \left(\begin{array}{rr|r} 1 & 0 & -3 \\ 0 & 1 & -5 \end{array}\right) \end{array}$$

The two lines intersect in a single point $(x_1, x_2) = (-3, -5)$.



Lay, 1.1.11

Carlos Oscar Sorzano, Aug. 31st, 2013

Solve the equation system

$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + 1x_3 &= -2 \end{aligned}$$

Solution: Let us construct the augmented system matrix

$$\left(\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{array}\right)$$

Now, we apply row operations to solve it

$$\begin{array}{cc} \mathbf{r}_2 \leftrightarrow \mathbf{r}_1 & \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -2 \end{array}\right) \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_1 & \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 2 \end{array}\right) \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2 & \left(\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -2 \end{array}\right) \end{array}$$

Last row represents the equation $0 = -2$ which is non-sense and, therefore, there is no solution of the system. The equation system is incompatible.

Lay, 1.1.12

Clara Susana Rey Abad, Oct. 29, 2013

Solve the equation system:

$$\begin{array}{rcr} x_1 - 5x_2 + 4x_3 &=& -3 \\ 2x_1 - 7x_2 + 3x_3 &=& -2 \\ -2x_1 + x_2 + 7x_3 &=& -1 \end{array}$$

Solution: Let us construct the augmented system matrix

$$\left(\begin{array}{rrr|r} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{array}\right)$$

Now, we apply row operations to solve it

$$\begin{array}{cc}
\mathbf{r}_2 \leftarrow \mathbf{r}_2 + \mathbf{r}_3 & \left(\begin{array}{rrr|r} 1 & -5 & 4 & -3 \\ 0 & -6 & 10 & -3 \\ -2 & 1 & 7 & -1 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_1 & \left(\begin{array}{rrr|r} 1 & -5 & 4 & -3 \\ 0 & -6 & 10 & -3 \\ 0 & -9 & 15 & -7 \end{array}\right) \\
\mathbf{r}_2 \leftarrow \mathbf{r}_2 \div 3 & \left(\begin{array}{rrr|r} 1 & -5 & 4 & -3 \\ 0 & -2 & 10/3 & -1 \\ 0 & -9 & 15 & -7 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 \div 3 & \left(\begin{array}{rrr|r} 1 & -5 & 4 & -3 \\ 0 & -2 & 10/3 & -1 \\ 0 & -3 & 5 & -7/3 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 \cdot 2 - \mathbf{r}_2 \cdot 3 & \left(\begin{array}{rrr|r} 1 & -5 & 4 & -3 \\ 0 & -2 & 10/3 & -1 \\ 0 & 0 & 0 & -5/3 \end{array}\right)
\end{array}$$

Last row represents the equation $0 = -5/3$ which is non-sense and, therefore, there is no solution of the system. The equation system is incompatible.

Lay, 1.1.13

Clara Susana Rey Abad, Oct. 30, 2013

Solve the equation system:

$$\begin{array}{rcr} x_1 - 3x_3 &=& 8 \\ 2x_1 + 2x_2 + 9x_3 &=& 7 \\ x_2 + 5x_3 &=& -2 \end{array}$$

Solution: Let us construct the augmented system matrix

$$\left(\begin{array}{rrr|r} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right)$$

Now, we apply row operations to solve it

$$
\begin{array}{cc}
\mathbf{r}_2 \leftrightarrow \mathbf{r}_3 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 2 & 2 & 9 & 7 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_1 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_2 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right) \\
\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \mathbf{r}_3 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 5 & -5 \end{array}\right) \\
\begin{array}{c} \mathbf{r}_3 \leftarrow \mathbf{r}_3 + \div 5 \\ \mathbf{r}_1 \leftarrow \mathbf{r}_1 + 3\mathbf{r}_3 \end{array} & \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)
\end{array}
$$

Whe can deduce from the reduced echelon form that

$$
\begin{array}{rcr}
x_1 & = & 5 \\
x_2 & = & 3 \\
x_3 & = & -1
\end{array}
$$

Therefore, there is a unique solution of the system. The equation system is compatible determinate.

Lay, 1.1.14

Clara Susana Rey Abad, Nov. 4, 2013

Solve the equation system:

$$
\begin{array}{rcr}
2x_1 - 6x_3 & = & -8 \\
x_2 + 2x_3 & = & 3 \\
3x_1 + 6x_2 - 2x_3 & = & -4
\end{array}
$$

Solution: Let us construct the augmented system matrix

$$
\left(\begin{array}{ccc|c} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right)
$$

Now, we apply row operations to solve it

$$\begin{array}{cc}
\mathbf{r}_1 \leftarrow \mathbf{r}_1 \div 2 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_1 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 4 & 8 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 \div 2 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 4 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_2 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & -5 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 \div 4 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -5/4 \end{array}\right) \\
\mathbf{r}_2 \leftarrow \mathbf{r}_2 + 2\mathbf{r}_3 & \left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & -5/4 \end{array}\right) \\
\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 3\mathbf{r}_3 & \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & -5/4 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 \div -1 & \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 5/4 \end{array}\right)
\end{array}$$

Whe can deduce from the reduced echelon form that

$$\begin{array}{rcr} x_1 & = & -1/4 \\ x_2 & = & 1/2 \\ x_3 & = & 5/4 \end{array}$$

Therefore, there is a unique solution of the system. The equation system is compatible determinated.

Lay, 1.1.15

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine whether the following system is consistent (do not fully solve the system).

$$\begin{array}{rrrrcr} x_1 & -6x_2 & & & = & 5 \\ & x_2 & -4x_3 & +x_4 & = & 0 \\ -x_1 & +6x_2 & +x_3 & +5x_4 & = & 3 \\ & -x_2 & +5x_3 & +4x_4 & = & 0 \end{array}$$

Solution: Let us construct the augmented system matrix

$$\left(\begin{array}{rrrr|r} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ -1 & 6 & 1 & 5 & 3 \\ 0 & -1 & 5 & 4 & 0 \end{array}\right)$$

Now, we apply row operations to solve it

$$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_1 \qquad \left(\begin{array}{rrrr|r} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & -1 & 5 & 4 & 0 \end{array}\right)$$

$$\mathbf{r}_4 \leftarrow \mathbf{r}_4 + \mathbf{r}_2 \qquad \left(\begin{array}{rrrr|r} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 5 & 0 \end{array}\right)$$

$$\mathbf{r}_4 \leftrightarrow \mathbf{r}_4 - \mathbf{r}_3 \qquad \left(\begin{array}{rrrr|r} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & -8 \end{array}\right)$$

The system is incompatible since the last row implies the equation $0 = -8$.

Lay, 1.1.16

Clara Susana Rey Abad, Nov. 4, 2013

Determine whether the following system is consistent (do not fully solve the system).

$$\begin{array}{rrrrcr} 2x_1 & & & -4x_4 & = & -10 \\ & +3x_2 & +3x_3 & & = & 0 \\ & & +x_3 & +4x_4 & = & -1 \\ -3x_1 & +2x_2 & +3x_3 & +x_4 & = & 5 \end{array}$$

Solution: Let us construct the augmented system matrix

$$\left(\begin{array}{rrrr|r} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ -3 & 2 & 3 & 1 & 5 \end{array}\right)$$

Now, we apply row operations to solve it

$$\mathbf{r}_1 \leftarrow \mathbf{r}_1 \div 2 \qquad \left(\begin{array}{rrrr|r} 1 & 0 & 0 & -2 & -5 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ -3 & 2 & 3 & 1 & 5 \end{array}\right)$$

$$\mathbf{r}_2 \leftarrow \mathbf{r}_2 \div 3 \qquad \left(\begin{array}{rrrr|r} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ -3 & 2 & 3 & 1 & 5 \end{array}\right)$$

$$\mathbf{r}_4 \leftarrow \mathbf{r}_4 + 3\mathbf{r}_1 \qquad \left(\begin{array}{rrrr|r} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 2 & 3 & -5 & -10 \end{array}\right)$$

$$\mathbf{r}_4 \leftarrow \mathbf{r}_4 - 2\mathbf{r}_2 \qquad \left(\begin{array}{rrrr|r} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -5 & -10 \end{array}\right)$$

$$\mathbf{r}_4 \leftarrow \mathbf{r}_4 - \mathbf{r}_3 \qquad \left(\begin{array}{rrrr|r} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & -9 & -9 \end{array}\right)$$

The system is compatible since there are four equations and four leading entries.

Lay, 1.1.17

Andrea Santos Cortés, Oct. 20th., 2014

Do the three lines $2x_1 + 3x_2 = -1$, $6x_1 + 5x_2 = 0$ and $2x_1 - 5x_2 = 7$ have a common point of intersection? Explain.

Solution: Let us construct the augmented system matrix

$$\left(\begin{array}{rr|r} 2 & 3 & -1 \\ 6 & 5 & 0 \\ 2 & -5 & 7 \end{array}\right)$$

Now, we apply row operations to determine whether it is compatible or not

$$\begin{array}{cc}
\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1 & \left(\begin{array}{rr|r} 2 & 3 & -1 \\ 6 & 5 & 0 \\ 0 & 8 & -8 \end{array}\right) \\
\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1 & \left(\begin{array}{rr|r} 2 & 3 & -1 \\ 0 & -4 & 3 \\ 0 & 8 & -8 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_2 & \left(\begin{array}{rr|r} 2 & 3 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & -2 \end{array}\right)
\end{array}$$

The system is incompatible and consequently the three lines do not intersect at a common point.

Lay, 1.1.18

Carlos Oscar Sorzano, Aug. 31st, 2013

Do the three planes $2x_1 + 4x_2 + 4x_3 = 4$, $x_2 - 2x_3 = -2$ and $2x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

Solution: Let us construct the augmented system matrix

$$\left(\begin{array}{rrr|r} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{array}\right)$$

Now, we apply row operations to determine whether it is compatible or not

$$\begin{array}{cc}
\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1 & \left(\begin{array}{rrr|r} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{array}\right) \\
\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2 & \left(\begin{array}{rrr|r} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{array}\right)
\end{array}$$

The system is compatible determinate and consequently the three planes intersect at a single point.

Lay, 1.1.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system.

$$\left(\begin{array}{rrr|l} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{array}\right)$$

Solution: We apply row operations to reduce this augmented matrix

$$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_1 \quad \left(\begin{array}{rrr|r} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & 2g+k \end{array}\right)$$

$$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2 \quad \left(\begin{array}{rrr|r} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & 2g+k+h \end{array}\right)$$

The system is compatible only if $2g + k + h = 0$. In this case, the system has infinite solutions since it is compatible indeterminate.

Lay, 1.1.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose the system below is compatible for all possible values of f and g. What can you say about the coefficients c and d?

$$\begin{array}{rcl} 2x_1 + 4x_2 & = & f \\ cx_1 + dx_2 & = & g \end{array}$$

Solution: Let us construct the augmented matrix and apply row operations to reduce it

$$\left(\begin{array}{cc|c} 2 & 4 & f \\ c & d & g \end{array}\right)$$

$$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \frac{c}{2}\mathbf{r}_1 \quad \left(\begin{array}{cr|r} 2 & 4 & f \\ 0 & d-2c & g - \frac{cf}{2} \end{array}\right)$$

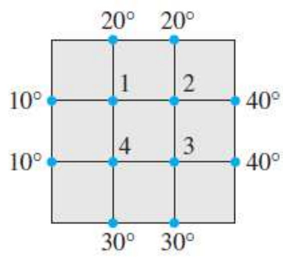
If the system is compatible for any value of f and g, then it must be that the coefficient $d - 2c$ is different from 0 (if it were 0, then there would be combinations of f and g for which the system would be incompatible).

Lay, 1.1.33

Carlos Oscar Sorzano, Aug. 31st, 2013

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin-plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let T_1, ..., T_4 denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes (to the left, below, right and above). For instance,

$$T_1 = \frac{1}{4}(10 + 20 + T_2 + T_4)$$



Write a system of four equations whose solution gives estimates for the temperatures T_1, ..., T_4

Solution: The following equations express the temperatures at each node as the average of the four surrounding nodes.

$$\begin{aligned}
T_1 &= \tfrac{1}{4}(10 + 20 + T_2 + T_4)\\
T_2 &= \tfrac{1}{4}(20 + 40 + T_1 + T_3)\\
T_3 &= \tfrac{1}{4}(30 + 40 + T_2 + T_4)\\
T_4 &= \tfrac{1}{4}(10 + 30 + T_1 + T_3)
\end{aligned}$$

We may rewrite this equation system as

$$\begin{array}{rrrrcl}
T_1 & -\frac{1}{4}T_2 & & -\frac{1}{4}T_4 & = & 7.5\\
-\frac{1}{4}T_1 & +T_2 & -\frac{1}{4}T_3 & & = & 15\\
& -\frac{1}{4}T_2 & +T_3 & -\frac{1}{4}T_4 & = & 17.5\\
-\frac{1}{4}T_1 & & -\frac{1}{4}T_3 & +T_4 & = & 10
\end{array}$$

Lay, 1.2.1

Ignacio Sanchez Lopez, Jan. 12th, 2015

Determine which of the following matrices are in reduced echelon form and which others are only in echelon form.

a $\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$

d $\begin{pmatrix} 1 & 1 & 0 & 1 & 1\\ 0 & 2 & 0 & 2 & 2\\ 0 & 0 & 0 & 3 & 3\\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$

Solution:

a It is in reduced echelon form.

b It is in echelon form .

c It is not in echelon form nor in reduced echelon form because there is a row full of zeros above another row with elements diferent from zero .

d It is in echelon form.

Lay, 1.2.2

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine which of the following matrices are in reduced echelon form and which others are only in echelon form.

a $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

c $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

d $\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: Let's remind the conditions to be in reduced echelon form.

1. Within each row, the first element different from zero (called the leading entry) is in a column to the right of the leading entry of the previous row.
2. Within each column, all values below a leading entry are zero.
3. All rows without a leading entry (i.e., they only have zeros) are below all the rows in which at least one element is not zero.
4. The leading entry of each row is 1.
5. The leading entry is the only 1 in its column.

Those matrices meeting only 1-3 are said to be in echelon form. Looking at the matrices of the exercise.

a It is in reduced echelon.

b It is in echelon form because there is a leading entry in the second column but it is not 1.

c It is not in echelon form nor in reduced echelon form because the first row is full of zeroes, and there are rows with leading entries below.

d It is in echelon form because the leading entries in each row are not the only non-zero values in their columns.

Lay, 1.2.7

Ignacio Sanchez Lopez, Jan. 14th, 2015

Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array}\right)$$

Solution: The augmented matrix is row equivalent to

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

that represents the equations

$$\begin{aligned} x_1 &= -3x_2 + 3 \\ x_3 &= 3 \end{aligned}$$

and there is no constraint for x_2 Therefore, the set of solutions of the equation system is

$$S = \{(-3x_2 + 3, x_2, 3) \quad \forall x_2 \in \mathbb{R}\}$$

Lay, 1.2.8

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the general solution of the system whose augmented matrix is

$$\begin{pmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{pmatrix}$$

Solution: The augmented matrix is row equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \end{pmatrix}$$

That represents the equations

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 3 \end{aligned}$$

and there is no constraint for x_3. Therefore, the set of solutions of the equation system is

$$S = \{(4, 3, x_3) \quad \forall x_3 \in \mathbb{R}\}$$

Lay, 1.2.9

Ignacio Sanchez Lopez,Jan. 14th, 2015

Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & 6 \end{array}\right)$$

Solution: The augmented matrix is row equivalent to

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & 1 & -6 & 5 \end{array}\right)$$

that represents the equations

$$\begin{aligned} x_1 &= -5x_3 + 16 \\ x_2 &= 6x_3 + 5 \end{aligned}$$

and there is no constraint for x_3. Therefore, the set of solutions of the equation system is

$$S = \{(-5x_3 + 16, 6x_3 + 5, x_3) \quad \forall x_3 \in \mathbb{R}\}$$

Lay, 1.2.10

Ignacio Sanchez Lopez, Jan. 14th, 2015

Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{array}\right)$$

Solution: The augmented matrix is row equivalent to

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & -7 \end{array}\right)$$

that represents the equations

$$\begin{aligned} x_1 &= 2x_2 - 4 \\ x_3 &= -7 \end{aligned}$$

and there is no constraint for x_2. Therefore, the set of solutions of the equation system is

$$S = \{(2x_2 - 4, x_2, -7) \quad \forall x_2 \in \mathbb{R}\}$$

Lay, 1.2.11

Ignacio Sanchez Lopez, Jan. 17th, 2015

Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{array}\right)$$

Solution: The augmented matrix is row equivalent to

$$\left(\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

That represents the equation

$$x_1 = 4/3x_2 - 2/3x_3$$

and there are no constraints for x_2, x_3. Therefore, the set of solutions of the equation system is

$$S = \{(4/3x_2 - 2/3x_3, x_2, x_3) \quad \forall x_2, x_3 \in \mathbb{R}\}$$

Lay, 1.2.18

Carlos Oscar Sorzano, June, 14th 2014

Determine h such that the augmented matrix

$$\begin{pmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{pmatrix} \tag{1}$$

corresponds to a consistent linear system.
Solution: If we subtract h times the first row from the second row we get the augmented matrix

$$\begin{pmatrix} 1 & -3 & 1 \\ 0 & 6+3h & -2-h \end{pmatrix} \tag{2}$$

If the system must be consistent then

1. Either $6 + 3h \neq 0 \Rightarrow h \neq -2$, or
2. $6 + 3h = 0$ and $-2 - h = 0$. These two equations are satisfied by $h = -2$

Consequently, it does not matter the value h takes, the equation system is always consistent.

Lay, 1.2.19

Carlos Oscar Sorzano, Aug. 31st, 2013

In the following equation system

$$\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$$

choose values for h and k such that it has (a) no solution, (b) a unique solution, and (c) many solutions.
Solution: The augmented matrix of the equation system is

$$\begin{pmatrix} 1 & h & 2 \\ 4 & 8 & k \end{pmatrix}$$

Let's reduce it

$$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 4\mathbf{r}_1 \quad \begin{pmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{pmatrix}$$

a If $8 - 4h = 2$ and $k - 8 \neq 0$, then the equation system has no solution. Two specific values are $h = \frac{3}{2}$ and $k = 0$.

b If $8 - 4h \neq 2$, then there is a unique solution. In particular, for $h = k = 0$, the equation system has a unique solution.

c If $8 - 4h = 2$ and $k - 8 = 0$, there are infinite solutions. Particularly, this happens for $h = \frac{3}{2}$ and $k = 8$.

Lay, 1.2.25

Carlos Oscar Sorzano, Nov. 4th, 2014

Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

Solution: Tne only way in which a equation system is inconsistent is if there is a row in which there is no pivot while the corresponding independent term is not 0. If the coefficient matrix has a pivot in every row, then the system cannot be inconsistent, and it is consistent, consequently.

Lay, 1.2.33

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data $(1, 6)$, $(2, 15)$, and $(3, 28)$.

Solution: We need to find a_0, a_1 and a_2 such that

$$\begin{array}{rcl} a_0 + a_1(1) + a_2(1)^2 & = & 6 \\ a_0 + a_1(2) + a_2(2)^2 & = & 15 \\ a_0 + a_1(3) + a_2(3)^2 & = & 28 \end{array} \Rightarrow \begin{pmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \\ 28 \end{pmatrix}$$

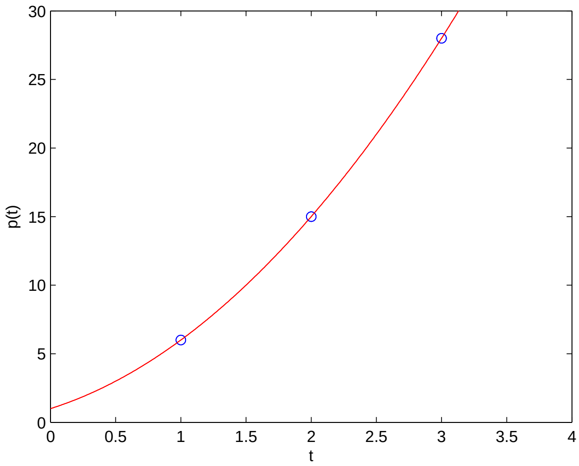
The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1^2 & 6 \\ 1 & 2 & 2^2 & 15 \\ 1 & 3 & 3^2 & 28 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Consequently, $a_0 = 1$, $a_1 = 3$ and $a_2 = 2$. The interpolating polynomial is

$$p(t) = 1 + 3t + 2t^2$$

The data points as well as the polynomial are represented below



Lay, 1.2.34

Carlos Oscar Sorzano, Aug. 31st, 2013

In a wind tunnel, the force on a projectile due to air resistance was measured at different velocities:

Velocity (100 ft/s)	0	2	4	6	8	10
Force (100 lb)	0	2.90	14.8	39.6	74.3	119

Find an interpolating polynomial for these data and estimate the force on the projectile when it is travelling at 750 ft/s. Use $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$. What happens if you try to use a polynomial of degree 3?

Solution: Similarly to the previous problem, the equation system is

$$\begin{pmatrix} 1 & 0 & 0^2 & 0^3 & 0^4 & 0^5 \\ 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 1 & 4 & 4^2 & 4^3 & 4^4 & 4^5 \\ 1 & 6 & 6^2 & 6^3 & 6^4 & 6^5 \\ 1 & 8 & 8^2 & 8^3 & 8^4 & 8^5 \\ 1 & 10 & 10^2 & 10^3 & 10^4 & 10^5 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.9 \\ 14.8 \\ 39.6 \\ 74.3 \\ 119 \end{pmatrix}$$

Its solution is

$$f(t) = 1.7125t - 1.1948t^2 + 0.6615t^3 - 0.0701t^4 + 0.0026t^5$$

At a velocity of 750 ft/s, the force on the projectile is

$$f(7.50) = 1.7125(7.50) - 1.1948(7.50)^2 + 0.6615(7.50)^3 - 0.0701(7.50)^4 + 0.0026(7.50)^5 = 64.6(100lb)$$

If we try to solve the same equation system with a polynomial of degree 3,

$$\begin{pmatrix} 1 & 0 & 0^2 & 0^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 6 & 6^2 & 6^3 \\ 1 & 8 & 8^2 & 8^3 \\ 1 & 10 & 10^2 & 10^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2.9 \\ 14.8 \\ 39.6 \\ 74.3 \\ 119 \end{pmatrix}$$

we find that there is no solution of the equation system.

Lay, 1.3.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Compute $\mathbf{u}+\mathbf{v}$ with $\mathbf{u}=\begin{pmatrix}-1\\2\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}-3\\-1\end{pmatrix}$

Solution:

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}-1\\2\end{pmatrix}+\begin{pmatrix}-3\\-1\end{pmatrix}=\begin{pmatrix}-4\\1\end{pmatrix}$$

Lay, 1.3.2

Marta Monsalve Buendía, Oct. 13th, 2014

Compute $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-2\mathbf{v}$ with $\mathbf{u}=\begin{pmatrix}3\\2\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}2\\-1\end{pmatrix}$

Solution:

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix}$$

$$\mathbf{u}-2\mathbf{v}=\begin{pmatrix}3\\2\end{pmatrix}-2\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}-1\\4\end{pmatrix}$$

Lay, 1.3.3

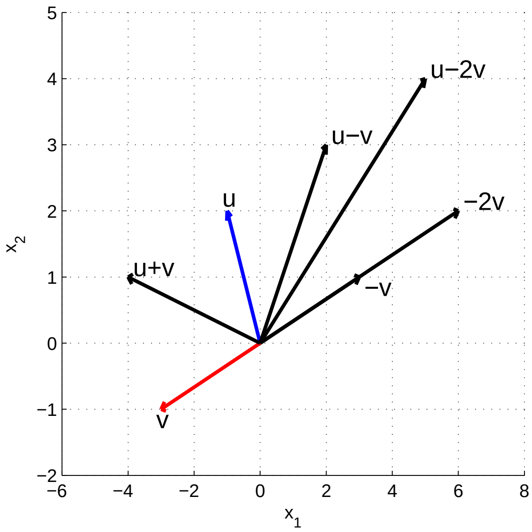
Carlos Oscar Sorzano, Aug. 31st, 2013

Draw in a graph $\mathbf{u}$, $\mathbf{v}$, $-\mathbf{v}$, $-2\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $\mathbf{u}-2\mathbf{v}$ with $\mathbf{u}=\begin{pmatrix}-1\\2\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}-3\\-1\end{pmatrix}$

Solution: Let's make first all these calculations:

$$\begin{array}{rcl}
\mathbf{u} & = & \begin{pmatrix}-1 & 2\end{pmatrix}\\
\mathbf{v} & = & \begin{pmatrix}-3 & -1\end{pmatrix}\\
-\mathbf{v} & = & -\begin{pmatrix}-3 & -1\end{pmatrix}=\begin{pmatrix}3 & 1\end{pmatrix}\\
-2\mathbf{v} & = & -2\begin{pmatrix}-3 & -1\end{pmatrix}=\begin{pmatrix}6 & 2\end{pmatrix}\\
\mathbf{u}+\mathbf{v} & = & \begin{pmatrix}-1 & 2\end{pmatrix}+\begin{pmatrix}-3 & -1\end{pmatrix}=\begin{pmatrix}-4 & 1\end{pmatrix}\\
\mathbf{u}-\mathbf{v} & = & \begin{pmatrix}-1 & 2\end{pmatrix}-\begin{pmatrix}-3 & -1\end{pmatrix}=\begin{pmatrix}2 & 3\end{pmatrix}\\
\mathbf{u}-2\mathbf{v} & = & \begin{pmatrix}-1 & 2\end{pmatrix}-2\begin{pmatrix}-3 & -1\end{pmatrix}=\begin{pmatrix}-1 & 2\end{pmatrix}+\begin{pmatrix}6 & 2\end{pmatrix}=\begin{pmatrix}5 & 4\end{pmatrix}
\end{array}$$

The following figure shows these vectors



Lay, 1.3.5

Yolanda Manrique Marcos, Dec. 17th, 2013

Write an equation system that is equivalent to the vector equation:

$$x_1 \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$

Solution: We may write the following equation system (in matrix form):

$$\begin{pmatrix} 3 & 5 \\ -2 & 0 \\ 8 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$

Lay, 1.3.6

Carlos Oscar Sorzano, Aug. 31st, 2013

Write an equation system that is equivalent to the vector equation:

$$x_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

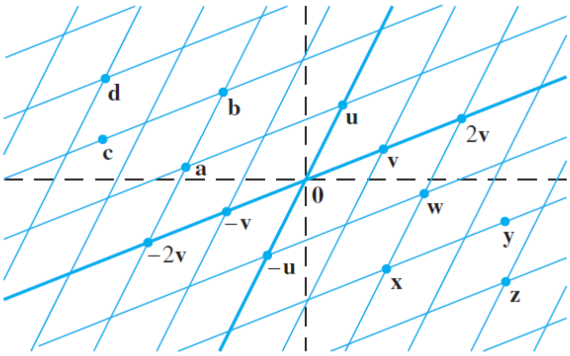
Solution: We may write the following equation system (in matrix form):

$$\begin{pmatrix} 3 & 7 & -2 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Lay, 1.3.7

Carlos Oscar Sorzano, Aug. 31st, 2013

In the following figure



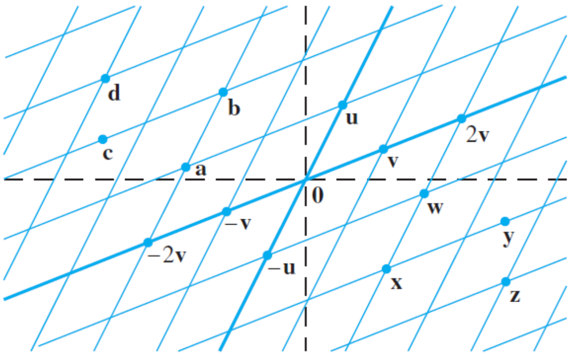
write **a**, **b**, **c** and **d** as a function of **u** and **v**.
Solution:

$$\begin{aligned}\mathbf{a} &= \mathbf{u} - 2\mathbf{v}\\ \mathbf{b} &= 2\mathbf{u} - 2\mathbf{v}\\ \mathbf{c} &= 2\mathbf{u} - 3.5\mathbf{v}\\ \mathbf{c} &= 3\mathbf{u} - 4\mathbf{v}\end{aligned}$$

Lay, 1.3.8

Marta Monsalve Buendía, Oct. 13th, 2014

In the following figure



write **w**, **x**, **y** and **z** as a function of **u** and **v**.
Solution:

$$\begin{aligned}\mathbf{w} &= -\mathbf{u} + 2\mathbf{v}\\ \mathbf{x} &= -2\mathbf{u} + 2\mathbf{v}\\ \mathbf{y} &= -2\mathbf{u} + 3.5\mathbf{v}\\ \mathbf{z} &= -3\mathbf{u} + 4\mathbf{v}\end{aligned}$$

Lay, 1.3.13

Andrea Santos Cortés, Oct. 15th, 2014

Let $A = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix}$. Determine if $\mathbf{b}$ is a linear combination of the vectors formed from the columns of the matrix A

Solution: The vectors that form the columns of the matrix A are $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix}$ and $\mathbf{a}_3 = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$. To check whether $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$,$\mathbf{a}_2$ and $\mathbf{a}_3$ all we have to do is to find coefficients x_1, x_2, and x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

or what is the same

$$\begin{pmatrix} x_1 - 4x_2 + 2x_3 \\ 3x_2 + 5x_3 \\ -2x_1 + 8x_2 - 4x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix}$$

This is an inconsistent system of equations and, consequently,

$$\mathbf{b} \notin \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$$

Lay, 1.3.22

Carlos Oscar Sorzano, Nov. 4th, 2014

Construct a 3×3 matrix A, with nonzero entries, and a vector $\mathbf{b} \in \mathbb{R}^3$ such that $\mathbf{b}$ is not in the set spanned by the columns of A.

Solution: Let $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$. Obviously $\mathbf{b}$ is not in the space spanned by the columns of A because the column space of A are all vectors of the form (x_1, x_1, x_1).

Lay, 1.3.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}$. Denote the columns of A as $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, and let $W = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

a Is $\mathbf{b}$ in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vector are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?

b Is $\mathbf{b}$ in W? How many vectors are in W?

c Show that $\mathbf{a}_1$ is in W.

Solution:

a No, $\mathbf{b}$ is not equal to any of the columns of A. In $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ there are only 3 vectors.

b In W there are infinite vectors. To check whether $\mathbf{b}$ is in W all we have to do is to find coefficients x_1, x_2, and x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

or what is the same

$$\begin{pmatrix} x_1 - 4x_2 \\ 3x_2 - 2x_3 \\ -2x_1 + 6x_2 + 3x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}$$

Solving the equation system we find: $x_1 = -4$, $x_2 = -1$, $x_3 = -2$, i.e.

$$\mathbf{b} = -4\mathbf{a}_1 - \mathbf{a}_2 - 2\mathbf{a}_3 = -4\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} - 2\begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}$$

and, consequently, $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

c It is enough to observe that

$$\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$$

Lay, 1.3.27

Carlos Oscar Sorzano, Aug. 31st, 2013

A mining company has two mines. One's day operation at mine #1 produces ore that contains 30 metric tons of copper and 600 kg of silver, while one day's operation at mine #2 produces ore that containes 4 metric tones of copper and 380 kg of silver. Let $\mathbf{v}_1 = \begin{pmatrix} 30 \\ 600 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 40 \\ 380 \end{pmatrix}$. Then, $\mathbf{v}_1$ and $\mathbf{v}_2$ represent the output per day at mines #1 and #2, respectively.

a What physical interpretation can be given to the vector $5\mathbf{v}_1$?

b Suppose the company operates mine #1 for x_1 days, and mine #2 for x_2 days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 240 tons of copper and 2824 kg of silver.

c Solve the previous equation

Solution:

a $5\mathbf{v}_1$ is the production of copper and silver of mine #1 after 5 days of operation.

b The vector equation sought is

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{pmatrix} 240 \\ 2824 \end{pmatrix}$$

or what is the same

$$\begin{pmatrix} 30x_1 + 40x_2 \\ 600x_1 + 380x_2 \end{pmatrix} = \begin{pmatrix} 240 \\ 2824 \end{pmatrix}$$

c The solution of this equation is $x_1 = 1.7270$ and $x_2 = 4.7048$, as can be easily checked

$$\begin{pmatrix} 30 \cdot 1.7270 + 40 \cdot 4.7048 \\ 600 \cdot 1.7270 + 380 \cdot 4.7048 \end{pmatrix} = \begin{pmatrix} 240 \\ 2824 \end{pmatrix}$$

Lay, 1.3.29

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ be points in $\mathbb{R}^3$ and suppose that for $j = 1, 2, ..., k$ an object of mass m_j is located at point $\mathbf{v}_j$. Physicists call such objects as *point masses*. The total mass of the system of point masses is

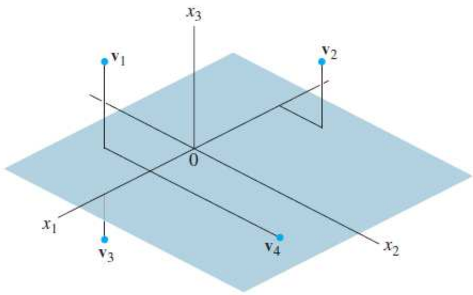
$$m = m_1 + m_2 + ... + m_k$$

The *center of gravity* (or *center of mass*) of the system is:

$$\overline{\mathbf{v}} = \frac{1}{m}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + ... + m_k\mathbf{v}_k)$$

Compute the center of mass of the system consisting of the following point masses (see figure):

Point	Mass
$\mathbf{v}_1 = (2, -2, 4)$	4 g
$\mathbf{v}_2 = (-4, 2, 3)$	2 g
$\mathbf{v}_3 = (4, 0, -2)$	3 g
$\mathbf{v}_4 = (1, -6, 0)$	5 g



Solution: Let us calculate the total mass

$$m = m_1 + m_2 + m_3 + m_4 = 4 + 2 + 3 + 5 = 14g$$

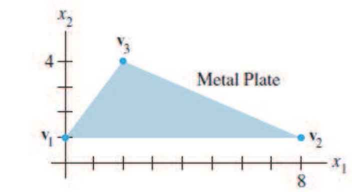
Now, the center of mass

$$\overline{\mathbf{v}} = \frac{1}{m}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + m_4\mathbf{v}_4) =$$
$$\frac{1}{14}\left(4\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} + 2\begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + 5\begin{pmatrix} 1 \\ -6 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \frac{17}{14} \\ -\frac{17}{7} \\ \frac{8}{7} \end{pmatrix}$$

Lay, 1.3.31

Carlos Oscar Sorzano, Aug. 31st, 2013

A thin triangular, metal plate of uniform density and thickness has vertices $\mathbf{v}_1 = (0,1)$, $\mathbf{v}_2 = (8,1)$, and $\mathbf{v}_3 = (2,4)$



and the mass of the plate is 3g.

a Find the (x,y)-coordinates of the center of mass of the plate. This "balanced point" of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.

b Determine how to distribute an additional mass of 6g at the three vertices to move the balance point of the plate to (2,2).

Solution:

a Let us calculate the total mass

$$m = m_1 + m_2 + m_3 = 1 + 1 + 1 = 3g$$

Now, the center of mass

$$\overline{\mathbf{v}} = \frac{1}{m}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3) = \frac{1}{3}\left(1\begin{pmatrix}0\\1\end{pmatrix} + 1\begin{pmatrix}8\\1\end{pmatrix} + 1\begin{pmatrix}2\\4\end{pmatrix}\right) = \begin{pmatrix}\frac{10}{3}\\2\end{pmatrix}$$

b If we now want to shift the center of masses, let us define as w_1, w_2 and w_3 the masses to be added to each one of the vertices, with the constraint

$$w_1 + w_2 + w_3 = 6$$

The new center of masses will be

$$\begin{gathered}\overline{\mathbf{v}} = \frac{1}{m+6}((m_1+w_1)\mathbf{v}_1 + (m_2+w_2)\mathbf{v}_2 + (m_3+w_3)\mathbf{v}_3) = \\ \frac{1}{9}\left(\begin{pmatrix}0\\1+w_1\end{pmatrix} + \begin{pmatrix}8(1+w_2)\\1+w_2\end{pmatrix} + \begin{pmatrix}2(1+w_3)\\4(1+w_3)\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix} \\ \begin{pmatrix}\frac{10+8w_2+2w_3}{9}\\\frac{6+w_1+w_2+4w_3}{9}\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}\end{gathered}$$

which gives us the equation system

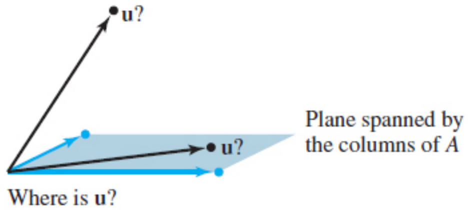
$$\begin{aligned} w_1 + w_2 + w_3 &= 6 \\ 8w_2 + 2w_3 &= 8 \\ w_1 + w_2 + 4w_3 &= 12 \end{aligned}$$

whose solution is $w_1 = 3.5$g, $w_2 = 0.5$g, $w_3 = 2$g.

Lay, 1.4.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u} = (0, 4, 4)$ and $A = \begin{pmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{pmatrix}$. Is $\mathbf{u}$ in the plane spanned by the columns of A? Why or why not?



Solution: We need to solve the vector equation

$$\mathbf{u} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2$$

or what is the same, the equation system represented by the augmented matrix below

$$\left(\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & -5 & 0 \\ 0 & \frac{8}{3} & 4 \\ 0 & 0 & 0 \end{array}\right)$$

The system is compatible determinate, meaning that there exist c_1 and c_2 so that the vector equation is satisfied and, therefore, $\mathbf{u}$ belongs to the plane spanned by the columns of A.

Lay, 1.4.18

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $B = \begin{pmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{pmatrix}$. Can every row of $\mathbb{R}^4$ be written as a linear combination of the columns of B? Do the columns of B span $\mathbb{R}^3$?

Solution: Let's see if every column of B has a pivot element. For doing so, we will compute a row-equivalent matrix by applying row elementary operations:

$$B \sim \begin{pmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Not all the columns have a pivot element, for instance, column 3 has not, therefore, the columns of B cannot spane $\mathbb{R}^4$. The columns of B do not span $\mathbb{R}^3$ because they are vectors of $\mathbb{R}^4$ and not vectors of $\mathbb{R}^3$.

Lay, 1.4.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u} = \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$. It can be shown that $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact (and no row operations) to find x_1 and x_2 that satisfy the equation:

$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$$

Solution: We note that the first column of the matrix in the system equation is $\mathbf{u}$, the second column is $\mathbf{v}$ and the vector of independent terms is $\mathbf{w}$. Consequently, the equation system is trying to find x_1 and x_2 such that

$$x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$$

Comparing this equation with the fact of the statement

$$2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0} \Rightarrow 2\mathbf{u} - 3\mathbf{v} = \mathbf{w}$$

we deduce that $x_1 = 2$ and $x_2 = -3$.

Lay, 1.4.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using $\mathbf{v}_1$, $\mathbf{v}_2$, ... for the vectors and c_1, c_2, ... for scalars. Define what each symbol represents using the data given in the matrix equation.

$$\begin{pmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ -11 \end{pmatrix}$$

Solution: Let us define $\mathbf{v}_1$ as the first column of the matrix (i.e., $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$), $\mathbf{v}_2$ as the second column ($\mathbf{v}_2 = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$), ... Let us also define c_1 as the first coefficient in the vector in the left-hand side of the equation ($c_1 = -3$), c_2 as the second coefficient ($c_2 = 1$), ...

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_5\mathbf{v}_5 = \begin{pmatrix} 11 \\ -11 \end{pmatrix}$$

Lay, 1.4.32

Carlos Oscar Sorzano, Aug. 31st, 2013

Could a set of 3 vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

Solution: None of the two situations is possible. To span all $\mathbb{R}^4$ one need at least 4 vectors (in fact it is enough with 4 linearly independent vectors). The same happens with $\mathbb{R}^m$, one needs at least m vectors. Any smaller number of vectors cannot fully span $\mathbb{R}^m$.

Lay, 1.4.35

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be a 5×3 matrix, let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and let $\mathbf{z}$ be a vector in $\mathbb{R}^5$. Suppose $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = 5\mathbf{z}$ is consistent?

Solution: If $A\mathbf{y} = \mathbf{z}$, then multiplying the equation by 5 we get

$$5(A\mathbf{y}) = 5\mathbf{z}$$

Using the properties of scalar, matrix and vector multiplications, we may rearrange the equation as

$$A(5\mathbf{y}) = 5\mathbf{z}$$

Now, simply calling $\mathbf{x} = 5\mathbf{y}$ we get the equation proposed in the problem:

$$A\mathbf{x} = 5\mathbf{z}$$

whose solution is obviously $\mathbf{x} = 5\mathbf{y}$.

Lay, 1.4.37

Sarah Rance Lopez, Jan. 12th, 2015

Determine if the columns of the following matrix span all $\mathbb{R}^4$

$$A = \begin{pmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{pmatrix}$$

Solution: By applying row operations we reach

$$A \sim \begin{pmatrix} 7 & 2 & -5 & 8 \\ 0 & -11 & 3 & -23 \\ 0 & 0 & 50 & -189 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This latter matrix has 3 pivot columns (1, 2, 3), therefore, it does not span $\mathbb{R}^4$.

Lay, 1.4.38

Sarah Rance Lopez, Jan. 12th, 2015

Determine if the columns of the following matrix span all $\mathbb{R}^4$

$$A = \begin{pmatrix} 4 & -5 & -1 & 8 \\ 3 & -7 & -4 & 2 \\ 5 & -6 & -1 & 4 \\ 9 & 1 & 10 & 7 \end{pmatrix}$$

Solution: By applying row operations we reach

$$A \sim \begin{pmatrix} 4 & -5 & -1 & 8 \\ 0 & -1 & -1 & 24 \\ 0 & 0 & 72 & 1084 \\ 0 & 0 & 0 & 328 \end{pmatrix}$$

This latter matrix has 4 pivot columns, therefore, it spans $\mathbb{R}^4$.

Lay, 1.4.39

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine if the columns of the following matrix span all $\mathbb{R}^4$

$$A = \begin{pmatrix} 10 & -7 & 1 & 4 & 6 \\ -8 & 4 & -6 & -10 & -3 \\ -7 & 11 & -5 & -1 & -8 \\ 3 & -1 & 10 & 12 & 12 \end{pmatrix}$$

Solution: By applying row operations we reach

$$A \sim \begin{pmatrix} 10 & -7 & 1 & 4 & 6 \\ 0 & -1.6 & -5.2 & -6.8 & 1.8 \\ 0 & 0 & -24.125 & -24.125 & 3.0625 \\ 0 & 0 & 0 & 0 & 12.215 \end{pmatrix}$$

This latter matrix has 4 pivot columns (1, 2, 3 and 5), therefore, it spans $\mathbb{R}^4$.

Lay, 1.4.40

Sarah Rance Lopez, Jan. 12th 2015

Determine if the columns of the following matrix span all $\mathbb{R}^4$

$$A = \begin{pmatrix} 5 & 11 & -6 & -7 & 12 \\ -7 & -3 & -4 & 6 & -9 \\ 11 & 5 & 6 & -9 & -3 \\ -3 & 4 & 7 & 2 & 7 \end{pmatrix}$$

Solution: By applying row operations we reach

$$A \sim \begin{pmatrix} 5 & 11 & -6 & -7 & 12 \\ 0 & 1 & -0.87 & -0.3 & 0.02 \\ 0 & 0 & -1 & -0.99 & -0.77 \\ 0 & 0 & 0 & -0.6 & 0.07 \end{pmatrix}$$

This latter matrix has 4 pivot columns (1, 2, 3 and 4), therefore, it spans $\mathbb{R}^4$.

Lay, 1.5.7

Sarah Rance Lopez,Jan. 12th, 2015

Describe all solutions of $A\mathbf{x} = \mathbf{0}$ where A is row equivalent to

$$B = \begin{pmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{pmatrix}$$

Solution: Last equation implies that $x_2 = 4x_3 - 5x_4$. The first equation implies

$$\begin{aligned} x_1 &= -3x_2 + 3x_3 - 7x_4 \\ x_2 &= 4x_3 - 5x_4 \end{aligned}$$

Considering the last equation we may simplify the first equation:

$$x_1 = -3(4x_3 - 5x_4) + 3x_3 - 7x_4 = -9x_3 + 8x_4$$

Gathering all this information we deduce that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{pmatrix} \quad \forall x_3, x_4 \in \mathbb{R}$$

Note that the free variables (x_3, x_4) are the non-pivot columns in matrix B.

Lay, 1.5.10

Ignacio Sanchez Lopez, Jan. 12th, 2015

Describe all solutions of $A\mathbf{x} = \mathbf{0}$ where A is row equivalent to

$$B = \begin{pmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & 8 \end{pmatrix}$$

Solution: We can clearly see that row 1 and row to are equivalents(row 2 is 2 times row 1), and we can easily deduce

$$x_1 = -3x_2 + 4x_4$$

Gathering all this information we deduce that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} -3x_2 + 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \forall x_2, x_3, x_4 \in \mathbb{R}$$

Note that the free variables (x_2, x_3, x_4) are the non-pivot columns in matrix B.

Lay, 1.5.11

Carlos Oscar Sorzano, Aug. 31st, 2013

Describe all solutions of $A\mathbf{x} = \mathbf{0}$ where A is row equivalent to

$$B = \begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Last equation implies that $x_5 - 4x_6 = 0$ or what is the same $x_5 = 4x_6$. Similarly, the first three equations imply

$$\begin{aligned} x_1 = 4x_2 + 2x_3 - 3x_5 + 5x_6 \\ x_3 = x_6 \\ x_5 = 4x_6 \end{aligned}$$

Considering the last two equations we may simplify the first equation:

$$x_1 = 4x_2 + 2x_6 - 3(4x_6) + 5x_6 = 4x_2 - 5x_6$$

Gathering all this information we deduce that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{pmatrix} \quad \forall x_2, x_4, x_6 \in \mathbb{R}$$

Note that the free variables (x_2, x_4, x_6) are the non-pivot columns in matrix B.

Lay, 1.5.12

Sarah Rance Lopez, Jan. 12th, 2015

Describe all solutions of $A\mathbf{x} = \mathbf{0}$ where A is row equivalent to

$$B' = \begin{pmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: We can further reduce the system matrix to

$$B = \begin{pmatrix} 1 & -2 & 3 & 0 & 29 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This set of equations can be rewritten as

$$\begin{array}{rcl} x_1 & = & 2x_2 - 3x_3 - 29x_5 \\ x_4 & = & -4x_5 \\ x_6 & = & 0 \end{array}$$

Gathering all this information we deduce that the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} 2x_2 - 3x_3 - 29x_5 \\ x_2 \\ x_3 \\ -4x_5 \\ x_5 \\ 0 \end{pmatrix} \quad \forall x_2, x_3, x_5 \in \mathbb{R}$$

Note that the free variables (x_2, x_3, x_5) are the non-pivot columns in matrix B.

Lay, 1.5.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with x_3 free. Use vectors to describe this set as a line in $\mathbb{R}^3$.

Solution: The general solution of the system of linear equations is

$$\mathbf{x} = \begin{pmatrix} 5 + 4x_3 \\ -2 - 7x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}$$

This is a line that passes through the point $\mathbf{x}_0 = (5, -2, 0)$ and whose direction vector is $(4, -7, 1)$.

Lay, 1.5.14

Sarah Rance Lopez, Jan 12th, 2015

Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5x_4$, $x_2 = 3 - 2x_4$, $x_3 = 2 + 5x_4$ with x_4 free. Use vectors to describe this set as a "line" in $\mathbb{R}^3$.

Solution: The general solution of the system of linear equations is

$$\mathbf{x} = \begin{pmatrix} 5x_4 \\ 3 - 2x_4 \\ 2 + 5x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 5 \\ -2 \\ 5 \\ 1 \end{pmatrix}$$

This is a line that passes through the point $\mathbf{x}_0 = (0, 3, 2, 0)$ and whose direction vector is $(5, -2, 5, 1)$.

Lay, 1.5.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine the parametric equation of the line through $\mathbf{a} = (-2, 0)$ and parallel to $\mathbf{b} = (-5, 3)$.

Solution: The requested line can be expressed as a function of a free parameter $t \in \mathbb{R}$

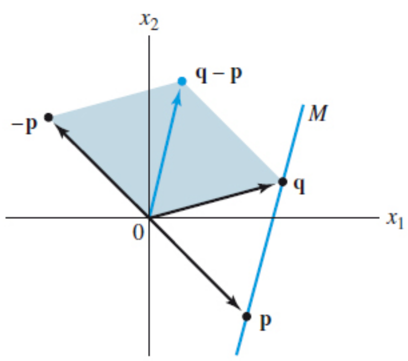
$$\mathbf{l}(t) = \mathbf{a} + t\mathbf{b} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 - 5t \\ 3t \end{pmatrix}$$

Lay, 1.5.21

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine the parametric equation of the line M through $\mathbf{p} = (3, -3)$ and $\mathbf{q} = (4, 1)$.

Solution: In the following figure, we show how the direction vector of the requested line is $\mathbf{q} - \mathbf{p}$ (or $\mathbf{p} - \mathbf{q}$).



The requested line can be expressed as a function of a free parameter $t \in \mathbb{R}$

$$\mathbf{l}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t \left(\begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \end{pmatrix} \right) = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Note that this line is at $\mathbf{p}$ for $t = 0$ and at $\mathbf{q}$ for $t = 1$.

Lay, 1.5.25

Carlos Oscar Sorzano, Aug. 31st, 2013

a Suppose $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{p} = \mathbf{b}$. Let $\mathbf{v}_h$ be any solution of the homogeneous equation $A\mathbf{v}_h = \mathbf{0}$ and let $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. Show that $\mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

b Let $\mathbf{w}$ be a any solution of $A\mathbf{x} = \mathbf{b}$, and define $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Show that $\mathbf{v}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$. This shows that every solution of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ with $\mathbf{p}$ a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ a solution of $A\mathbf{x} = \mathbf{0}$.

Solution:

a Let us check whether $A\mathbf{w} = \mathbf{b}$

$$\begin{array}{rcll} A\mathbf{w} & = & A(\mathbf{p} + \mathbf{v}_h) & \text{By definition of } \mathbf{w} \\ & = & A\mathbf{p} + A\mathbf{v}_h & \text{By distributivy of matrix product} \\ & = & \mathbf{b} + \mathbf{0} & \text{By definition of } \mathbf{p} \text{ and } \mathbf{v}_h \\ & = & \mathbf{b} & \text{Because } \mathbf{0} \text{ is the neutral of vector addition} \end{array}$$

b By definition of $\mathbf{w}$ and $\mathbf{p}$ we have

$$A\mathbf{w} = \mathbf{b}$$
$$A\mathbf{p} = \mathbf{b}$$

If we subtract both equations, we obtain

$$A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b}$$
$$A(\mathbf{w} - \mathbf{p}) = \mathbf{0}$$

Taking into account that $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$, this means

$$A\mathbf{v}_h = \mathbf{0}$$

As required by the exercise, we have proven that $\mathbf{v}_h$ is a solution of the equation $A\mathbf{x} = \mathbf{0}$.

Lay, 1.5.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose A is the 3×3 zero matrix. Describe the solution set of the equation $A\mathbf{x} = \mathbf{0}$

Solution: The set of solutions is $S = \mathbb{R}^3$ since for any vector $\mathbf{x} \in \mathbb{R}^3$ we have

$$A\mathbf{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

Lay, 1.5.36

Carlos Oscar Sorzano, Aug. 31st, 2013

Given $A = \begin{pmatrix} 3 & -2 \\ -6 & 4 \\ 12 & -8 \end{pmatrix}$, find one nontrivial solution of the equation $A\mathbf{x} = \mathbf{0}$.

Solution: We note that the second and third rows of A are multiples of the first one. So any solution of the form (given by the first row)

$$3x_1 - 2x_2 = 0$$

is a solution. In particular $x_1 = 2$ and $x_2 = 3$ is a solution. We can check that

$$2\begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 4 \\ -8 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \\ 24 \end{pmatrix} + \begin{pmatrix} -6 \\ 12 \\ -24 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Lay, 1.5.39

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be a $m \times n$ matrix, and let $\mathbf{v}$ and $\mathbf{w}$ be vectors with the property that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. Explain why $A(\mathbf{v} + \mathbf{w}) = \mathbf{0}$. Then, explain why $A(c\mathbf{v} + d\mathbf{w}) = \mathbf{0}$ for each pair of scalars c and d.

Solution: We know that

$$\begin{aligned} A\mathbf{v} &= \mathbf{0} \\ A\mathbf{w} &= \mathbf{0} \end{aligned}$$

Adding both equations

$$A(\mathbf{v} + \mathbf{w}) = \mathbf{0}$$

As stated by the problem. For showing that $A(c\mathbf{v} + d\mathbf{w}) = \mathbf{0}$ we may follow a different strategy

$$\begin{array}{rcll} A(c\mathbf{v} + d\mathbf{w}) & = & A(c\mathbf{v}) + A(d\mathbf{w}) & \text{By distributivity of matrix multiplication} \\ & = & c(A\mathbf{v}) + d(A\mathbf{w}) & \text{By scalar product property of matrix multiplication} \\ & = & c(\mathbf{0}) + d(\mathbf{0}) & \text{By definition of } \mathbf{v} \text{ and } \mathbf{w} \\ & = & \mathbf{0} & \end{array}$$

Lay, 1.6.5

Carlos Oscar Sorzano, Aug. 31st, 2013

An economy has four sectors: Agriculture, Manufacturing, Services and Transportation. Agriculture sells 20% of its output to Manufacturing, 30% to Services, 30% to Transportation, and retains the rest. Manufacturing sells 35% of its output to Agriculture, 35% to Services, 20% to Transportation, and retains the rest. Services sells 10% of its output to Agriculture, 20% to Manufacturing, 20% to Transportation, and retains the rest. Transportation sells 20% of its output to Agriculture, 30% to Manufacturing, 20% to Services and retains the rest.

a. Construct the exchange table of this economy.

b. Find a set of equilibrium prices for the economy if the value of Transportation is \$10,00 per unit.

c. The Services sector launches a successful "eat farm fresh" campaign, and increases its share of the output from the Agricultural sector to 40%, whereas the share of the Agricultural production going to Manufacturing falls to 10%. Construct the exchange table for this new economy.

d. Find a set of equilibrium prices for this new economy if the value of Transportation is still \$10,00 per unit. What effect has the "eat farm fresh" campaign had on the equilibrium prices for the sectors of this economy?

Solution:

a. The exchange matrix is given by

$$E = \begin{pmatrix} 0.20 & 0.20 & 0.30 & 0.30 \\ 0.35 & 0.10 & 0.25 & 0.20 \\ 0.10 & 0.20 & 0.50 & 0.20 \\ 0.20 & 0.30 & 0.20 & 0.30 \end{pmatrix}$$

First row implies that the output of Agriculture is sold 20% to Agriculture, 20% to Manufacturing, 30% to Services and 30% to Transportation.

b. In equilibrium the expenses of any of the sectors are equal to its incomes. If we construct the vector of sector values $\mathbf{v}$, we may express this relationship as

$$\mathbf{v} = E\mathbf{v} \Rightarrow (I - E)\mathbf{v} = \mathbf{0}$$

Expanding the different elements

$$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.20 & 0.20 & 0.30 & 0.30 \\ 0.35 & 0.10 & 0.35 & 0.20 \\ 0.10 & 0.20 & 0.50 & 0.20 \\ 0.20 & 0.30 & 0.20 & 0.30 \end{pmatrix} \right) \begin{pmatrix} v_A \\ v_M \\ v_S \\ v_T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\left(\begin{array}{rrrr|r} 0.80 & -0.20 & -0.30 & -0.30 & 0 \\ -0.35 & 0.90 & -0.35 & -0.20 & 0 \\ -0.10 & -0.20 & 0.50 & -0.20 & 0 \\ -0.20 & -0.30 & -0.20 & 0.70 & 0 \end{array} \right) \sim \left(\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The solution of this homogeneous system is

$$v_A = v_M = v_S = v_T \quad \forall v_T \in \mathbb{R}$$

In particular, since the problem states that $v_T = 10$, we have $v_A = v_M = v_S = v_T = 10$.

c. The new exchange table is

$$E = \begin{pmatrix} 0.20 & 0.10 & 0.40 & 0.30 \\ 0.35 & 0.10 & 0.25 & 0.20 \\ 0.10 & 0.20 & 0.50 & 0.20 \\ 0.20 & 0.30 & 0.20 & 0.30 \end{pmatrix}$$

d. The new augmented matrix is

$$\left(\begin{array}{rrrr|r} 0.80 & -0.10 & -0.40 & -0.30 & 0 \\ -0.35 & 0.90 & -0.35 & -0.20 & 0 \\ -0.10 & -0.20 & 0.50 & -0.20 & 0 \\ -0.20 & -0.30 & -0.20 & 0.70 & 0 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Again the solution is

$$v_A = v_M = v_S = v_T \quad \forall v_T \in \mathbb{R}$$

So, the campaign has had no effect on the different prices.

Lay, 1.6.7

Carlos Oscar Sorzano, Aug. 31st, 2013

Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water and carbon dioxide (gas):

$$NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_20 + CO_2$$

Balance this chemical equation.
Solution: Let's assign a number of molecules to each one of the compounds

$$x_1NaHCO_3 + x_2H_3C_6H_5O_7 \rightarrow x_3Na_3C_6H_5O_7 + x_4H_2O + x_5CO_2$$

Now let's count the number of atoms of each kind

$$\begin{array}{ll} Na: & x_1 = 3x_3 \\ H: & x_1 + 3x_2 + 5x_2 = 5x_3 + 2x_4 \\ C: & x_1 + 6x_2 = 6x_3 + x_5 \\ O: & 3x_1 + 7x_2 = 7x_3 + x_4 + 2x_5 \end{array}$$

The augmented matrix of this equation system is

$$\left(\begin{array}{rrrrr|r} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

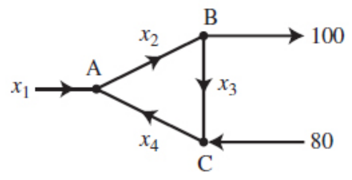
Letting $x_5 = 3$, we have $x_1 = x_5 = 3$, $x_2 = \frac{1}{3}x_5 = 1$, $x_3 = \frac{1}{3}x_5 = 1$, $x_4 = x_5 = 3$.
Finally, the balanced chemical reaction is

$$3NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + 3H_2O + 3CO_2$$

Lay, 1.6.12

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the smallest possible value for x_4?



Solution: To analyze this network we note that at each node the inputs must be equal to its outputs. Consequently:

$$\begin{array}{cc} \text{A} & x_1 + x_4 = x_2 \\ \text{B} & x_2 = x_3 + 100 \\ \text{C} & x_3 + 80 = x_4 \end{array}$$

The augmented matrix of this equation system is

$$\left(\begin{array}{rrrr|r} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -80 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & -1 & 20 \\ 0 & 0 & 1 & -1 & -80 \end{array}\right)$$

The general solution of this equation system is

$$(x_1, x_2, x_3, x_4) = (20, 20 + x_4, -80 + x_4, x_4)$$

If all flows must be nonnegative, x_4 must be at least 80, because otherwise, x_3 would be negative.

Lay, 1.7.9

Carlos Oscar Sorzano, Aug. 31st, 2013

Given the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 9 \\ -6 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 5 \\ -7 \\ h \end{pmatrix}$. For which value of h is the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent.

Solution: We need to solve the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$$

and find a non-trivial solution. The augmented matrix of this equation system is

$$\left(\begin{array}{rrr|r} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This equation system is compatible indeterminate for any value of h, meaning that the set of vectors is linearly dependent disregarding the value of h. This is because $\mathbf{v}_2 = -3\mathbf{v}_1$.

Lay, 1.7.39

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose A is a $m \times n$ matrix with the property that for all $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Use the definition of linear independence to explain why the columns of A must be linearly independent.

Solution: If $A\mathbf{x} = \mathbf{b}$ has at most one solution for all $\mathbf{b}$, then in particular for $\mathbf{b} = \mathbf{0}$, the equation $A\mathbf{x} = \mathbf{0}$ has at most one solution. But we already know that $\mathbf{x} = \mathbf{0}$ is a solution (the trivial solution). So it must be its only solution. Let us refer to the columns of A as $\mathbf{a}_i$ ($i = 1, 2, ..., n$). The equation $A\mathbf{x} = \mathbf{0}$ can be rewritten as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0}$$

Because the trivial solution is its only solution, then the set of column vectors of A is linearly independent.

Lay, 1.7.40

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose an $m \times n$ matrix has n pivot columns. Explain why for each $\mathbf{b} \in \mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. [Hint: Explain why $A\mathbf{x} = \mathbf{b}$ cannot have infinitely many solutions.

Solution: In order to have infinite solutions, we need to have free variables that correspond to non-pivot columns of the matrix A. If A has n pivot columns, then there are no free variables, and there cannot be an infinite number of solutions.

Lay, 1.8.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$.

a. Show that f is a linear transformation when $b = 0$.

b. Find a property of linear transformations that is violated when $b \neq 0$.

c. Why is f called a linear function?

Solution:

a. We need to show that $\forall x_1, x_2 \in \mathbb{R}$, $\forall c \in \mathbb{R}$

- $f(x_1 + x_2) = f(x_1) + f(x_2)$
  In this particular case:
  $$f(x_1 + x_2) = m(x_1 + x_2) = mx_1 + mx_2 = f(x_1) + f(x_2)$$
- $f(cx_1) = cf(x_1)$
  In this particular case:
  $$f(cx_1) = m(cx_1) = c(mx_1) = cf(x_1)$$

b. When $b \neq 0$ none of the two properties is fulfilled. Let's see an example with the second one:

$$f(cx_1) = m(cx_1) + b = cmx_1 + b \neq cmx_1 + cb = c(mx_1 + b) = cf(x_1)$$

c. f is called a linear function because its graph $(x, f(x))$ is a line the 2D plane. However, to be a linear transformation the line needs to pass through the origin. If $b \neq 0$ the line defined by f does not pass through the origin.

Lay, 1.8.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Given $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{p}$ in $\mathbb{R}^n$, the line through $\mathbf{p}$ in the direction of $\mathbf{v}$ has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Show that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps this line onto another line or onto a single point (*a degenerate line*).
Solution: Let's define $\mathbf{y} = T(\mathbf{x})$ and check whether it is a line or not:

$$\begin{array}{rcll} \mathbf{y} & = & T(\mathbf{x}) & \text{By definition of } \mathbf{x} \\ & = & T(\mathbf{p} + t\mathbf{v}) & \text{By linearity of } T \\ & = & T(\mathbf{p}) + tT(\mathbf{v}) & \end{array}$$

If $T(\mathbf{v}) \neq \mathbf{0}$, then $\mathbf{y}$ describes a line that goes through $T(\mathbf{p})$ in the direction of $T(\mathbf{v})$. If $T(\mathbf{v}) = \mathbf{0}$, then $\mathbf{y}$ is a single point.

Lay, 1.8.26

Carlos Oscar Sorzano, Aug. 31st, 2013

a. Show that the line through the vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$ may be written in parametric form as $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$.

b. The line segment from $\mathbf{p}$ to $\mathbf{q}$ is the set of points of the form $(1-t)\mathbf{p}+t\mathbf{q}$ with $0 \leq t \leq 1$ (as shown in the figure below). Show that a linear transformation maps this line segment onto a line segment or onto a single point.

Solution:

a. It is obvious that the line $\mathbf{x} = (1-t)\mathbf{p}+t\mathbf{q}$ goes through $\mathbf{p}$ (substitute $t = 0$) and by $\mathbf{q}$ (substitute $t = 1$). We need to show that the locus of all these points is a line. To do so we rewrite it as

$$\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$$

that is the parametric form of a line.

b. If we transform the points in the segment $\mathbf{y} = T(\mathbf{x})$ we have

$$\begin{array}{rcll} \mathbf{y} & = & T(\mathbf{x}) & \text{By definition of } \mathbf{x} \\ & = & T(\mathbf{p} + t(\mathbf{q} - \mathbf{p})) & \text{By linearity of } T \\ & = & T(\mathbf{p}) + tT(\mathbf{q} - \mathbf{p}) & \text{By linearity of } T \\ & = & T(\mathbf{p}) + tT(\mathbf{q}) - tT(\mathbf{p}) & \\ & = & (1 - t)T(\mathbf{p}) + tT(\mathbf{q}) & \end{array}$$

If $T(\mathbf{q}-\mathbf{p}) \neq \mathbf{0}$, then $\mathbf{y}$ describes a line that goes through $T(\mathbf{p})$ in the direction of $T(\mathbf{q}-\mathbf{p})$. If $T(\mathbf{q}-\mathbf{p}) = \mathbf{0}$, then $\mathbf{y}$ is a single point.

Lay, 1.8.30

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, 2, ..., p$. Show that T is the zero transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$

Solution: If $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_p$ span $\mathbb{R}^n$, then any vector $\mathbf{x} \in \mathbb{R}^n$ can be expressed as a linear combination of $\mathbf{v}_i$'s:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p$$

Applying the transformation T to $\mathbf{x}$ we get

$$\begin{array}{rcll} T(\mathbf{x}) & = & T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p) & \text{By definition of } \mathbf{x} \\ & = & c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + ... + c_pT(\mathbf{v}_p) & \text{By linearity of } T \\ & = & c_1\mathbf{0} + c_2\mathbf{0} + ... + c_p\mathbf{0} & \text{As stated by the problem} \\ & = & \mathbf{0} & \end{array}$$

Lay, 1.8.34

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the transformation that reflects each vector $\mathbf{x} = (x_1, x_2, x_3)$ through the plane $x_3 = 0$ onto $T(\mathbf{x}) = (x_1, x_2, -x_3)$. Show that T is a linear transformation.

Solution: We need to show that $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\forall c \in \mathbb{R}$

- $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$
  In this particular case:

  $$T(\mathbf{u} + \mathbf{v}) = T((u_1 + v_1, u_2 + v_2, u_3 + v_3)) = (u_1 + v_1, u_2 + v_2, -u_3 - v_3)$$

  On the other hand:

  $$T(\mathbf{u}) + T(\mathbf{v}) = (u_1, u_2, -u_3) + (v_1, v_2, -v_3) = (u_1 + v_1, u_2 + v_2, -u_3 - v_3)$$

  which are obviously equal.

- $T(c\mathbf{u}) = cT(\mathbf{u})$
  In this particular case:

  $$T(c\mathbf{u}) = T((cu_1, cu_2, cu_3)) = (cu_1, cu_2, -cu_3) = c(u_1, u_2, -u_3) = cT(\mathbf{u})$$

Lay, 1.9.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the standard matrix of $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$, when $T(\mathbf{e}_1) = (3, 1, 3, 1)$ and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$ where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

Solution: The standard matrix of T is

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{pmatrix}$$

Lay, 1.9.2

Yolanda Manrique Marcos, December 17th, 2013

Find the standard matrix of $T : \mathbb{R}^3 \to \mathbb{R}^2$, when $T(\mathbf{e}_1) = (1, 4)$, $T(\mathbf{e}_2) = (-2, 9)$ and $T(\mathbf{e}_3) = (3, -8)$ where $\mathbf{e}_1$,$\mathbf{e}_2$ and $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.
Solution: The standard matrix of T is

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{pmatrix}$$

Lay, 1.9.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the standard matrix of $T : \mathbb{R}^2 \to \mathbb{R}^2$, when T is a vertical shear that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 3\mathbf{e}_2$, but leaves $\mathbf{e}_2$ unchanged.
Solution: The standard matrix of T is

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

Lay, 1.9.7

Carlos Oscar Sorzano, Nov. 4th, 2014

$T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $-\frac{3\pi}{4}$ radians (clockwise) and then reflects points through the horizontal x_1-axis.
Solution: Let us construct a matrix of the form

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix}$$

After rotating the vector $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by $-\frac{3\pi}{4}$, the vector becomes $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$. If we reflect through the x_1-axis, we have $T(\mathbf{e}_1) = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Doing the same with $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by $-\frac{3\pi}{4}$, the vector becomes $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$. If we reflect through the x_1-axis, we have $T(\mathbf{e}_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Finally, the transformation matrix becomes

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Lay, 1.9.17

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, 0, 2x_2 + x_4, x_2 - x_4)$. Show that T is a linear transformation by finding a matrix that implements the mapping.
Solution: If we define $\mathbf{x} = (x_1, x_2, x_3, x_4)$, then we may define T as

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Since this transformation is a matrix transformation of the form $T(\mathbf{x}) = A\mathbf{x}$, then it is a linear transformation.

Lay, 1.9.23

Carlos Oscar Sorzano, Nov. 11th, 2013

For each of the following statements determine if they are True or False. Justify your answer.

1. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the matrix $n \times n$ identity matrix.
2. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin through an angle ϕ, then T is a linear transformation.
3. When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.
4. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $\mathbf{x}$ in $\mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.
5. If A is a 3×2 matrix, then the transformation $\mathbf{x} \to A\mathbf{x}$ cannot be one-to-one.

Solution:

1. True. Since the columns of I_n are a basis of $\mathbb{R}^n$, then any vector in $\mathbb{R}^n$ can be expressed as:

   $$\mathbf{x} = x_1\mathbf{e}_1 + ... + x_n\mathbf{x}_n$$

   Since T is a linear transformation, then

   $$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + ... + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + ... + x_nT(\mathbf{e}_n)$$

   That is, to calculate $T(\mathbf{x})$ we only need to know how to transform the columns of the identity matrix of size $n \times n$.

2. True. As shown in Example 1.9.3, such a rotating transformation can be expressed as

   $$T(\mathbf{x}) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

That is, it is of the form $T(\mathbf{x}) = A\mathbf{x}$, that is a linear transformation.

3. False. The composition of two linear transformations is also a linear transformation as shown below:
Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ be two linear transformations, and $T_{12} : \mathbb{R}^n \to \mathbb{R}^p$ be defined as $(T_1 \circ T_2)(\mathbf{x}) = T_2(T_1(\mathbf{x}))$

   - We need to show that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, it is verified that $T_{12}(\mathbf{u}+\mathbf{v}) = T_{12}(\mathbf{u}) + T_{12}(\mathbf{v})$

$$\begin{array}{rcll} T_{12}(\mathbf{u}+\mathbf{v}) & = & T_2(T_1(\mathbf{u}+\mathbf{v})) & \text{By definition} \\ & = & T_2(T_1(\mathbf{u}) + T_1(\mathbf{v})) & T_1 \text{ is a linear transformation} \\ & = & T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) & T_2 \text{ is a linear transformation} \\ & = & T_{12}(\mathbf{u}) + T_{12}(\mathbf{v}) & \text{By definition} \end{array}$$

4. False. For instance the mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x_1, x_2) = (x_1, x_2, 0)$ produces a vector in $\mathbb{R}^3$ for every vector in $\mathbb{R}^2$. However, the transformation is not onto $\mathbb{R}^3$ because there are vectors in this space that are not coming from any input vector (for instance, the vector $(0, 0, 1)$ is not the image of any of the vectors in $\mathbb{R}^2$).

5. False. Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. The transformation T is, then, defined as $T(x_1, x_2) = (x_1, x_2, 0)$. Let us analyze how many input vectors map onto each vector $\mathbf{y}$ in $\mathbb{R}^3$. For doing so, let us analyze the equation system $A\mathbf{x} = \mathbf{y}$

$$\left(\begin{array}{cc|c} 1 & 0 & y_1 \\ 0 & 1 & y_2 \\ 0 & 0 & y_3 \end{array}\right)$$

This equation system has no solution if $y_3 \neq 0$, and a unique solution if $y_3 = 0$ (the unique solution is $x_1 = y_1$ and $x_2 = y_2$). Therefore, the transformation is one-to-one.

Lay, 1.9.33

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

In fact,

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_n) \end{pmatrix}$$

where $\mathbf{e}_i$ is the i-th column of the $n \times n$ identity matrix. Show that A is unique.
Solution: Let us assume A is not unique. That is, there exists another matrix $A' \neq A$ such that $\forall \mathbf{x} \in \mathbb{R}^n$

$$T(\mathbf{x}) = A'\mathbf{x}$$

If we now subtract the two equations ($T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = A'\mathbf{x}$), we have

$$\begin{aligned} T(\mathbf{x}) - T(\mathbf{x}) &= A\mathbf{x} - A'\mathbf{x} \\ \mathbf{0} &= (A - A')\mathbf{x} \end{aligned}$$

If this is true for all $\mathbf{x}$ is because $A - A' = 0$, or what is the same, $A = A'$. But this is a contradiction with our hypothesis that $A \neq A'$ and, consequently A is unique.

Lay, 1.9.34

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that the mapping $\mathbf{x} \to T(S(\mathbf{x}))$ is a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^m$.

Solution: We need to show that $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}$, $\forall c \in \mathbb{R}$

- $T(S(\mathbf{x}_1 + \mathbf{x}_2)) = T(S(\mathbf{x}_1)) + T(S(\mathbf{x}_2))$

$$\begin{array}{rcll} T(S(\mathbf{x}_1 + \mathbf{x}_2)) &=& T(S(\mathbf{x}_1) + S(\mathbf{x}_2)) & \text{Because S is linear} \\ &=& T(S(\mathbf{x}_1)) + T(S(\mathbf{x}_2)) & \text{Because T is linear} \end{array}$$

- $T(S(c\mathbf{x}_1)) = cT(S(\mathbf{x}_1))$

$$\begin{array}{rcll} T(S(c\mathbf{x}_1)) &=& T(cS(\mathbf{x}_1)) & \text{Because S is linear} \\ &=& cT(S(\mathbf{x}_1)) & \text{Because T is linear} \end{array}$$

Lay, 1.9.37

Carlos Oscar Sorzano, Aug. 31st, 2013

Let T be a linear transformation whose standard matrix is given by $A = \begin{pmatrix} -5 & 6 & -5 & 6 \\ 8 & 3 & -3 & 8 \\ 2 & 9 & 5 & -12 \end{pmatrix}$. Is T a one-to-one transformation?

Solution: The standard matrix is row-equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0.38 \\ 0 & 1 & 0 & -0.32 \\ 0 & 0 & 1 & -1.97 \end{pmatrix}$$

The transformation is not one-to-one (injective) because the columns of the standard matrix are not linearly independent (the fourth column can be expressed as $0.38\mathbf{a}_1 - 0.32\mathbf{a}_2 - 1.97\mathbf{a}_3$).

Lay, 1.9.39

Carlos Oscar Sorzano, Aug. 31st, 2013

Let T be a linear transformation whose standard matrix is given by

$$A = \begin{pmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{pmatrix}$$

Does T map $\mathbb{R}^5$ onto $\mathbb{R}^5$?
Solution: The standard matrix is row-equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The transformation is not onto because there are only 4 pivot columns, i.e., only 4 linearly independent vectors and we need 5 to span $\mathbb{R}^5$.

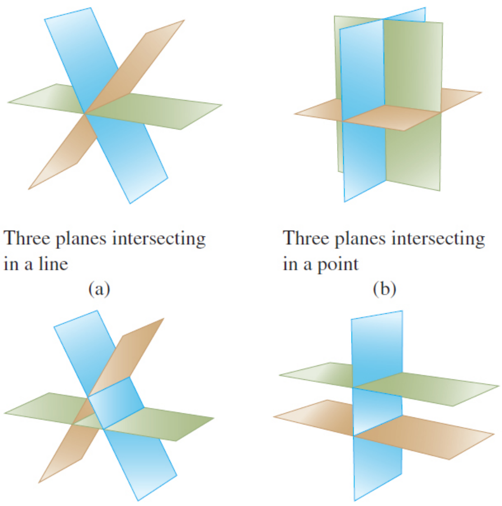
Lay, 1.Suppl.3

Carlos Oscar Sorzano, Jan. 19th 2015

The solutions (x, y, z) of a single linear equation

$$ax + by + cz = d$$

form a plane in $\mathbb{R}^3$ when a, b and c are not all zero. Construct sets of three linear equations whose graphs (a) intersect in a single line, (b) intersect in a single point, and (c) have no points in common. Typical graphs are illustrated in the figure.



Three planes intersecting in a line
(a)

Three planes intersecting in a point
(b)

Three planes with no intersection
(c)

Three planes with no intersection
(c')

Solution: The following equation systems show an example of each one of the situations.
<u>Case a</u>:

$$\begin{array}{rcl} -y+z &=& 0 \\ y+z &=& 0 \\ z &=& 0 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow y = z = 0$$

Any vector of the form $(x,0,0)$ is solution of the equation system. Since the set of solutions

$$S=\{(x,0,0) \quad \forall x\in\mathbb{R}\}$$

is defined by a single free variable, the set of solutions is a straight line.

Case b:

$$\begin{array}{rcl} x &=& 0 \\ y &=& 0 \\ z &=& 0 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow x=y=z=0$$

The only solution of the equation system is the point $(0,0,0)$.

Case c:

$$\begin{array}{rcl} -y+z &=& 1 \\ y+z &=& 1 \\ z &=& 0 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

The system is incompatible because the last equation states $0=-1$.

Case c':

$$\begin{array}{rcl} y &=& 0 \\ z &=& 1 \\ z &=& 0 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

The system is incompatible because the last equation states $0=-1$.

Burgos, 1.1.a

Carlos Oscar Sorzano, Nov. 4th, 2014

Consider the equation system

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad (i=1,2,...,m)$$

Assume that $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,...,\alpha_n)$ and $\boldsymbol{\beta}=(\beta_1,\beta_2,...,\beta_n)$ are solutions of the equation system. Show that

1. $\alpha-\beta$ is a solution of the equation system

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \quad (i=1,2,...,m)$$

2. $\boldsymbol{\alpha}+\lambda(\boldsymbol{\alpha}-\boldsymbol{\beta})$ is also a solution for any $\lambda\in\mathbb{R}$

Solution:

1. Let us substitute $\boldsymbol{\alpha}-\boldsymbol{\beta}$ into the homogeneous equation system. For any $i=1,2,...,m$ we have

$$\sum_{j=1}^{n} a_{ij}(\alpha_j-\beta_j) = \sum_{j=1}^{n} a_{ij}\alpha_j - \sum_{j=1}^{n} a_{ij}\beta_j = b_i - b_i = 0$$

2. Let us substitute $\boldsymbol{\alpha}+\lambda(\boldsymbol{\alpha}-\boldsymbol{\beta})$ into the nonhomogeneous equation system. For any $i=1,2,...,m$ we have

$$\sum_{j=1}^{n} a_{ij}(\alpha_j+\lambda(\alpha_j-\beta_j)) = \sum_{j=1}^{n} a_{ij}\alpha_j + \lambda\sum_{j=1}^{n} a_{ij}(\alpha_j-\beta_j) = b_i+0=b_i$$

2 Chapter 2

Lay, 2.1.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} 2 & -5 \\ 3 & -2 \end{pmatrix}$. Calculate $3I_2 - A$ and $(3I_2)A$

Solution:

$$\begin{aligned} 3I_2 - A &= 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & -5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -3 & 5 \end{pmatrix} \\ (3I_2)A &= \left(3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 2 & -5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 6 & -15 \\ 9 & -6 \end{pmatrix} \end{aligned}$$

Lay, 2.1.4

Carlos Oscar Sorzano, Aug. 31st, 2013

Compute $A - 5I_3$ and $(5I_3)A$, where

$$A = \begin{pmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{pmatrix}$$

Solution:

$$\begin{aligned} A - 5I_3 &= \begin{pmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -6 & -2 \\ -9 & -2 & -11 \\ -8 & -4 & -3 \end{pmatrix} \\ (5I_3)A &= \left(5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 25 & -5 & 15 \\ -20 & 15 & -30 \\ -15 & 5 & 10 \end{pmatrix} \end{aligned}$$

Lay, 2.1.5

Andrea Santos Cortés, Oct, 7th, 2014

Given the matrices

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$$

Compute the product AB in two ways:(a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row-column rule for computing AB.

Solution:

$$A\mathbf{b}_1 = \begin{pmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -10 \\ 2 \\ 26 \end{pmatrix}$$

$$A\mathbf{b}_2 = \begin{pmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \\ -19 \end{pmatrix}$$

$$AB = \begin{pmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -10 & 11 \\ 2 & 8 \\ 26 & -19 \end{pmatrix}$$

Lay, 2.1.6

Andrea Santos Cortés, Oct, 13th, 2014

Given the matrices

$$A = \begin{pmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 4 \\ 3 & -2 \end{pmatrix}$$

Compute the product AB in two ways:(a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row-column rule for computing AB.

Solution:

$$A\mathbf{b}_1 = \begin{pmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 12 \\ 3 \end{pmatrix}$$

$$A\mathbf{b}_2 = \begin{pmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 22 \\ -22 \\ -2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{pmatrix}$$

Lay, 2.1.7

Andrea Santos Cortés, Oct. 13th, 2014

If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

Solution: B is a matrix of size 3×7.

Lay, 2.1.8

Carlos Oscar Sorzano, Aug. 31st, 2013

How many rows does B have if BC is a 5 x 4 matrix?

Solution:

Let's say $BC = D$.

D will be a 5 x 4 matrix if and only if $B_{5,i}$ and $C_{i,4}$.

Therefore, B has 5 rows.

Lay, 2.1.10

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -3 & -5 \\ 2 & 1 \end{pmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

Solution:

$$AB = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -21 & -21 \\ 7 & 7 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 1 \end{pmatrix} = AC$$

Lay, 2.1.12

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B

Solution: We search for a matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ such that

$$AB = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 3b_{11} - 6b_{21} & 3b_{12} - 6b_{22} \\ 4b_{21} - 2b_{11} & 4b_{22} - 2b_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This matrix equation gives us 4 equations

$$\begin{aligned} 3b_{11} - 6b_{21} &= 0 \\ 3b_{12} - 6b_{22} &= 0 \\ 4b_{21} - 2b_{11} &= 0 \\ 4b_{22} - 2b_{12} &= 0 \end{aligned}$$

The augmented matrix of this equation system is

$$\left(\begin{array}{cccc|c} 3 & 0 & -6 & 0 & 0 \\ 0 & 3 & 0 & -6 & 0 \\ -2 & 0 & 4 & 0 & 0 \\ 0 & -2 & 0 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Consequently, $b_{11} = 2b_{21}$ and $b_{12} = 2b_{22}$. That is, any matrix of the form

$$B = \begin{pmatrix} 2b_{21} & 2b_{22} \\ b_{21} & b_{22} \end{pmatrix}$$

yields $AB = 0$. One such example is $B = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

Lay, 2.1.17

Ana Peña Gil, Jan. 19th, 2014

If $A = \begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix}$ and $AB = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}$, determine the first and second columns of B.

Solution: We search for a matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ such that

$$AB = \begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} - 3b_{21} & b_{12} - 3b_{22} \\ 5b_{21} - 3b_{11} & 5b_{22} - 3b_{12} \end{pmatrix} = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}$$

This matrix equation gives us 4 equations

$$
\begin{aligned}
b_{11} - 3b_{21} &= -3 \\
b_{12} - 3b_{22} &= -11 \\
5b_{21} - 3b_{11} &= 1 \\
5b_{22} - 3b_{12} &= 17
\end{aligned}
$$

The augmented matrix of this equation system is

$$
\left(\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & -3 \\ 0 & 1 & 0 & -3 & -11 \\ -3 & 0 & 5 & 0 & 1 \\ 0 & -3 & 0 & 5 & 17 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & -3 \\ 0 & 1 & 0 & -3 & -11 \\ 0 & 0 & -4 & 0 & -8 \\ 0 & 0 & 0 & -4 & -16 \end{array}\right)
$$

Consequently, $b_{11} = 3$, $b_{12} = 1$, $b_{21} = 2$ and $b_{22} = 4$. The first and second columns of B are:

$$
B = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}
$$

Lay, 2.1.18

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose the third column of B is all zeros. What can be said about the third column of AB?

Solution: Let us consider the different columns of B

$$
B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & ... \end{pmatrix}
$$

The product of AB is

$$
AB = A \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & ... \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 & ... \end{pmatrix}
$$

If $\mathbf{b}_3 = \mathbf{0}$, then

$$
A\mathbf{b}_3 = A\mathbf{0} = \mathbf{0}
$$

So, the third column is also $\mathbf{0}$.

Lay, 2.1.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose the third column of B is the sum of the first two columns. What can be said about the third column of the product AB?

Solution: Let us consider the different columns of B

$$
B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & ... \end{pmatrix}
$$

The product of AB is

$$
AB = A \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & ... \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 & ... \end{pmatrix}
$$

If $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$, then

$$
A\mathbf{b}_3 = A(\mathbf{b}_1 + \mathbf{b}_2) = A\mathbf{b}_1 + A\mathbf{b}_2
$$

That is, the third column of AB is also the sum of the first and second columns of AB.

Lay, 2.1.20

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose that the first two columns, $\mathbf{b}_1$ and $\mathbf{b}_2$, of B are equal. What can be said about the columns of AB? Why?

Solution: Let us consider the different columns of B

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & ... \end{pmatrix}$$

The product of AB is

$$AB = A \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & ... \end{pmatrix} = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 & ... \end{pmatrix}$$

If $\mathbf{b}_1 = \mathbf{b}_2$, then

$$A\mathbf{b}_1 = A\mathbf{b}_2$$

So, both columns are also equal. Additionally, we may say that the columns of AB are not linearly independent because there exists a linear combination of them that produces the vector $\mathbf{0}$.

$$1(A\mathbf{b}_1) - 1(A\mathbf{b}_2) + 0(A\mathbf{b}_3) + 0(A\mathbf{b}_4) + ... = \mathbf{0}$$

Lay, 2.1.22

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if the columns of B are linearly independent, so are the columns of AB.

Solution: This statement is not true. For instance, the columns of

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are linearly independent. However, given $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, the columns of

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

are not linearly independent because the second column is twice the first one.

AB is linearly independent if the columns of A and B are linearly independent.

Lay, 2.1.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $m \times n$ matrix. Suppose there exists an $n \times m$ matrix C such that $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

Solution: If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then

$$CA\mathbf{x} = C(A\mathbf{x}) = C\mathbf{0} = \mathbf{0}.$$

But on the other side

$$CA\mathbf{x} = (CA)\mathbf{x} = I_n\mathbf{x} = \mathbf{x}.$$

Consequently, $\mathbf{x} = \mathbf{0}$. This shows that the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. A requirement for this is that there are not more columns than rows.

Lay, 2.1.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose A is a $3 \times n$ matrix whose columns span $\mathbb{R}^3$. Explain how to construct an $n \times 3$ matrix D such that $AD = I_3$.

Solution: Let us define a generic matrix D

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ ... & ... & ... \\ d_{n1} & d_{n2} & d_{n3} \end{pmatrix}$$

We need that $AD = I_3$. This gives us 9 $(= 3 \cdot 3)$ equations to find the matrix D. If the columns of A span $\mathbb{R}^3$ and $n > 3$, the system is compatible indeterminate and there will be infinite solutions to the problem. If $n = 3$, there is a single solution to the problem. In the case that the columns of A did not span $\mathbb{R}^3$, there would not be any solution to the problem.

Lay, 2.1.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose A is an $m \times n$ matrix and there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$. [*Hint*: think of the product CAD.]

Solution: Let us compute

$$(CA)D = I_nD = D$$

On the other side, let us compute

$$C(AD) = CI_m = C$$

But we know that matrix multiplication is associative and, consequently, $C = D$.

By Exercise Lay 2.1.23 we know that A cannot have more columns than rows, and by Exercise Lay 2.1.26 we know that A cannot have more rows than columns. Consequently, the number of rows and columns must be the same and $m = n$.

Lay, 2.1.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $m \times n$ matrix. Suppose there exists an $n \times m$ matrix D such that $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why A cannot have more rows than columns.

Solution: Let us consider the rows of $D = \begin{pmatrix} \mathbf{d}_1 & \mathbf{d}_2 & ... & \mathbf{d}_m \end{pmatrix}$. The product AD is

$$AD = \begin{pmatrix} A\mathbf{d}_1 & A\mathbf{d}_2 & ... & A\mathbf{d}_m \end{pmatrix} = I_m = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & ... & \mathbf{e}_m \end{pmatrix}$$

where $\mathbf{e}_i$ is the i-th column of I_m. For a particular column, we have

$$A\mathbf{d}_i = \mathbf{e}_i$$

The columns of I_m form a basis of $\mathbb{R}^m$. Therefore, for any $\mathbf{b} \in \mathbb{R}^m$ it can be expressed as a linear combination of the $\mathbf{e}_i$ vectors

$$\mathbf{b} = \sum_{i=1}^{m} b_i \mathbf{e}_i = \sum_{i=1}^{m} b_i A\mathbf{d}_i = A\left(\sum_{i=1}^{m} b_i \mathbf{d}_i\right)$$

So we deduce, there exists a solution to the equation $A\mathbf{x} = \mathbf{b}$ that is

$$\mathbf{x} = \sum_{i=1}^{m} b_i \mathbf{d}_i$$

If A had more rows than columns, then it would not have a solution for every $\mathbf{b}$ because there would be $\mathbf{b}$'s for which the reduced echelon form has rows full of zeros and the independent terms are not 0.

Lay, 2.1.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u} = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$.

Solution:

$$\begin{aligned}
\mathbf{u}^T\mathbf{v} &= \begin{pmatrix} -3 & 2 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -3a + 2b - 5c \\
\mathbf{v}^T\mathbf{u} &= \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix} = -3a + 2b - 5c \\
\mathbf{u}\mathbf{v}^T &= \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix} \begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} -3a & -3b & -3c \\ 2a & 2b & 2c \\ -5a & -5b & -5c \end{pmatrix} \\
\mathbf{v}\mathbf{u}^T &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} -3 & 2 & -5 \end{pmatrix} = \begin{pmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{pmatrix}
\end{aligned}$$

Lay, 2.2.7

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} 1 & 2 \\ 5 & 12 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, $\mathbf{b}_3 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, and $\mathbf{b}_4 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

a. Find A^{-1} and use it to solve the four equations $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, $A\mathbf{x} = \mathbf{b}_3$, $A\mathbf{x} = \mathbf{b}_4$.

b. The four equations in part (a) can be solved by the *same* set of row operations, since the coefficients matrix is the same in each case. Solve the four equations in part (a) by reducing the augmented matrix $\left(\begin{array}{c|cccc} A & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{array} \right)$.

Solution:

a. To find A^{-1} we will apply row operations on the augmented matrix exploiting that $\left(\begin{array}{c|c} A & I \end{array} \right) \sim \left(\begin{array}{c|c} I & A^{-1} \end{array} \right)$.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 5 & 12 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \end{array}\right)$$

Now we use this inverse matrix to solve the linear equations

$$\begin{array}{c}
\left(\begin{array}{cc} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{array}\right)\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -9 \\ 4 \end{pmatrix} \\
\left(\begin{array}{cc} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{array}\right)\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \end{pmatrix} \\
\left(\begin{array}{cc} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{array}\right)\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix} \\
\left(\begin{array}{cc} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{array}\right)\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ -5 \end{pmatrix}
\end{array}$$

b. Now, we will apply row operations on the augmented matrix suggested by the problem

$$\left(\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{array}\right) \sim \left(\begin{array}{cc|cccc} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{array}\right)$$

Lay, 2.2.11

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $X = A^{-1}B$.

Solution: Consider the columns of X and B:

$$\begin{array}{c} X = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_p \end{pmatrix} \\ B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{pmatrix} \end{array}$$

The matrix equation $AX = B$ is a simultaneous set of equations:

$$\begin{array}{c} A\mathbf{x}_1 = \mathbf{b}_1 \\ A\mathbf{x}_2 = \mathbf{b}_2 \\ \dots \\ A\mathbf{x}_p = \mathbf{b}_p \end{array}$$

Since A is invertible, each equation has a unique solution given by

$$\begin{array}{c} \mathbf{x}_1 = A^{-1}\mathbf{b}_1 \\ \mathbf{x}_2 = A^{-1}\mathbf{b}_2 \\ \dots \\ \mathbf{x}_p = A^{-1}\mathbf{b}_p \end{array}$$

Or what is the same

$$\begin{array}{rcl} X & = & \begin{pmatrix} A^{-1}\mathbf{b}_1 & A^{-1}\mathbf{b}_2 & \dots & A^{-1}\mathbf{b}_p \end{pmatrix} \\ & = & A^{-1}\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{pmatrix} \\ & = & A^{-1}B \end{array}$$

Lay, 2.2.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, if A is not invertible?
Solution: If A is invertible we multiply on the left by A^{-1} to obtain

$$\begin{gathered} A^{-1}(AB) = A^{-1}(AC) \\ (A^{-1}A)B = (A^{-1}A)C \\ I_n B = I_n C \\ B = C \end{gathered}$$

If A is not invertible, then the statement is not generally true. For example, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = AC$$

Lay, 2.2.16

Carlos Oscar Sorzano, Nov. 4th 2014

Suppose A and B are $n \times n$ matrices, B is invertible, and AB is invertible. Show that A is invertible. [*Hint*: Let $C = AB$ and solve this equation for A]
Solution: If we solve for A in

$$C = AB \Rightarrow A = CB^{-1} = (AB)B^{-1}$$

This matrix is well constructed because AB and B^{-1} are both $n \times n$ matrices and can, therefore, be multiplied. Also B^{-1} exists because B is invertible.

The inverse of A would be:

$$A^{-1} = ((AB)B^{-1})^{-1} = B(AB)^{-1}$$

Since AB is invertible (see problem statement), then A^{-1} is also well constructed and it exists.

Lay, 2.2.17

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose A, B and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)D = I = D(ABC)$
Solution: The sought matrix D is

$$D = C^{-1}B^{-1}A^{-1}$$

Let us check that this matrix is actually the inverse of ABC.

$$\begin{aligned} (ABC)D &= (ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} \\ &= ABB^{-1}A^{-1} = AA^{-1} = I \\ D(ABC) &= (C^{-1}B^{-1}A^{-1})(ABC) = C^{-1}B^{-1}(A^{-1}A)BC \\ &= C^{-1}B^{-1}BC = C^{-1}C = I \end{aligned}$$

Lay, 2.2.19

Carlos Oscar Sorzano, Aug. 31st, 2013

If A, B and C are invertible $n \times n$ matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution, X? If so, find it.

Solution: If B and C are invertible, so are B^{-1} and C^{-1}, and their inverses are B and C, respectively. In this way, we may multiply on the left by C and on the right by B to obtain

$$\begin{gathered} CC^{-1}(A + X)B^{-1}B = CI_nB \\ A + X = CB \\ X = CB - A \end{gathered}$$

Lay, 2.2.21

Carlos Oscar Sorzano, Aug. 31st, 2013

Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.

Solution: If A is invertible we have shown (see Theorem 2.2, Chapter 3, Biomedical Engineering Notes) that for every $\mathbf{b} \in \mathbb{R}^n$, there is a unique solution of the equation $A\mathbf{x} = \mathbf{b}$. In particular, there exists a solution for the equation $A\mathbf{x} = \mathbf{0}$ that is $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$. Since the only solution of this problem is the trivial one, then by Theorem 6.1, Chapter 2, Biomedical Engineering Notes, the columns of A are linearly independent.

Lay, 2.2.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that if $ad - bc = 0$, then the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution. Why does this imply that A is not invertible?

Solution: Let us reduce the augmented matrix $\left(\begin{array}{c} A|\mathbf{b} \end{array} \right)$.

$$\left(\begin{array}{ccc} a & b & b_1 \\ c & d & b_2 \end{array} \right) \sim \left(\begin{array}{ccc} a & b & b_1 \\ 0 & ad - bc & ab_2 - cb_1 \end{array} \right)$$

In fact if $ad - bc = 0$, the matrix equation may have infinite solutions (if $ab_2 - cb_1 = 0$) or no solution at all (if $ab_2 - cb_1 \neq 0$). This implies that A is not invertible because if it were invertible for any $\mathbf{b} \in \mathbb{R}^2$ the equation $A\mathbf{x} = \mathbf{b}$ would have a single solution.

Lay, 2.2.34

Use the algorithm from this section to calculate the inverse of

$$A = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 2 & 2 & 0 & ... & 0 \\ 3 & 3 & 3 & ... & 0 \\ ... & ... & ... & ... & ... \\ n & n & n & ... & n \end{pmatrix}$$

Solution: Let us apply row operations to the matrix $\left(\begin{array}{c|c} A & I \end{array}\right)$ to reduce it to $\left(\begin{array}{c|c} I & A^{-1} \end{array}\right)$

$$\left(\begin{array}{cccccc|ccccccc}
1 & 0 & 0 & 0 & ... & 0 & 1 & 0 & 0 & 0 & ... & 0 & 0 \\
2 & 2 & 0 & 0 & ... & 0 & 0 & 1 & 0 & 0 & ... & 0 & 0 \\
3 & 3 & 3 & 0 & ... & 0 & 0 & 0 & 1 & 0 & ... & 0 & 0 \\
4 & 4 & 4 & 4 & ... & 0 & 0 & 0 & 0 & 1 & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
n & n & n & n & ... & n & 0 & 0 & 0 & 0 & ... & 0 & 1
\end{array}\right) \sim$$

$$\left(\begin{array}{cccccc|ccccccc}
1 & 0 & 0 & 0 & ... & 0 & 1 & 0 & 0 & 0 & ... & 0 & 0 \\
0 & 1 & 0 & 0 & ... & 0 & -1 & \frac{1}{2} & 0 & 0 & ... & 0 & 0 \\
3 & 3 & 3 & 0 & ... & 0 & 0 & 0 & 1 & 0 & ... & 0 & 0 \\
4 & 4 & 4 & 4 & ... & 0 & 0 & 0 & 0 & 1 & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & & \\
n & n & n & n & ... & n & 0 & 0 & 0 & 0 & ... & 0 & 1
\end{array}\right) \sim$$

$$\left(\begin{array}{cccccc|ccccccc}
1 & 0 & 0 & 0 & ... & 0 & 1 & 0 & 0 & 0 & ... & 0 & 0 \\
0 & 1 & 0 & 0 & ... & 0 & -1 & \frac{1}{2} & 0 & 0 & ... & 0 & 0 \\
0 & 0 & 1 & 0 & ... & 0 & 0 & -\frac{1}{2} & \frac{1}{3} & 0 & ... & 0 & 0 \\
4 & 4 & 4 & 4 & ... & 0 & 0 & 0 & 0 & 1 & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
n & n & n & n & ... & n & 0 & 0 & 0 & 0 & ... & 0 & 1
\end{array}\right) \sim$$

$$\left(\begin{array}{cccccc|ccccccc}
1 & 0 & 0 & 0 & ... & 0 & 1 & 0 & 0 & 0 & ... & 0 & 0 \\
0 & 1 & 0 & 0 & ... & 0 & -1 & \frac{1}{2} & 0 & 0 & ... & 0 & 0 \\
0 & 0 & 1 & 0 & ... & 0 & 0 & -\frac{1}{2} & \frac{1}{3} & 0 & ... & 0 & 0 \\
0 & 0 & 0 & 1 & ... & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{4} & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
n & n & n & n & ... & n & 0 & 0 & 0 & 0 & ... & 0 & 1
\end{array}\right) \sim$$

$$\left(\begin{array}{cccccc|ccccccc}
1 & 0 & 0 & 0 & ... & 0 & 1 & 0 & 0 & 0 & ... & 0 & 0 \\
0 & 1 & 0 & 0 & ... & 0 & -1 & \frac{1}{2} & 0 & 0 & ... & 0 & 0 \\
0 & 0 & 1 & 0 & ... & 0 & 0 & -\frac{1}{2} & \frac{1}{3} & 0 & ... & 0 & 0 \\
0 & 0 & 0 & 1 & ... & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{4} & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & 0 & 0 & ... & 1 & 0 & 0 & 0 & 0 & ... & -\frac{1}{n-1} & \frac{1}{n}
\end{array}\right)$$

Lay, 2.2.36

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} -25 & -9 & -27 \\ 536 & 185 & 537 \\ 154 & 52 & 143 \end{pmatrix}$. Find the second and third columns of A^{-1} without computing the first column.

Solution: Let us reduce the augmented matrix $\left(\begin{array}{c|cc} A & \mathbf{e}_2 & \mathbf{e}_3 \end{array}\right)$.

$$\left(\begin{array}{ccc|cc}
-25 & -9 & -27 & 0 & 0 \\
536 & 185 & 537 & 1 & 0 \\
154 & 52 & 143 & 0 & 1
\end{array}\right) \sim \left(\begin{array}{ccc|cc}
1 & 0 & 0 & 0.1126 & -0.1559 \\
0 & 1 & 0 & -0.5611 & 1.0077 \\
0 & 0 & 1 & 0.0828 & -0.1915
\end{array}\right)$$

The last two columns of the latter matrix are the two columns required by the problem.

Lay, 2.3.13

Carlos Oscar Sorzano, Aug. 31st, 2013

An $m \times n$ upper triangular matrix is one whose entries below the main diagonal are 0's. When is a square upper triangular matrix invertible?
Solution: An upper triangular matrix is already in echelon form. It is row-equivalent to I_n, and hence invertible, if its diagonal elements are different from 0. If any of the diagonal entries is zero, then there would be free variables in the equation system $A\mathbf{x} = \mathbf{b}$ and the matrix would not be invertible.

Lay, 2.3.15

Ana Peña gil, Jan. 19th 2014

Is it possible for a 4x4 matrix to be invertible when its columns do not span $\mathbb{R}^4$? Why or why not?

Solution: This is not possible. If an 4x4 matrix does not span $\mathbb{R}^4$, then it means that their columns are not linearly independent. If the columns are linearly dependent, the determinant of the matrix is 0, so the matrix cannot be invertible.

Lay, 2.3.16

Carlos Oscar Sorzano, Aug. 31st, 2013

If an $n \times n$ matrix A is invertible, then the columns of A^T are linearly independent. Explain why.
Solution: By the Invertible Matrix Theorem, if A is invertible, so is A^T. if A^T is invertible, then by the same theorem, the columns of A^T are linearly independent.

Lay, 2.3.17

Carlos Oscar Sorzano, Aug. 31st, 2013

Can a square matrix with two identical columns be invertible? Why or why not?
Solution: It cannot be invertible because the two columns are not linearly independent, and by the Invertible Matrix Theorem, if a matrix is invertible, then its columns are linearly independent.

Lay, 2.3.18

Ana Peña Gil, Jan. 19th 2014

Can a square matrix with two identical columns be invertible? Why or why not?

Solution: When the 2x2 matrix has identical columns, we say their columns are linearly dependent, so its determinant is zero. Because of that, the matrix is not invertible.

Lay, 2.3.33

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$. Show that T is invertible and find a formula for T^{-1}.

Solution: We may write the transformation as

$$T(x_1, x_2) = \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

By defining the matrix $A = \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix}$ and computing its inverse $A^{-1} = \begin{pmatrix} -0.0986 & 0.1268 \\ 0.0563 & 0.0704 \end{pmatrix}$, we may write the inverse transformation as

$$T(x_1, x_2) = \begin{pmatrix} -0.0986 & 0.1268 \\ 0.0563 & 0.0704 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Lay, 2.3.41

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose an experiment leads to the following system of equations

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.249 \\ 1.6x_1 + 1.1x_2 &= 6.843 \end{aligned}$$

a. Solve the previous equation system, and then, the equation system below in which the data on the right has been rounded to two decimal places.

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.25 \\ 1.6x_1 + 1.1x_2 &= 6.84 \end{aligned}$$

b. The entries in the rounded system of equations differ from those of the exact system by less than 0.05%. Find the percentage error when using the solution of the rounded equation system as an approximation to the solution of the exact system.

Solution:

a. The solution of the exact equation system is

$$\mathbf{x}_{\text{exact}} = A^{-1} \begin{pmatrix} 19.249 \\ 6.843 \end{pmatrix} = \begin{pmatrix} 3.94 \\ 0.49 \end{pmatrix}$$

. The solution of the rounded equation system is

$$\mathbf{x}_{\text{rounded}} = A^{-1} \begin{pmatrix} 19.25 \\ 6.84 \end{pmatrix} = \begin{pmatrix} 2.90 \\ 2.00 \end{pmatrix}$$

b. The error percentage is given for each variable as

$$\begin{aligned} \epsilon_1 = 100\tfrac{|x_{1,\text{exact}} - x_{1,\text{rounded}}|}{|x_{1,\text{exact}}|} &= 100\tfrac{|3.94-2.90|}{|3.94|} = 26.40\% \\ \epsilon_2 = 100\tfrac{|x_{2,\text{exact}} - x_{2,\text{rounded}}|}{|x_{2,\text{exact}}|} &= 100\tfrac{|0.49-2.00|}{|0.49|} = 308.16\% \end{aligned}$$

Lay, 2.4.1

Ignacio Sánchez López, Dec, 15th, 2014

Assume that the matrices are patitions conformably for block multiplication. Compute the following product:

$$A = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix}$$

$$B = \begin{pmatrix} A & B \\ -C & D \end{pmatrix}$$

Solution: Let us compute AB

$$AB = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ EA + C & EB + D \end{pmatrix}$$

Lay, 2.4.2

Ignacio Sanchez Lopez, Dec, 29th, 2014

Assume that the matrices are patitioned conformably for block multiplication. Compute the following product:

$$A = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}$$

$$B = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

Solution: Let us compute AB

$$AB = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} EP & EQ \\ FR & FS \end{pmatrix}$$

Lay, 2.4.3

Ignacio Sanchez Lopez, Dec, 29th, 2014

Assume that the matrices are patitioned conformably for block multiplication. Compute the following product:

$$A = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Solution: Let us compute AB

$$AB = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$$

Lay, 2.4.4

Ignacio Sanchez Lopez, Dec, 29th, 2014

Assume that the matrices are patitioned conformably for block multiplication. Compute the following product:

$$A = \begin{pmatrix} I & 0 \\ -E & I \end{pmatrix}$$

$$B = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

Solution: Let us compute AB

$$AB = \begin{pmatrix} I & 0 \\ -E & I \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} W & X \\ Y - EW & Z - EX \end{pmatrix}$$

Lay, 2.4.15

Carlos Oscar Sorzano, Aug. 31st, 2013

When a deep space probe is launched, corrections may be necessary to place the probe on a precisely calculated trajectory. Radio telemetry provides a stream of vectors, $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_k$, giving information at different times about how the probe's position compares with its planned trajectory. Let X_k be the matrix $\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & ... & \mathbf{x}_k \end{pmatrix}$. The matrix $G_k = X_k X_k^T$ is computed as the radar data is analyzed. When $\mathbf{x}_{k+1}$ arrives a new G_{k+1} must be computed. Since the data vectors arrive at high speed, the computational burden could be severe. But partitioned matrix multiplication helps tremendously. Compute the column-row expansions of G_k and G_{k+1}, and describe what must be computed in order to *update* G_k to form G_{k+1}.
Solution: Let's analyze first G_k:

$$G_k = X_k X_k^T = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & ... & \mathbf{x}_k \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ ... \\ \mathbf{x}_k^T \end{pmatrix} = \sum_{i=1}^{k} \mathbf{x}_i \mathbf{x}_i^T$$

Similarly

$$G_{k+1} = X_{k+1} X_{k+1}^T = \sum_{i=1}^{k+1} \mathbf{x}_i \mathbf{x}_i^T = \left(\sum_{i=1}^{k} \mathbf{x}_i \mathbf{x}_i^T \right) + \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T = G_k + \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T$$

Thus, it suffices to compute $\mathbf{x}_{k+1} \mathbf{x}_{k+1}^T$ and add it to the previous matrix G_k.

Lay, 2.4.16

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. If A_{11} is invertible, then the matrix $S = A_{22} - A_{21} A_{11}^{-1} A_{12}$ is called the Schur complement of A_{11}. Likewise, if A_{22} is invertible, the matrix $A_{11} - A_{12} A_{22}^{-1} A_{21}$ is called the Schur complement of A_{22}. Suppose A_{11} is invertible. Find X and Y such that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}$$

Solution: Let us multiply the matrices on the right

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ XA_{11} & S \end{pmatrix}\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y + S \end{pmatrix}$$

Comparing this product to A we derive the following equations:

$$\begin{aligned} A_{11}Y &= A_{12} \\ XA_{11} &= A_{21} \\ XA_{11}Y + S &= A_{22} \end{aligned}$$

That are solved like

$$\begin{aligned} Y &= A_{11}^{-1}A_{12} \\ X &= A_{21}A_{11}^{-1} \end{aligned}$$

We need to check that the last equation is verified

$$\begin{aligned} XA_{11}Y + S &= A_{22} \\ (A_{21}A_{11}^{-1})A_{11}(A_{11}^{-1}A_{12}) + (A_{22} - A_{21}A_{11}^{-1}A_{12}) &= A_{22} \\ A_{21}A_{11}^{-1}A_{12} + A_{22} - A_{21}A_{11}^{-1}A_{12} &= A_{22} \\ A_{22} &= A_{22} \end{aligned}$$

Lay, 2.4.18

Carlos Oscar Sorzano, Aug. 31st, 2013

Let X be an $m \times n$ data matrix such that X^TX is invertible, and let $M = I_m - X(X^TX)^{-1}X^T$. Add a column $\mathbf{x}_0$ to the data to form $W = \begin{pmatrix} X & \mathbf{x}_0 \end{pmatrix}$. Compute W^TW. The (1,1)-entry is X^TX. Show that the Schur complement (Exercise Lay 2.4.16) of X^TX can be written in the form $\mathbf{x}_0^TM\mathbf{x}_0$. It can be shown that $(\mathbf{x}_0^TM\mathbf{x}_0)^{-1}$ is the (2,2)-entry in $(W^TW)^{-1}$. This entry has a useful statistical interpretation under apropriate hypotheses.
Solution:

$$W^TW = \begin{pmatrix} X^T \\ \mathbf{x}_0^T \end{pmatrix}\begin{pmatrix} X & \mathbf{x}_0 \end{pmatrix} = \begin{pmatrix} X^TX & X^T\mathbf{x}_0 \\ \mathbf{x}_0^TX & \mathbf{x}_0^T\mathbf{x}_0 \end{pmatrix}$$

The Schur complement is defined as $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$, that in this particular case is

$$S = \mathbf{x}_0^T\mathbf{x}_0 - \mathbf{x}_0^TX(X^TX)^{-1}X^T\mathbf{x}_0 = \mathbf{x}_0^T(I_m - X(X^TX)^{-1}X^T)\mathbf{x}_0 = \mathbf{x}_0^TM\mathbf{x}_0$$

Lay, 2.4.19

Carlos Oscar Sorzano, Aug. 31st, 2013

In the study of engineering control of physical systems, a standard set of differential equations is transformed by Laplace transforms into the following system of linear equations:

$$\begin{pmatrix} A - sI_n & B \\ C & I_m \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix}$$

where A is $n \times n$, B is $n \times m$, C is $m \times n$ and s is a variable. The vector $\mathbf{u} \in \mathbb{R}^m$ is the "input" to the system, $\mathbf{y} \in \mathbb{R}^m$ is the "output" of the system, and $\mathbf{x} \in \mathbb{R}^n$ is the "state" vector. Actually, the vectors $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{y}$ are functions of s, but this does not affect the algebraic calculations of this exercise.

Assume $A - sI_n$ is invertible and view the previous equation as a system of two matrix equations. Solve the top equation for $\mathbf{x}$ and substitute in the bottom equation. The result is an equation of the form $W(s)\mathbf{u} = \mathbf{y}$, where $W(s)$ is a matrix that depends on s. $W(s)$ is called the *transfer function* of the system because it transforms the input $\mathbf{u}$ into the output $\mathbf{y}$. Find $W(s)$ and describe how it is related to the partitioned system matrix of the equation above.

Solution: The first equation gives us

$$(A - sI_n)\mathbf{x} + B\mathbf{u} = \mathbf{0} \Rightarrow \mathbf{x} = -(A - sI_n)^{-1}B\mathbf{u}$$

Now we go with the second equation and substitute this value into it

$$\begin{gathered}
C\mathbf{x} + \mathbf{u} = \mathbf{y} \\
C(-(A - sI_n)^{-1}B\mathbf{u}) + \mathbf{u} = \mathbf{y} \\
(-C(A - sI_n)^{-1}B + I_m)\mathbf{u} = \mathbf{y} \\
(I_m - C(A - sI_n)^{-1}B)\mathbf{u} = \mathbf{y}
\end{gathered}$$

So, the transfer function is given by the matrix $W(s) = I_m - C(A - sI_n)^{-1}B$.

Lay, 2.5.Practice

Carlos Oscar Sorzano, Aug. 31st, 2013

Find an LU factorization of the matrix $A = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix}$

Solution: We apply row operations on A to reduce it to an upper triangular matrix and annotate the different matrices that we needed

$$A = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix} \quad \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1 \quad E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_1A = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix} \quad \mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1 \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_2E_1A = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix} \quad \mathbf{r}_4 \leftarrow \mathbf{r}_4 - 2\mathbf{r}_1 \quad E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_3E_2E_1A = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ -6 & 3 & 3 & 4 \end{pmatrix} \quad \mathbf{r}_5 \leftarrow \mathbf{r}_5 + 3\mathbf{r}_1 \quad E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} E_4 \\ E_3E_2E_1A \end{matrix} = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{pmatrix} \quad \mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2 \quad E_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} E_5E_4 \\ E_3E_2E_1A \end{matrix} = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{pmatrix} \quad \mathbf{r}_4 \leftarrow \mathbf{r}_4 - 2\mathbf{r}_2 \quad E_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} E_6E_5E_4 \\ E_3E_2E_1A \end{matrix} = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & -9 & -3 & 13 \end{pmatrix} \quad \mathbf{r}_5 \leftarrow \mathbf{r}_5 + 3\mathbf{r}_2 \quad E_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} E_7 \\ E_6E_5E_4 \\ E_3E_2E_1A \end{matrix} = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad \mathbf{r}_4 \leftarrow \mathbf{r}_4 + \mathbf{r}_3 \quad E_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} E_8E_7 \\ E_6E_5E_4 \\ E_3E_2E_1A \end{matrix} = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad \mathbf{r}_5 \leftarrow \mathbf{r}_5 - 2\mathbf{r}_3 \quad E_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} E_9E_8E_7 \\ E_6E_5E_4 \\ E_3E_2E_1A \end{matrix} = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This latter matrix is U and L is

$$\begin{aligned} L &= (E_9E_8E_7E_6E_5E_4E_3E_2E_1)^{-1} = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ -4 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 2 & 1 & -2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{pmatrix} \end{aligned}$$

Finally, we have

$$A = LU \Rightarrow \begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lay, 2.5.9

Carlos Oscar Sorzano, Aug. 31st, 2013

Find an LU factorization of the matrix $A = \begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}$

Solution: We apply row operations on A to reduce it to an upper triangular matrix and annotate the different matrices that we needed

$$\begin{array}{rlll} A = & \begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix} & \mathbf{r}_2 \leftarrow \mathbf{r}_2 + 3\mathbf{r}_1 & E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ E_1A = & \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 9 & 9 & 14 \end{pmatrix} & \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_1 & E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \\ E_2E_1A = & \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 6 & 8 \end{pmatrix} & \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_1 & E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \\ E_3E_2E_1A = & \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix} & & \end{array}$$

This latter matrix is U and L is

$$\begin{aligned} L &= (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \end{aligned}$$

Finally, we have

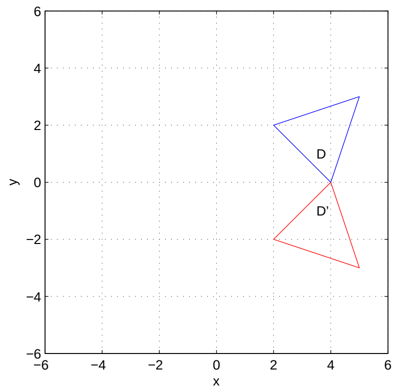
$$A = LU \Rightarrow \begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Lay, 2.7.2

Carlos Oscar Sorzano, Aug. 31st, 2013

Use matrix multiplication to find the image of the triangle with data matrix $D = \begin{pmatrix} 4 & 2 & 5 \\ 0 & 2 & 3 \end{pmatrix}$ under the transformation that reflects a point through the y-axis. Sketch both the original triangle and its image.
Solution: The referred to transformation is the one whose matrix is $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$D' = AD = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 5 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 5 \\ 0 & -2 & -3 \end{pmatrix}$$



Lay, 2.7.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the 3×3 matrix that translate by $(2, 1)$ and then rotate by $90°$ about the origin in 2D using homogeneous coordinates.
Solution: The required transformation is

$$\tilde{A} = \begin{pmatrix} \cos(90°) & \sin(90°) & 0 \\ -\sin(90°) & \cos(90°) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Lay, 2.7.10

Carlos Oscar Sorzano, Aug. 31st, 2013

Consider the following geometric 2D transformations: D, a dilation (in which the x and y coordinates are scaled by the same factor); R, a rotation; and T, a translation. Does D commute with R? That is $(D(R(\mathbf{x})) = R(D(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^2$? Does D commute with T? Does T commute with R?

Solution: The three proposed transformations can be written as matrix transformations in homogeneous coordinates

$$
\begin{aligned}
D(\mathbf{x}) &= \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{x}} \\
R(\mathbf{x}) &= \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{x}} \\
T(\mathbf{x}) &= \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{x}}
\end{aligned}
$$

Now we need to check whether $D(R(\mathbf{x})) = R(D(\mathbf{x}))$

$$
\begin{aligned}
D(R(\mathbf{x})) &= D\left(\begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right) \\
&= D\left(\begin{pmatrix} \cos(\alpha)x + \sin(\alpha)y \\ -\sin(\alpha)x + \cos(\alpha)y \\ 1 \end{pmatrix} \right) = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\alpha)x + \sin(\alpha)y \\ -\sin(\alpha)x + \cos(\alpha)y \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} r\cos(\alpha)x + r\sin(\alpha)y \\ -r\sin(\alpha)x + r\cos(\alpha)y \\ 1 \end{pmatrix}
\end{aligned}
$$

On the other side

$$
\begin{aligned}
R(D(\mathbf{x})) &= R\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right) \\
&= R\left(\begin{pmatrix} rx \\ ry \\ 1 \end{pmatrix} \right) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} rx \\ ry \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} r\cos(\alpha)x + r\sin(\alpha)y \\ -r\sin(\alpha)x + r\cos(\alpha)y \\ 1 \end{pmatrix}
\end{aligned}
$$

So $D(R(\mathbf{x})) = R(D(\mathbf{x}))$ and rotation commutes with dilation.
If we repeat the same exercise with dilations and translations

$$
\begin{aligned}
D(T(\mathbf{x})) &= D\left(\begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right) \\
&= D\left(\begin{pmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{pmatrix} \right) = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} rx + r\Delta x \\ ry + r\Delta y \\ 1 \end{pmatrix}
\end{aligned}
$$

On the other side

$$\begin{aligned}
T(D(\mathbf{x})) &= T\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}\right) \\
&= T\left(\begin{pmatrix} rx \\ ry \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} rx \\ ry \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} rx + \Delta x \\ ry + \Delta y \\ 1 \end{pmatrix}
\end{aligned}$$

So $D(T(\mathbf{x})) \neq T(D(\mathbf{x}))$ and translation does not commute with dilation. Repeting once more the exercise with rotation and translation we would reach the conclusion that they do not commute.

Lay, 2.7.12

Carlos Oscar Sorzano, Aug. 31st, 2013

A rotation in $\mathbb{R}^2$ usually requires four multiplications. Compute the product below and show that the matrix for a rotation can be factored into three shear transformations (each of which requires only one multiplication).

$$\begin{pmatrix} 1 & \tan(\frac{\phi}{2}) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -\sin(\phi) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \tan(\frac{\phi}{2}) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: Multiplying the three matrices we get

$$\begin{pmatrix} 1-\tan(\frac{\phi}{2})\sin(\phi) & \tan(\frac{\phi}{2})(2-\tan(\frac{\phi}{2})\sin(\phi)) & \\ -\sin(\phi) & 1-\tan(\frac{\phi}{2})\sin(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

At this point, we make use of the trigonometric identity

$$\tan(\tfrac{\phi}{2}) = \tfrac{\sin(\phi)}{1+\cos(\phi)} = \tfrac{1-\cos(\phi)}{\sin(\phi)}$$

Then

$$\begin{aligned}
1-\tan(\tfrac{\phi}{2})\sin(\phi) &= 1-\tfrac{\sin(\phi)}{1+\cos(\phi)}\sin(\phi) = \tfrac{1+\cos(\phi)-\sin^2(\phi)}{1+\cos(\phi)} = \tfrac{\cos^2(\phi)+\cos(\phi)}{1+\cos(\phi)} \\
&= \cos(\phi)\tfrac{\cos(\phi)+1}{1+\cos(\phi)} = \cos(\phi)
\end{aligned}$$

Let us simplify now $\tan(\frac{\phi}{2})(2-\tan(\frac{\phi}{2})\sin(\phi))$:

$$\begin{aligned}
\tan(\tfrac{\phi}{2})(2-\tan(\tfrac{\phi}{2})\sin(\phi)) &= \tan(\tfrac{\phi}{2})\left(2-\tfrac{1-\cos(\phi)}{\sin(\phi)}\sin(\phi)\right) \\
&= \tan(\tfrac{\phi}{2})(2-(1-\cos(\phi))) = \tan(\tfrac{\phi}{2})(1+\cos(\phi)) \\
&= \tfrac{\sin(\phi)}{1+\cos(\phi)}(1+\cos(\phi)) = \sin(\phi)
\end{aligned}$$

In summary

$$\begin{pmatrix} 1-\tan(\frac{\phi}{2})\sin(\phi) & \tan(\frac{\phi}{2})(2-\tan(\frac{\phi}{2})\sin(\phi)) & \\ -\sin(\phi) & 1-\tan(\frac{\phi}{2})\sin(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

That is, the multiplication of the three matrices above is the same as the application of a rotation matrix. But applying a rotation matrix involves 4 multiplications, while the application of the three matrices requires only 3.

Lay, 2.7.22

Carlos Oscar Sorzano, Aug. 31st, 2013

The signal broadcast by commercial television describes each color by a vector (Y, I, Q). If the screen is black and white, only the Y coordinate is used (this gives a better monochrome picture than using CIE data for colors). The correspondence between YIQ and a "standard" RGB color is given by

$$\begin{pmatrix} Y \\ I \\ Q \end{pmatrix} = \begin{pmatrix} 0.299 & 0.587 & 0.114 \\ 0.596 & -0.275 & -0.321 \\ 0.212 & -0.528 & 0.311 \end{pmatrix} \begin{pmatrix} R \\ G \\ B \end{pmatrix}$$

(A screen manufacturer would change the matrix entries to work for its RGB entries.) Find the equation that converts the YIQ data transmitted by the television station to the RGB data needed for the television screen.

Solution: If we consider the equation above to be

$$\begin{pmatrix} Y \\ I \\ Q \end{pmatrix} = A \begin{pmatrix} R \\ G \\ B \end{pmatrix},$$

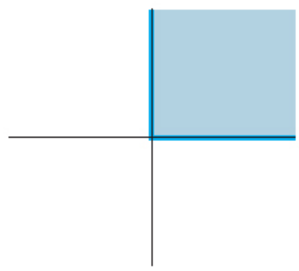
then

$$\begin{pmatrix} R \\ G \\ B \end{pmatrix} = A^{-1} \begin{pmatrix} Y \\ I \\ Q \end{pmatrix} = \begin{pmatrix} 1.0031 & 0.9548 & 0.6179 \\ 0.9968 & -0.2707 & -0.6448 \\ 1.0085 & -1.1105 & 1.6996 \end{pmatrix} \begin{pmatrix} Y \\ I \\ Q \end{pmatrix}$$

Lay, 2.8.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Given the set H represented below (bold lines imply that those points belong to H)



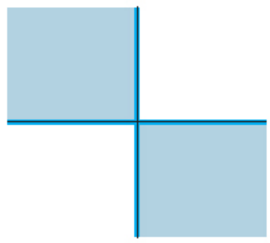
Give a specific reason of why the set is not a subspace of $\mathbb{R}^2$

Solution: For instance $\mathbf{x} = (1, 0)$ belongs to H, but $-\mathbf{x} = (-1, 0)$ does not.

Lay, 2.8.2

Carlos Oscar Sorzano, Aug. 31st, 2013

Given the set H represented below (bold lines imply that those points belong to H)



Give a specific reason of why the set is not a subspace of $\mathbb{R}^2$

Solution: For instance $\mathbf{x}_1 = (-1, 1)$ and $\mathbf{x}_2 = (2, 0)$ belong to H, but $\mathbf{x}_1 + \mathbf{x}_2 = (1, 1)$ does not.

Lay, 2.8.5

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{v}_1 = (1, 3, -4)$, $\mathbf{v}_2 = (-2, -3, 7)$, and $\mathbf{w} = (-3, -3, 10)$. Determine if $\mathbf{w}$ is in the subspace of $\mathbb{R}^3$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: If $\mathbf{w}$ is in the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$, then there must exists two constants c_1 and c_2 such that

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

We may solve this problem through the augmented matrix

$$\left(\begin{array}{rr|r} 1 & -2 & -3 \\ 3 & -3 & -3 \\ -4 & 7 & 10 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & -2 & -3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right)$$

The equation system is compatible determinate existing a single solution, and consequently, $\mathbf{w}$ belongs to the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Lay, 2.8.11

Carlos Oscar Sorzano, June, 6th 2014

Let A be the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{pmatrix} \tag{3}$$

Give the values of p and q such that Nul$\{A\}$ is a subspace of $\mathbb{R}^p$ and Col$\{A\}$ is a subspace of $\mathbb{R}^q$.

Solution: Since A is a 3×4 matrix, the transformation $T(\mathbf{x}) = A\mathbf{x}$ takes vectors of $\mathbb{R}^4$ and transforms them into vectors of $\mathbb{R}^3$. Nul$\{A\}$ is formed by those vectors in the input space that map onto the $\mathbf{0}$ vector of the output space. Consequently, $p = 4$. In the same way, Col$\{A\}$ is the subspace of the output space formed by all those vectors that can be reached by the transformation. Consquently, $q = 3$.

Lay, 2.8.13

Carlos Oscar Sorzano, June, 6th 2014

Let A be the matrix in Exercise 2.8.11. Find a non-zero vector of Nul$\{A\}$ and a non-zero vector of Col$\{A\}$.
Solution: To find a non-zero vector of Col$\{A\}$ we simply take the first column of A

$$\mathrm{Col}\{A\} \ni \mathbf{y} = \begin{pmatrix} 3 \\ -9 \\ 9 \end{pmatrix} \tag{4}$$

This vector is achieved by multiplying A by $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

To find a non-zero vector of Nul$\{A\}$ we need to solve the equation system $A\mathbf{x} = \mathbf{0}$

$$\left(\begin{array}{cccc|c} 3 & 2 & 1 & -5 & 0 \\ -9 & -4 & 1 & 7 & 0 \\ 9 & 2 & -5 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & -4 & -8 & 16 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim$$
$$\left(\begin{array}{cccc|c} 3 & 0 & -3 & 3 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{5}$$

We see that x_3 and x_4 are free variables, and that x_1 and x_2 can be calculated as

$$\begin{array}{rcl} x_1 & = & x_3 - x_4 \\ x_2 & = & -2x_3 + 4x_4 \end{array} \tag{6}$$

If we substitute $x_3 = 1$ and $x_4 = 0$ we get the vector

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \tag{7}$$

Lay, 2.8.28

Carlos Oscar Sorzano, Nov. 11th, 2013

Construct a 3×3 matrix A and a vector $\mathbf{b}$ such that $\mathbf{b}$ is not in Col$\{A\}$.
Solution: Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Obviously, $\mathbf{b} \notin$ Col$\{A\}$ because there is no way that we can combine the columns of the matrix A to obtain a 1 in the third component.

Lay, 2.8.35

Carlos Oscar Sorzano, Nov. 4th 2014

If B is a 5×5 matrix and Nul$\{B\}$ is not the zero subspace, what can be said about Col$\{B\}$?
Solution: If Nul$\{B\} \neq \{\mathbf{0}\}$, then the columns of B are not linearly independent, and consequently we can assess that Col$\{B\} \neq \mathbb{R}^5$ because a basis of

Col$\{B\}$ has less than 5 elements, but to span $\mathbb{R}^5$ we need 5 linearly independent vectors.

Lay, 2.8.36

Carlos Oscar Sorzano, Nov. 4th 2014

What can be said about Nul$\{C\}$ when C is a 6×4 matrix with linearly independent columns?

Solution: If the columns are linearly independent the only solution of the problem $C\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Consequently, Nul$\{C\} = \{\mathbf{0}\}$.

Lay, 2.9.1

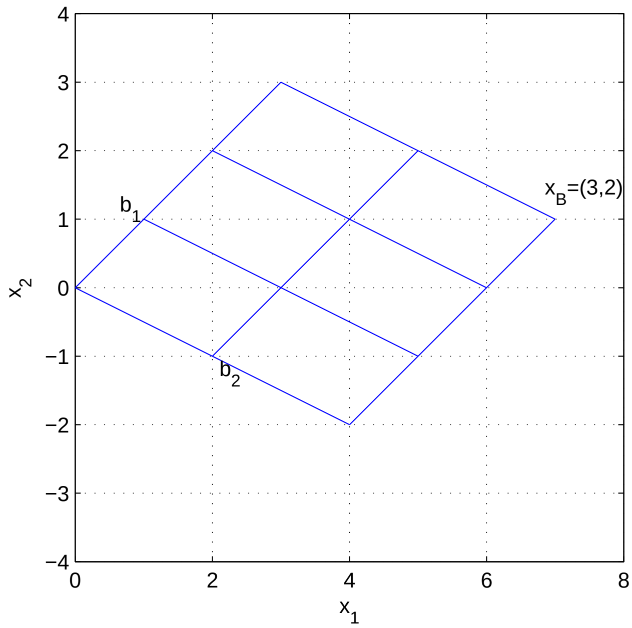
Carlos Oscar Sorzano, Aug. 31st, 2013

Given the basis $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$ and $[\mathbf{x}]_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find $\mathbf{x}$ and illustrate your answer.

Solution: Using the coordinates of $\mathbf{x}$ in the basis B we find

$$\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2 = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

The following figure illustrates this situation



Lay, 2.9.3

Carlos Oscar Sorzano, Aug. 31st, 2013

$\mathbf{x} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$ is in a subspace H whose basis is $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ with $\mathbf{b}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$. Find the coordinates of $\mathbf{x}$ in the basis B.

Solution: Let us look for the coordinates that satisfy

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$$

For this, we will use the augmented matrix

$$\left(\begin{array}{rr|r} 2 & -1 & 0 \\ -3 & 5 & 7 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

So, the coordinates of $\mathbf{x}$ in the basis B are $[\mathbf{x}]_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Lay, 2.9.9

Carlos Oscar Sorzano, Aug. 31st, 2013

Consider $A = \begin{pmatrix} 1 & 3 & 2 & -6 \\ 3 & 9 & 1 & 5 \\ 2 & 6 & -1 & 9 \\ 5 & 15 & 0 & 14 \end{pmatrix}$ and its echelon form $\begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
Find bases for Nul$\{A\}$ and Col$\{A\}$.
Solution: The basis of Nul$\{A\}$ is found by the equation system $A\mathbf{x} = \mathbf{0}$ whose augmented matrix is row-equivalent to

$$\left(\begin{array}{rrrr|r} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 5 & -7 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We may calculate its reduced echelon form

$$\left(\begin{array}{rrrr|r} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 5 & -7 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = B$$

This implies the following equations:

$$\begin{aligned} x_1 &= -3x_2 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

So the basis of Nul$\{A\}$ is given by the non-pivot columns of B, i.e.,

$$\text{Basis}\{\text{Nul}\{A\}\} = \{(-3, 1, 0, 0)\}$$

The basis of Col$\{A\}$ is given by the pivot columns of B. The basis of the column space of B is given by its first, third and fourth columns ($\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_4\}$). Similarly, the basis of the column space of A is given by its first, third and fourth columns ($\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$), i.e.,

$$\text{Basis}\{\text{Col}\{A\}\} = \{(1, 3, 2, 5), (2, 1, -1, 0), (-6, 5, 9, 14)\}$$

Lay, 2.9.19

Carlos Oscar Sorzano, Aug. 31st, 2013

If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of 3 vectors and if A is a 5×7 matrix, what is the rank of A.
Solution: According to the rank theorem

$$\text{Rank}\{A\} + \dim\{\text{Nul}\{A\}\} = n$$

where n is the number of columns of A. In this particular case,

$$\text{Rank}\{A\} + 3 = 7 \Rightarrow \text{Rank}\{A\} = 4$$

Lay, 2.9.23

Carlos Oscar Sorzano, Nov. 11th, 2013

If possible, construct a 3×5 matrix such that $\dim\{\text{Nul}\{A\}\} = 3$ and $\dim\{\text{Col}\{A\}\} = 2$.

Solution: Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. The dimension of its column space is given by the number of pivot columns (columns 1 and 4 are pivot columns), while the dimension of its null space is given by the number of non-pivot columns (columns 2, 3 and 5 are non-pivot).

Lay, 2.9.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose vectors $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_p$ span a subspace W, and let $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_q\}$ by any set in W containing more than p vectors. Fill in the details of the following argument to show that $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_q\}$ must be linearly dependent. First, let $B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & ... & \mathbf{b}_p \end{pmatrix}$ and $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & ... & \mathbf{a}_q \end{pmatrix}$.

a. Explain why for each vector $\mathbf{a}_j$, there exists a vector $\mathbf{c}_j$ in $\mathbb{R}^p$ such that $\mathbf{a}_j = B\mathbf{c}_j$.

b. Let $C = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & ... & \mathbf{c}_q \end{pmatrix}$. Explain why there is a non-zero vector $\mathbf{u}$ such that $C\mathbf{u} = \mathbf{0}$.

c. Use B and C to show that $A\mathbf{u} = \mathbf{0}$. This shows that the columns of A are linearly dependent.

Solution:

a. Each vector $\mathbf{a}_j$ is in W, that is spanned by the $\mathbf{b}_i$ vectors. That means that there exist some coefficients c_{ji} such that

$$\mathbf{a}_j = c_{j1}\mathbf{b}_1 + c_{j2}\mathbf{b}_2 + ... + c_{jp}\mathbf{b}_p$$

or what is the same

$$\mathbf{a}_j = B\mathbf{c}_j$$

b. Note that the $\mathbf{c}_j$ vectors are in $\mathbb{R}^p$ since they have p components. The problem stated that $q > p$, that is there are more $\mathbf{c}_j$ vectors than p (their dimension). By Theorem 6.2 of Chapter 2, we have that this set of equations is linearly dependent, that is, there exist some coefficients (not all of them zero) such that

$$u_1\mathbf{c}_1 + u_2\mathbf{c}_2 + ... + u_p\mathbf{c}_p = \mathbf{0}$$

or

$$C\mathbf{u} = \mathbf{0}$$

c. Let us calculate $A\mathbf{u}$. From point a, we know that $A = BC$, therefore

$$A\mathbf{u} = (BC)\mathbf{u} = B(C\mathbf{u}) = B\mathbf{0} = \mathbf{0}$$

Lay, 2.9.28

Carlos Oscar Sorzano, Nov. 4th 2014

Use Exercise 2.9.27 to show that if $\mathcal{A}$ and $\mathcal{B}$ are bases for a subspace W of $\mathbb{R}^n$, then $\mathcal{A}$ cannot contain more vectors than $\mathcal{B}$, and, conversely $\mathcal{B}$ cannot contain more vectors than $\mathcal{A}$.

Solution: If $\mathcal{A}$ and $\mathcal{B}$ are bases of W, then

$$\text{Span}\{\mathcal{A}\} = \text{Span}\{\mathcal{B}\} = W$$

If $\mathcal{A}$ has more vectors than $\mathcal{B}$, then the set of vectors $\mathcal{A}$ would not be linearly independent, because there is a basis with fewer vectors. But this is a contradiction with the hypothesis that $\mathcal{A}$ is a basis, so $\mathcal{A}$ cannot have more vectors than $\mathcal{B}$.

Using the same reasoning we deduce that $\mathcal{B}$ cannot have more vectors than $\mathcal{A}$.

Lay, 2.Suppl.8

Carlos Oscar Sorzano, Jan. 19th 2015

Find a matrix A such that the transformation $\mathbf{x} \to A\mathbf{x}$ maps $(1,3)$ and $(2,7)$ into $(1,1)$ and $(3,1)$, respectively.

Solution: In general, matrix transformations of $\mathbb{R}^2$ into $\mathbb{R}^2$ respond to the equation

$$\mathbf{y} = A\mathbf{x} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

In this case we have

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \left\{ \begin{array}{c} 1 = a_{11} + 3a_{12} \\ 1 = a_{21} + 3a_{22} \end{array} \right.$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} \Rightarrow \left\{ \begin{array}{c} 3 = 2a_{11} + 7a_{12} \\ 1 = 2a_{21} + 7a_{22} \end{array} \right.$$

Or what is the same

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 2 & 7 & 0 & 0 \\ 0 & 0 & 2 & 7 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \\ -1 \end{pmatrix}$$

The matrix sought is

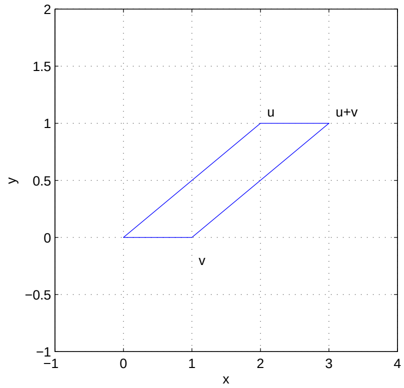
$$A = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}$$

3 Chapter 3

Lay, 3.1.42

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ 0 \end{pmatrix}$, where a, b, c are positive (for simplicity). Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $\mathbf{0}$, and compute the determinants of the matrices $\begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{v} & \mathbf{u} \end{pmatrix}$. Draw a picture and explain what you find.
Solution: The area of the parallelogram is base times height. In this case:



$$A = cb$$

The determinant of $\begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$ is

$$\begin{vmatrix} a & c \\ b & 0 \end{vmatrix} = a \cdot 0 - bc = -bc$$

The determinant of $\begin{pmatrix} \mathbf{v} & \mathbf{u} \end{pmatrix}$ is

$$\begin{vmatrix} c & a \\ 0 & b \end{vmatrix} = cb - a \cdot 0 = cb$$

We see that $A = \text{abs}\left(\left|\begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}\right|\right) = \text{abs}\left(\left|\begin{pmatrix} \mathbf{v} & \mathbf{u} \end{pmatrix}\right|\right)$

Lay, 3.2.14

Carlos Oscar Sorzano, Aug. 31st, 2013

Combine the methods of row reduction and cofactor expansion to compute the determinant $\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}$
Solution:

$$D = \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}$$

$$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_1 \quad D = \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -9 & 0 & 0 & 0 \\ 3 & -4 & 0 & 4 \end{vmatrix} = (-1)^{1+3}1 \begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix}$$

$$D = \begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix} = (-1)^{2+1}(-9) \begin{vmatrix} 3 & -3 \\ -4 & 4 \end{vmatrix}$$

$$D = 9(3 \cdot 4 - (-3) \cdot (-4)) = 0$$

Lay, 3.2.15

Carlos Oscar Sorzano, Aug. 31st, 2013

Assume $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Calculate $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$.

Solution:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = 5 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5 \cdot 7 = 35$$

Lay, 3.2.16

Marta Monsalve Buendía, Oct. 14th, 2014

Assume $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Calculate $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$.

Solution:

$$\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = 3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3 \cdot 7 = 21$$

Lay, 3.2.17

Marta Monsalve Buendía, Oct. 18th, 2014

Assume $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Calculate $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$.

Solution:

$$\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -7$$

Lay, 3.2.18

Carlos Oscar Sorzano, Aug. 31st, 2013

Assume $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Calculate $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$.

Solution:

$$\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Lay, 3.2.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Assume $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Calculate $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$.

Solution:

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2 \cdot 7 = 14$$

Lay, 3.2.20

Andrea Santos Cortés, Oct. 20th, 2014

Assume $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Calculate $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$.

Solution:

$$\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} d & e & f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Lay, 3.2.21

Andrea Santos Cortés, Oct. 20th, 2014

Use determinants to find out if the matrix is invertible

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}$$

Solution: Let's calculate the determinant

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = (2\cdot3\cdot1)+(1\cdot2\cdot0)+(3\cdot4\cdot1)-(0\cdot3\cdot1)-(3\cdot1\cdot1)-(2\cdot2\cdot4) = 7$$

The determinant is different from 0 and consequently, this matrix is invertible.

Lay, 3.2.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Use determinants to decide if the set of vectors $\begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$, $\begin{pmatrix} -7 \\ 0 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} -3 \\ -5 \\ 6 \end{pmatrix}$ is linearly independent.
Solution:

$$\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix} = 11$$

The three vectors are linearly independent because their determinant is different from 0.

Lay, 3.2.25

Andrea Santos Cortés, Oct. 21th, 2014

Use determinants to decide if the set of vectors $\begin{pmatrix} 7 \\ -4 \\ -6 \end{pmatrix}$, $\begin{pmatrix} -8 \\ 5 \\ 7 \end{pmatrix}$, and $\begin{pmatrix} 7 \\ 0 \\ -5 \end{pmatrix}$ is linearly independent.
Solution:

$$\begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{vmatrix} = -1$$

The three vectors are linearly independent because their determinant is different from 0.

Lay, 3.2.31

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if A is invertible, then $\det\{A^{-1}\} = \frac{1}{\det\{A\}}$
Solution: If A is invertible, then

$$AA^{-1} = I$$

Taking determinants on both sides

$$\begin{array}{c} \det\{AA^{-1}\} = \det\{I\} \\ \det\{A\}\det\{A^{-1}\} = 1 \\ \det\{A^{-1}\} = \frac{1}{\det\{A\}} \end{array}$$

Lay, 3.2.32

Carlos Oscar Sorzano, Aug. 31st, 2013

Find a formula for $\det\{rA\}$ when A is an $n \times n$ matrix.
Solution: Consider the column decomposition of A

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & ... & \mathbf{a}_n \end{pmatrix}$$

Then

$$rA = \begin{pmatrix} r\mathbf{a}_1 & r\mathbf{a}_2 & ... & r\mathbf{a}_n \end{pmatrix}$$

$$\begin{aligned}
\det\{rA\} &= \begin{vmatrix} r\mathbf{a}_1 & r\mathbf{a}_2 & ... & r\mathbf{a}_n \end{vmatrix} \\
&= r\begin{vmatrix} \mathbf{a}_1 & r\mathbf{a}_2 & ... & r\mathbf{a}_n \end{vmatrix} \\
&= r^2\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & ... & r\mathbf{a}_n \end{vmatrix} \\
&= r^n\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & ... & \mathbf{a}_n \end{vmatrix} \\
&= r^n \det\{A\}
\end{aligned}$$

Lay, 3.2.33

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A and B square matrices. Show that even though AB and BA may not be equal, it is always true that $\det\{AB\} = \det\{BA\}$

Solution: By applying properties of the determinants

$$\begin{gathered}
\det\{AB\} = \det\{BA\} \\
\det\{A\}\det\{B\} = \det\{B\}\det\{A\}
\end{gathered}$$

Lay, 3.2.34

Carlos Oscar Sorzano, Nov. 4th 2014

Let A and P be square matrices, with P invertible. Show that $\det\{PAP^{-1}\} = \det\{A\}$.

Solution: Since P is invertible, we have

$$\det\{P^{-1}\} = \frac{1}{\det\{P\}}$$

Then,

$$\det\{PAP^{-1}\} = \det\{P\}\det\{A\}\det\{P^{-1}\} = \det\{P\}\det\{A\}\frac{1}{\det\{P\}} = \det\{A\}$$

Lay, 3.2.35

Let U be a square matrix such that $U^TU = I$. Show that $\det(U) = \pm 1$.

Solution:

$U^TU = I$	$A = B \Rightarrow \det(A) = \det(B)$
$\det(U^TU) = \det(I)$	$\det(I) = 1$
$\det(U^TU) = 1$	For square matrices A and B, $\det(AB) = \det(A)\det(B)$
$\det(U^T)\det(U) = 1$	$\det(A^T) = \det(A)$
$\det(U)\det(U) = 1$	
$(\det(U))^2 = 1$	
$\det(U) = \pm 1$	

Lay, 3.2.36

Carlos Oscar Sorzano, Nov. 11th, 2013

Suppose that A is a square matrix such that $\det A^4 = 0$. Explain why A cannot be invertible.

Solution: We know that $|A^4| = 0 = |A|^4$. This means that $|A| = 0$ and consequently, it cannot be invertible.

Lay, 3.2.37

Ana Peña Gil, Jan. 19th 2014

Let $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$. Write $5A$. Is $\det\{5A\} = 5\det\{A\}$?

Solution:

$$5A = 5\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 5 \\ 20 & 10 \end{pmatrix}$$
$$\det\{5A\} = \begin{vmatrix} 15 & 5 \\ 20 & 10 \end{vmatrix} = 50$$
$$5\det\{A\} = 5\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 5 \cdot 2 = 10$$

We see that $\det\{5A\} \neq 5\det\{A\}$

Lay, 3.3.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Use Cramer's rule to solve the following equation system

$$\begin{aligned} 5x_1 + 7x_2 &= 3 \\ 2x_1 + 4x_2 &= 1 \end{aligned}$$

Solution:

$$x_1 = \frac{\begin{vmatrix} 3 & 7 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix}} = \frac{3\cdot4-7\cdot1}{5\cdot4-7\cdot2} = \frac{5}{6}$$
$$x_2 = \frac{\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & 4 \end{vmatrix}} = \frac{5\cdot1-3\cdot2}{5\cdot4-7\cdot2} = -\frac{1}{6}$$

Lay, 3.3.2

Laura Zarandieta, Oct. 29th 2013

Use Cramer's rule to solve the following equation system

$$\begin{aligned} 4x_1 + x_2 &= 6 \\ 5x_1 + 2x_2 &= 7 \end{aligned}$$

Solution:

$$x_1 = \frac{\begin{vmatrix} 6 & 1 \\ 7 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}} = \frac{6\cdot2-7\cdot1}{4\cdot2-1\cdot5} = \frac{5}{3}$$

$$x_2 = \frac{\begin{vmatrix} 4 & 6 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}} = \frac{4 \cdot 7 - 6 \cdot 5}{4 \cdot 2 - 1 \cdot 5} = \frac{-2}{3}$$

Lay, 3.3.3

Laura Zarandieta, Oct. 29th 2013

Use Cramer's rule to solve the following equation system

$$3x_1 - 2x_2 = 7$$
$$-5x_1 + 6x_2 = -5$$

Solution:

$$x_1 = \frac{\begin{vmatrix} 7 & -2 \\ -5 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix}} = \frac{7 \cdot 6 - (-2)(-5)}{3 \cdot 6 - (-2)(-5)} = \frac{32}{8} = 4$$

$$x_2 = \frac{\begin{vmatrix} 3 & 7 \\ -5 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix}} = \frac{3(-5) - 7(-5)}{3 \cdot 6 - (-2)(-5)} = \frac{20}{8} = \frac{10}{4} = 2.5$$

Lay, 3.3.4

Andrea Santos Cortés, Oct. 14th, 2014

Use Cramer's rule to solve the following equation system

$$-5x_1 + 3x_2 = 9$$
$$3x_1 - x_2 = -5$$

Solution:

We may write the following equation system (in matrix form):

$$\begin{pmatrix} -5 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 9 & 3 \\ -5 & -1 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{9 \cdot (-1) - 3 \cdot (-5)}{-5 \cdot (-1) - 3 \cdot 3} = -\frac{3}{2}$$

$$x_2 = \frac{\begin{vmatrix} -5 & 9 \\ 3 & -5 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{-5 \cdot (-5) - 9 \cdot 3}{-5 \cdot (-1) - 3 \cdot 3} = \frac{1}{2}$$

Lay, 3.3.7

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine the values of the parameter s for which the system below has a unique solution.

$$6sx_1 + 4x_2 = 5$$
$$9x_1 + 2sx_2 = -2$$

Solution: Applying Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 5 & 4 \\ -2 & 2s \end{vmatrix}}{\begin{vmatrix} 6s & 4 \\ 9 & 2s \end{vmatrix}} = \frac{5\cdot 2s - 4\cdot(-2)}{6s\cdot 2s - 4\cdot 9} = \frac{10s+8}{12s^2-36} = \frac{10(s+\frac{8}{10})}{12(s+\sqrt{3})(s-\sqrt{3})} = \frac{5(s+\frac{8}{10})}{6(s+\sqrt{3})(s-\sqrt{3})}$$

$$x_2 = \frac{\begin{vmatrix} 6s & 5 \\ 9 & -2 \end{vmatrix}}{\begin{vmatrix} 6s & 4 \\ 9 & 2s \end{vmatrix}} = \frac{6s\cdot(-2) - 5\cdot 9}{6s\cdot 2s - 4\cdot 9} = -\frac{12s+45}{12s^2-36} = -\frac{12(s+\frac{45}{12})}{12(s+\sqrt{3})(s-\sqrt{3})} =$$

$$-\frac{s+\frac{45}{12}}{(s+\sqrt{3})(s-\sqrt{3})}$$

This equation system has a unique solution if the denominator of the fractions above do not vanish, that is, $s \neq \pm\sqrt{3}$.

Lay, 3.3.11

Carlos Oscar Sorzano, Aug. 31st, 2013

Calculate the adjugate of the matrix $A = \begin{pmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$. Then, use it to calculate A^{-1}.

Solution: For calculating the adjugate of the matrix A we need to calculate all its cofactors

$$\begin{array}{rcrcl}
C_{11} & = & (-1)^{1+1} \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} & = & 0 \\
C_{12} & = & (-1)^{1+2} \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} & = & -3 \\
C_{13} & = & (-1)^{1+3} \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} & = & 3 \\
C_{21} & = & (-1)^{2+1} \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} & = & 1 \\
C_{22} & = & (-1)^{2+2} \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} & = & -1 \\
C_{23} & = & (-1)^{2+3} \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} & = & 2 \\
C_{31} & = & (-1)^{3+1} \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} & = & 0 \\
C_{32} & = & (-1)^{3+2} \begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} & = & -3 \\
C_{33} & = & (-1)^{3+3} \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} & = & 6
\end{array}$$

The adjoint is

$$A^* = \begin{pmatrix} 0 & -3 & 3 \\ 1 & -1 & 2 \\ 0 & -3 & 6 \end{pmatrix}$$

For calculating A^{-1} we need the determinant of A. We use the cofactor expansion along the second row

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 3 \cdot 1 = 3$$

Now

$$A^{-1} = \frac{1}{|A|}(A^*)^T = \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -1 & -\frac{1}{3} & -1 \\ 1 & \frac{2}{3} & 2 \end{pmatrix}$$

Lay, 3.3.19

Marta Monsalve Buendía, Oct. 18th, 2014

Find the area of the parallelogram whose vertices are (0,0), (5,2), (6,4), (11,6).
Solution:

Note that this parallelogram has a vertex at $\mathbf{0}$. Consequently, calling $\mathbf{x}_A = (5,2)^T$, $\mathbf{x}_B = (6,4)^T$, the sought area is the absolute value of the determinant of the vectors $\mathbf{x}_A$ and $\mathbf{x}_B$.

$$\text{abs}\left(\begin{vmatrix} \mathbf{x}_A & \mathbf{x}_B \end{vmatrix}\right) = \text{abs}\left(\begin{vmatrix} 5 & 6 \\ 2 & 4 \end{vmatrix}\right) = 8$$

Lay, 3.3.20

Carlos Oscar Sorzano, Aug. 31st, 2013

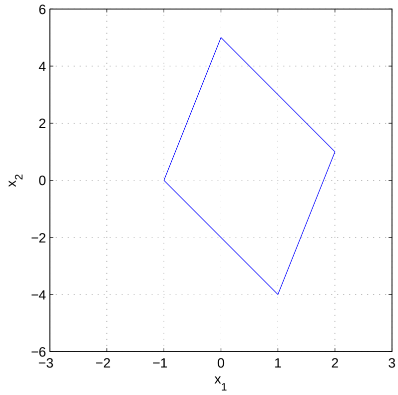
Find the area of the parallelogram whose vertices are (-1,0), (0,5), (1,-4), (2,1).
Solution: Let us draw the parallelogram: Calling $\mathbf{x}_A = (-1,0)$, $\mathbf{x}_B = (0,5)$, $\mathbf{x}_C = (1,-4)$, the sought area is the absolute value of the determinant of the vectors $\mathbf{x}_B - \mathbf{x}_A$ and $\mathbf{x}_C - \mathbf{x}_A$.

$$\begin{array}{c} \mathbf{x}_B - \mathbf{x}_A = \begin{pmatrix} 0 & 5 \end{pmatrix} - \begin{pmatrix} -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 \end{pmatrix} \\ \mathbf{x}_C - \mathbf{x}_A = \begin{pmatrix} 1 & -4 \end{pmatrix} - \begin{pmatrix} -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -4 \end{pmatrix} \\ \text{abs}\left(\begin{vmatrix} \mathbf{x}_B - \mathbf{x}_A & \mathbf{x}_C - \mathbf{x}_A \end{vmatrix}\right) = \text{abs}\left(\begin{vmatrix} 1 & 2 \\ 5 & -4 \end{vmatrix}\right) = 14 \end{array}$$

Lay, 3.3.21

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the area of the parallelogram whose vertices are (-1,0), (0,5), (1,-4), (2,1).
Solution: Let us draw the parallelogram:



Calling $\mathbf{x}_A = (-1, 0)$, $\mathbf{x}_B = (0, 5)$, $\mathbf{x}_C = (1, -4)$, the sought area is the absolute value of the determinant of the vectors $\mathbf{x}_B - \mathbf{x}_A$ and $\mathbf{x}_C - \mathbf{x}_A$.

$$\begin{aligned} \mathbf{x}_B - \mathbf{x}_A &= \begin{pmatrix} 0 & 5 \end{pmatrix} - \begin{pmatrix} -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 \end{pmatrix} \\ \mathbf{x}_C - \mathbf{x}_A &= \begin{pmatrix} 1 & -4 \end{pmatrix} - \begin{pmatrix} -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -4 \end{pmatrix} \\ \operatorname{abs}\left(\begin{vmatrix} \mathbf{x}_B - \mathbf{x}_A & \mathbf{x}_C - \mathbf{x}_A \end{vmatrix} \right) &= \operatorname{abs}\left(\begin{vmatrix} 1 & 2 \\ 5 & -4 \end{vmatrix} \right) = 14 \end{aligned}$$

Lay, 3.3.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Use the concept of volume to explain why the determinant of a 3×3 matrix is zero iff A is not invertible.

Solution: From the invertible matrix theorem, we know that a matrix is invertible iff its columns are linearly independent. So the statement of this problem can be restated as the determinant of a 3×3 matrix is zero iff the three columns of A are linearly dependent. On the other side interpreting the determinant of A as the volume of the parallelepiped formed by the three columns, the problem is "the volume of the parallelepiped formed by three vectors is zero iff the three columns of A are linearly dependent".

If the three vectors are linearly dependent, they span a subspace of dimension 2 or 1. In both cases, there is no real parallelepiped but a parallelogram or a segment and the volume of the parallelepiped is 0.

Let us show that if the volume of the parallepiped is zero, then three columns are linearly dependent. Let's assume they are linearly independent. Then, they would actually span a three-dimensiional space, and the volume of the parallelepiped formed by the three would not be zero. But this is a contradiction with our hypothesis. So the three vectors have to be linearly dependent.

Lay, 3.3.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\mathbf{p}$ be a vector and S a set in $\mathbb{R}^n$. Show that the image of $\mathbf{p} + S$ under T is the translated set $T(\mathbf{p}) + T(S)$ in $\mathbb{R}^m$.

Solution: Any vector of the set $\mathbf{p} + S$ is of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{s}$$

where $\mathbf{s} \in S$. If we apply T to $\mathbf{x}$ and exploiting the fact that T is a linear transformation, we get

$$T(\mathbf{x}) = T(\mathbf{p}+\mathbf{s}) = T(\mathbf{p}) + T(\mathbf{s})$$

The set of all vectors of the form $T(\mathbf{s})$ is actually $T(S)$, so we have that, as stated by the problem,

$$T(\mathbf{x}) \in T(\mathbf{p}) + T(S)$$

Lay, 3.3.29

Carlos Oscar Sorzano, June, 6th 2014

Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ in $\mathbb{R}^2$.

Solution: We know that the area of the parallelepiped formed by the vectors $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_1+\mathbf{v}_2$ is given by the determinant of the 2×2 matrix A (see Theorem 3.3.9)

$$A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}.$$

The area of the required triangle is just one half of this. In this way

$$\text{AreaOfTriangle} = \tfrac{1}{2}\det\left\{\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}\right\}.$$

Lay, 3.3.30

Carlos Oscar Sorzano, Nov. 4th 2014

Let R be the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3). Show that

$$\text{Area}\{R\} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

[*Hint*: Translate R to the origin by subtracting one of the vertices and use Exercise 3.3.29.]

Solution: Let us translate the triangle to the origin by subtracting the vertex (x_1, y_1). The new vertices of the triangle are $\mathbf{v}_1 = (x_2 - x_1, y_2 - y_1)$ and $\mathbf{v}_2 = (x_3 - x_1, y_3 - y_1)$. By applying Exercise 3.3.29, the area of this triangle is

$$\text{Area}\{R\} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & x_3 - x_2 \\ y_2 - y_1 & y_3 - y_2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_2 & y_3 - y_2 \end{vmatrix}$$

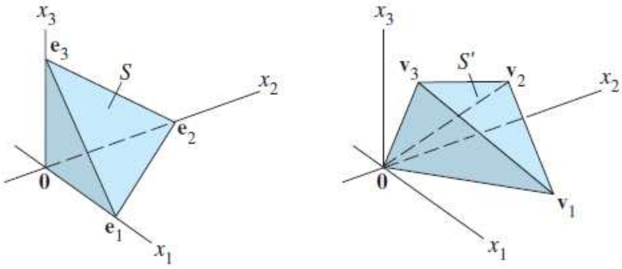
Let us now develop the determinant porposed

$$\begin{aligned}
\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} &= \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \\
&= \tfrac{1}{2}x_1(-1)^{1+1}\begin{vmatrix} y_2 - y_1 & 0 \\ y_3 - y_1 & 0 \end{vmatrix} + \\
&\quad \tfrac{1}{2}y_1(-1)^{1+2}\begin{vmatrix} x_2 - x_1 & 0 \\ x_3 - x_1 & 0 \end{vmatrix} + \\
&\quad \tfrac{1}{2}1(-1)^{1+3}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\
&= \tfrac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}
\end{aligned}$$

Lay, 3.3.32

Carlos Oscar Sorzano, Aug. 31st, 2013

Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}$, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ and let S' be the tetrahedron with vertices at vectors $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. See the figure.



a. Describe a linear transformation that maps S into S'.

b. Find a formula for the volume of the tetrahedron S' using the fact

$$\text{Volume of } S{=}\tfrac{1}{3} \text{ Area of the base} \cdot \text{Height.}$$

Solution:

a. Consider the matrix

$$A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$$

The tetrahedron S is formed by all those points that can be written in the form

$$\mathbf{x} = \lambda_0\mathbf{0} + \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3$$

with

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 \leq 1$$

If we consider now $A\mathbf{x}$, we have

$$\begin{array}{rcl} A\mathbf{x} &=& A(\lambda_0\mathbf{0} + \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3) \\ &=& \lambda_0 A\mathbf{0} + \lambda_1 A\mathbf{e}_1 + \lambda_2 A\mathbf{e}_2 + \lambda_3 A\mathbf{e}_3 \\ &=& \lambda_0\mathbf{0} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 \end{array}$$

So this is a point in the tetrahedron S' as required by the problem.

b. The base of the tetrahedron S is a triangle with vertices $\mathbf{0}$, $\mathbf{e}_1$ and $\mathbf{e}_2$, whose area is

$$\text{Area triangular base} = \frac{1}{2}\text{ Base} \cdot \text{Height} = \frac{1}{2}1 \cdot 1 = \frac{1}{2}.$$

The height of the tetrahedron is the length of $\mathbf{e}_3$, that is, 1. Finally

$$\text{Volume of } S{=}\frac{1}{3}\text{ Area of the base} \cdot \text{Height} = \frac{1}{3}\frac{1}{2}1 = \frac{1}{6}$$

According to Theorem 5.2 in Chapter 4, the volume of S' is

$$\text{Volume of } S'{=}|\det\{A\}|\text{ Volume of } S{=}\frac{1}{6}|\det\{A\}|$$

Lay, 3.Suppl.9

Carlos Oscar Sorzano, Nov. 11th, 2013

Let T be the Vandermonde matrix $T = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$. Use row operations to show that $|T| = (b-a)(c-a)(c-b)$

Solution: Let us calculate the determinant of T

$$\begin{array}{ll}
 & \det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = \\
\begin{array}{l}\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1\end{array} & \det \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} = \\
 & 1 \det \begin{pmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{pmatrix} = \\
\begin{array}{l}\mathbf{r}_2 \leftarrow \frac{1}{b-a}\mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \frac{1}{c-a}\mathbf{r}_3\end{array} & (b-a)(c-a) \det \begin{pmatrix} 1 & \frac{b^2-a^2}{b-a} \\ 1 & \frac{c^2-a^2}{c-a} \end{pmatrix} = \\
 & (b-a)(c-a) \det \begin{pmatrix} 1 & b+a \\ 1 & c+a \end{pmatrix} = \\
\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \mathbf{r}_1 & (b-a)(c-a) \det \begin{pmatrix} 1 & b+a \\ 0 & c-b \end{pmatrix} = \\
 & (b-a)(c-a)1 \det \begin{pmatrix} c-b \end{pmatrix} = \\
 & (b-a)(c-a)(c-b)
\end{array}$$

Lay, 3.Suppl.15

Carlos Oscar Sorzano, Jan. 19th 2015

Let A, B, C and D be $n \times n$ matrices with A invertible.

1. Find matrices X and Y to produce the block LU factorization

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & Y \end{pmatrix}$$

   and then show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$$

2. Show that if $AC = CA$, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB)$$

Solution:

1. Let us develop the right-hand side

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & Y \end{pmatrix} = \begin{pmatrix} A & B \\ XA & XB + Y \end{pmatrix}$$

By comparing to the left-hand side, we see that

$$XA = C \Rightarrow X = CA^{-1}$$

$$XB + Y = D \Rightarrow Y = D - XB = D - CA^{-1}B$$

That is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

The determinant of a triangular block matrix is the product of the determinants of its diagonal blocks. Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(I)\det(I)\det(A)\det(D - CA^{-1}B) = \det(A)\det(D - CA^{-1}B)$$

2. Let us multiply the block matrix from the left by

$$\begin{pmatrix} I & 0 \\ -C & A \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ -CA + AC & -CB + AD \end{pmatrix}$$

Since $AC = CA$, we have

$$\begin{pmatrix} I & 0 \\ -C & A \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & AD - CB \end{pmatrix}$$

Taking determinants on both sides

$$\det \begin{pmatrix} I & 0 \\ -C & A \end{pmatrix} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} A & B \\ 0 & AD - CB \end{pmatrix}$$

$$\det(A) \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(AD - CB)$$

Since A is invertible, its determinant is different from 0 and we can divide by it

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB)$$

Villa, 5.15

Carlos Oscar Sorzano, Nov. 4th, 2014

Calculate the determinant of order n

$$\begin{vmatrix} 1+x & 1 & 1 & ... & 1 \\ 1 & 1+x & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 1+x \end{vmatrix}$$

Solution: If we add all rows from 2 to n to the first row we have

$$\begin{aligned} \begin{vmatrix} 1+x & 1 & 1 & ... & 1 \\ 1 & 1+x & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 1+x \end{vmatrix} &= \begin{vmatrix} n+x & n+x & n+x & ... & n+x \\ 1 & 1+x & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 1+x \end{vmatrix} \\ &= (n+x) \begin{vmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & 1+x & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 1+x \end{vmatrix} \end{aligned}$$

We now subtract the first row from the rest of rows and develop the determinant by the cofactors of the first column:

$$\begin{aligned} (n+x) \begin{vmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & 1+x & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 1+x \end{vmatrix} &= (n+x) \begin{vmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & x & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & x \end{vmatrix} \\ &= (n+x)x^{n-1} \end{aligned}$$

4 Chapter 4

Lay, 4.1.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Let V be the first quadrant in the xy-plane; that is, let

$$V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x, y \geq 0 \right\}$$

a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u}+\mathbf{v}$ in V? Why?

b. Find a specific vector $\mathbf{u} \in V$ and a specific scalar c such that $c\mathbf{u}$ is not in V. (This is enough to show that V is not a vector space.)

Solution:

a. Let $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$, then

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} u_x + v_x \\ u_y + v_y \end{pmatrix}$$

If $u_x \geq 0$ and $v_x \geq 0$, then $u_x + v_x \geq 0$. Similarly for $u_y + v_y$. Consequently, $\mathbf{u}+\mathbf{v}$ is also in V.

b. Let $V \ni \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $c = -1$, then

$$c\mathbf{u} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

that is not in V.

Lay, 4.1.2

Andrea Santos Cortés, Nov. 11th, 2014

Let W be the union of the first and third quadrants in the xy-plane. That is, let

$$W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | xy \geq 0 \right\}$$

a. If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

b. Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u} + \mathbf{v}$ is not in W. This is enough to show that W is not a vector space.

Solution:

a. Let $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$, then

$$c\mathbf{u} = c\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} cu_x \\ cu_y \end{pmatrix}$$

If $u_x \geq 0$ and $u_y \geq 0$, then $c\mathbf{u}$ is also in W.(regardless of the sign of c)

b. Let $W \ni \mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $W \ni \mathbf{v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, then

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

that is not in W.

Lay, 4.1.3 (3rd ed.)

María Postigo Fliquete, Dec. 7th, 2014

Let H be the set of points inside and on the unit circle in the xy-plane. That is, let

$$H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x^2 + y^2 \leq 1 \right\}$$

Find a specific example-two vectors or a vector and a scalar-to show that H is not a subspace of R^2.

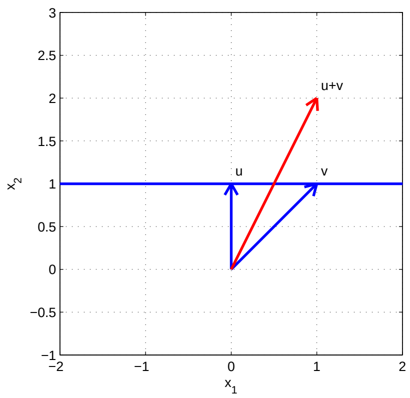
Solution: If $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and c=2, then $\mathbf{u}$ is in H, but $c\mathbf{u}$ isn't.

Lay, 4.1.4

Carlos Oscar Sorzano, Aug. 31st, 2013

Construct a geometric figure that illustrates why a line in $\mathbb{R}^2$ not through the origin is not closed under vector addition.

Solution:



In the figure above it is clear that vector $\mathbf{u}$ and $\mathbf{v}$ belong to the line, but $\mathbf{u} + \mathbf{v}$ does not.

Lay, 4.1.5

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine if the set of all polynomials of the form $p(t) = at^2 \quad \forall a \in \mathbb{R}$ are a subspace of $\mathbb{P}_2$.

Solution: Let $H = \{p(t) \in \mathbb{P}_2 | p(t) = at^2\}$. We need to show that this set meets the three requirements to be a subspace

- $\mathbf{0} \in H$
  This is true because for $a = 0$ we have $p(t) = 0t^2 = 0$.

- Given any two polynomials $p_1(t), p_2(t) \in H$, $p_1(t) + p_2(t) \in H$
  Assume $p_1(t) = a_1 t^2$ and $p_2(t) = a_2 t^2$, then

  $$p_1(t) + p_2(t) = a_1 t^2 + a_2 t^2 = (a_1 + a_2) t^2$$

  So $p_1(t) + p_2(t) \in H$

- Given any polynomial $p(t) \in H$ and $c \in \mathbb{R}$, $cp(t) \in H$

  $$cp(t) = c(at^2) = (ac)t^2$$

  So $cp(t) \in H$

Since H meets all properties, H is a subspace of $\mathbb{P}^2$.

Lay, 4.1.6

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine if the set of all polynomials of the form $p(t) = a + t^2 \quad \forall a \in \mathbb{R}$ are a subspace of $\mathbb{P}_2$.

Solution: Let $H = \{p(t) \in \mathbb{P}_2 | p(t) = a + t^2\}$. We need to show that this set meets the three requirements to be a subspace

- $0 \in H$
  But this is not true for H, there is no value of a such that $p(t) = a + t^2 = 0 \quad \forall t \in \mathbb{R}$

Since H does not meet one of the conditions to be subspace, it cannot be a subspace of $\mathbb{P}^2$.

Lay, 4.1.9

Ana Peña Gil, Jan. 19th 2014

Let H be the set of all vectors of the form $\begin{pmatrix} -2t \\ 5t \\ 3t \end{pmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

Solution:

$$\forall \mathbf{v} \in H \Rightarrow \mathbf{v} = \begin{pmatrix} -2t \\ 5t \\ 3t \end{pmatrix} = t \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}$$

So $H = \text{Span}\{(-2, 5, 3)\}$. Since H is generated by a set of vectors of $\mathbb{R}^3$, by Theorem 4.1, H is a vector subspace of $\mathbb{R}^3$.

Lay, 4.1.10

Ana Peña Gil, Jan. 19th 2014

Let H be the set of all vectors of the form $\begin{pmatrix} 3t \\ 0 \\ -7t \end{pmatrix}$, where t is any real number. Show that H is a subspace of $\mathbb{R}^3$.

Solution: Let $\mathbf{u} \in H$. Then, we can write $u = \begin{pmatrix} 3t \\ 0 \\ -7t \end{pmatrix} = t \begin{pmatrix} 3 \\ 0 \\ -7 \end{pmatrix}$. So $H = \text{Span}\{(3, 0, -7)\}$ and H is a vector subspace $\mathbb{R}^3$ because it can be generated by a vector of $\mathbb{R}^3$.

Lay, 4.1.11

Ana Peña Gil, Jan. 19th 2014

Let W be the set of all vectors of the form $\begin{pmatrix} 2b + 3c \\ -b \\ 2c \end{pmatrix}$, where b and c are arbitrary. Find the vectors $\mathbf{u}$ and $\mathbf{v}$ such that $H = \text{Span}\{\mathbf{u}, \mathbf{v})\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

Solution: Let $\mathbf{w} \in W$. We can write

$$\mathbf{w} = \begin{pmatrix} 2b+3c \\ -b \\ 2c \end{pmatrix} = b\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = b\mathbf{u} + c\mathbf{v}$$

So, $H = \text{Span}\{(\mathbf{u}, \mathbf{v})\} = \text{Span}\{(2,-1,0),(3,0,2)\}$. W is a vector subspace $\mathbb{R}^3$ because it can be generated by a set of vector of $\mathbb{R}^3$.

Lay, 4.1.11 (3rd ed.)

María Postigo Fliquete, Dec. 7th, 2014

Let W be the set of all vectors of the form

$$W = \begin{pmatrix} 5b+2c \\ b \\ c \end{pmatrix}$$

where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = Spam\{u, v\}$ Why does this show that W is a subspace of $\mathbb{R}^3$?

Solution: $\begin{pmatrix} 5b+2c \\ b \\ c \end{pmatrix} = b\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ So $W = Spam\{u, v\}$ where $u = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

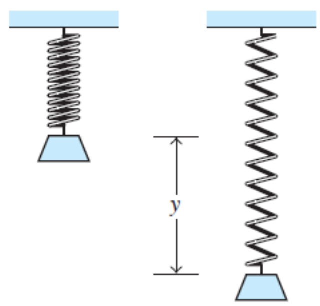
Lay, 4.1.19

Carlos Oscar Sorzano, Aug. 31st, 2013

If a mass m is placed at the end of a spring, and if the mass is pulled downward and released, the mass-spring system will begin to oscillate. The displacement y of the mass from its resting position is given by a function of the form

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

where ω is a constant that depends on the mass and the spring. (See the figure below.) Show that the set of all functions described above (with fixed ω and c_1, c_2 arbitrary) is a vector space.



Solution: Let us call V the set of all functions that can be expressed as

$$V = \{y(t)|y(t) = c_1\cos(\omega t) + c_2\sin(\omega t)\}$$

To show that V is a vector space we need to show that $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall c, d \in \mathbb{R}$

1. $\mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. $\exists \mathbf{0} \in V | \mathbf{u} + \mathbf{0} = \mathbf{u}$
5. $\forall \mathbf{u} \in V \quad \exists! \mathbf{w} \in V | \mathbf{u} + \mathbf{w} = \mathbf{0}$ (we normally write $\mathbf{w} = -\mathbf{u}$)
6. $c\mathbf{v} \in V$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Let's prove all these properties:

1. $\mathbf{u} + \mathbf{v} \in V$

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) + (c_{1v}\cos(\omega t) + c_{2v}\sin(\omega t)) \\
&= (c_{1u} + c_{1v})\cos(\omega t) + (c_{2u} + c_{2v})\sin(\omega t) \in V
\end{aligned}$$

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$$\begin{aligned}
\mathbf{v} + \mathbf{u} &= (c_{1v}\cos(\omega t) + c_{2v}\sin(\omega t)) + (c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) \\
&= (c_{1v} + c_{1u})\cos(\omega t) + (c_{2v} + c_{2u})\sin(\omega t) \\
&= (c_{1u} + c_{1v})\cos(\omega t) + (c_{2u} + c_{2v})\sin(\omega t) = \mathbf{u} + \mathbf{v}
\end{aligned}$$

3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ((c_{1v}\cos(\omega t) + c_{2v}\sin(\omega t)) + (c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t))) + \\
&\quad (c_{1w}\cos(\omega t) + c_{2w}\sin(\omega t)) \\
&= (c_{1u} + c_{1v})\cos(\omega t) + (c_{2u} + c_{2v})\sin(\omega t) + (c_{1w}\cos(\omega t) + c_{2w}\sin(\omega t)) \\
&= (c_{1u} + c_{1v} + c_{1w})\cos(\omega t) + (c_{2u} + c_{2v} + c_{2w})\sin(\omega t) \\
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) + \\
&\quad ((c_{1v}\cos(\omega t) + c_{2v}\sin(\omega t))) + (c_{1w}\cos(\omega t) + c_{2w}\sin(\omega t)) \\
&= (c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) + (c_{1v} + c_{1w})\cos(\omega t) + (c_{2v} + c_{2w})\sin(\omega t) \\
&= (c_{1u} + c_{1v} + c_{1w})\cos(\omega t) + (c_{2u} + c_{2v} + c_{2w})\sin(\omega t) \\
&= (\mathbf{u} + \mathbf{v}) + \mathbf{w}
\end{aligned}$$

4. $\exists \mathbf{0} \in V | \mathbf{u} + \mathbf{0} = \mathbf{u}$
The addition neutral element is $\mathbf{0} = 0\cos(\omega t) + 0\sin(\omega t)$. Let's see why

$$\begin{aligned}
\mathbf{u} + \mathbf{0} &= (c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) + (0\cos(\omega t) + 0\sin(\omega t)) \\
&= (c_{1u} + 0)\cos(\omega t) + (c_{2u} + 0)\sin(\omega t) \\
&= c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t) = \mathbf{u}
\end{aligned}$$

5. $\forall \mathbf{u} \in V \quad \exists! \mathbf{w} \in V | \mathbf{u} + \mathbf{w} = \mathbf{0}$ (we normally write $\mathbf{w} = -\mathbf{u}$)
If $\mathbf{u} = c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)$, its inverse with respect to addition is $-\mathbf{u} = -c_{1u}\cos(\omega t) - c_{2u}\sin(\omega t)$.

$$\begin{aligned}
\mathbf{u} + (-\mathbf{u}) &= (c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) + (-c_{1u}\cos(\omega t) - c_{2u}\sin(\omega t)) \\
&= (c_{1u} + (-c_{1u}))\cos(\omega t) + (c_{2u} + (-c_{2u}))\sin(\omega t) \\
&= 0\cos(\omega t) + 0\sin(\omega t) = \mathbf{0}
\end{aligned}$$

6. $c\mathbf{v} \in V$

$$\begin{aligned}
c\mathbf{u} &= c(c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) \\
&= (cc_{1u})\cos(\omega t) + (cc_{2u})\sin(\omega t) \in V
\end{aligned}$$

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

$$\begin{aligned}
c(\mathbf{u} + \mathbf{v}) &= c((c_{1u} + c_{1v})\cos(\omega t) + (c_{2u} + c_{2v})\sin(\omega t)) \\
&= (cc_{1u} + cc_{1v})\cos(\omega t) + (cc_{2u} + cc_{2v})\sin(\omega t) \\
c\mathbf{u} + c\mathbf{v} &= (cc_{1u}\cos(\omega t) + cc_{2u}\sin(\omega t)) + (cc_{1v}\cos(\omega t) + cc_{2v}\sin(\omega t)) \\
&= (cc_{1u} + cc_{1v})\cos(\omega t) + (cc_{2u} + cc_{2v})\sin(\omega t) = c(\mathbf{u} + \mathbf{v})
\end{aligned}$$

8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

$$\begin{aligned}
(c + d)\mathbf{u} &= (c + d)(c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) \\
&= (c + d)c_{1u}\cos(\omega t) + (c + d)c_{2u}\sin(\omega t) \\
&= (cc_{1u} + dc_{1u})\cos(\omega t) + (cc_{2u} + dc_{2u})\sin(\omega t) \\
&= cc_{1u}\cos(\omega t) + dc_{1u}\cos(\omega t) + cc_{2u}\sin(\omega t) + dc_{2u}\sin(\omega t) \\
&= (cc_{1u}\cos(\omega t) + cc_{2u}\sin(\omega t)) + (dc_{1u}\cos(\omega t) + dc_{2u}\sin(\omega t)) \\
&= c(c_{1u}\cos(\omega t) + cc_{2u}\sin(\omega t)) + d(c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) \\
&= c\mathbf{u} + d\mathbf{u}
\end{aligned}$$

9. $c(d\mathbf{u}) = (cd)\mathbf{u}$

$$\begin{aligned}
c(d\mathbf{u}) &= c(dc_{1u}\cos(\omega t) + dc_{2u}\sin(\omega t)) \\
&= cdc_{1u}\cos(\omega t) + cdc_{2u}\sin(\omega t) \\
&= cd(c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) \\
&= (cd)\mathbf{u}
\end{aligned}$$

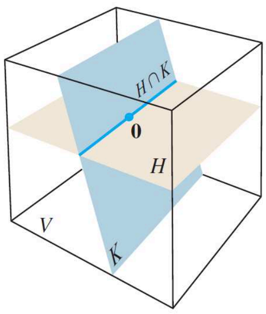
10. $1\mathbf{u} = \mathbf{u}$

$$\begin{aligned}
1\mathbf{u} &= 1(c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t)) \\
&= (1 \cdot c_{1u})\cos(\omega t) + (1 \cdot c_{2u})\sin(\omega t) \\
&= c_{1u}\cos(\omega t) + c_{2u}\sin(\omega t) \\
&= \mathbf{u}
\end{aligned}$$

Lay, 4.1.32

Carlos Oscar Sorzano, Aug. 31st, 2013

Let H and K be subspaces over a vector space V. The intersection of H and K, written as $H \cap K$, is the set of all vectors $\mathbf{v} \in V$ that belong to both H and K. Show that $H \cap K$ is a subspace of V. (See figure below.) Give an example in $\mathbb{R}^2$ to show that the union of subspaces is not, in general, a subspace.



Solution: We need to show that this $H \cap K$ meets the three requirements to be a subspace

- $\mathbf{0} \in H \cap K$
  This is true because for $\mathbf{0}$ belongs to both H and K since both of them are, in their turn, subspaces.

- Given any two vectors $\mathbf{u}, \mathbf{v} \in H \cap K$, $\mathbf{u} + \mathbf{v} \in H \cap K$
  $\mathbf{u}$ and $\mathbf{v}$ belong to both H and K. And these sets are subspaces, then

$$\mathbf{u} + \mathbf{v} \in H$$
$$\mathbf{u} + \mathbf{v} \in K$$

  So $\mathbf{u} + \mathbf{v} \in H \cap K$

- Given any vector $\mathbf{u} \in H \cap K$ and $c \in \mathbb{R}$, $c\mathbf{u} \in H \cap K$
  $\mathbf{u}$ belongs to both H and K. And these sets are subspaces, then

$$c\mathbf{u} \in H$$
$$c\mathbf{u} \in K$$

  So $c\mathbf{u} \in H \cap K$

Since $H \cap K$ meets all properties, $H \cap K$ is a subspace of V.
The union of subspaces is not, in general, a subspace. For instance in $\mathbb{R}^2$, the following sets are subspaces:

$$H = \{(x, 0) \in \mathbb{R}^2\}$$
$$K = \{(0, y) \in \mathbb{R}^2\}$$

but the union $H \cup K$ is not a subspace. For instance, $\mathbf{u} = (1, 0) \in H \cup K$ and $\mathbf{v} = (0, 1) \in H \cup K$, but $\mathbf{u} + \mathbf{v} = (1, 1) \notin H \cup K$.

Lay, 4.1.33 (3rd ed.)

María Postigo Fliquete, Dec. 7th, 2014

Given subspaces H and K of a vector space V, the sum of H and K, written as $H + K$, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K ; that is, $H + K = \{\mathbf{w} | \mathbf{w} = \mathbf{u} + \mathbf{v}\}$ for some $\mathbf{u}$ in H and some $\mathbf{v}$ in K

a. Show that $H + K$ is a subspace of V.

b. Show that H is a subspace of $H + K$ and K is a subspace of $H + K$.

Solution:

a. $H + K$ is a subspace of V because:

- $\mathbf{0} \in H + K$: since $\mathbf{0} \in H, K \Rightarrow \mathbf{0} + \mathbf{0} = \mathbf{0} \in H + K$.
- Let two vectors in $H + K$ be $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2$, with $\mathbf{u}_1, \mathbf{u}_2 \in H$ and $\mathbf{v}_1, \mathbf{v}_2 \in K$. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$$

  since $\mathbf{u}_1 + \mathbf{u}_2 \in H$ and $\mathbf{v}_1 + \mathbf{v}_2 \in K$ (because H and K are subspaces of V), then $\mathbf{w}_1 + \mathbf{w}_2 \in H + K$.
- Let $\mathbf{w} = \mathbf{u} + \mathbf{v}$ be a vector in $H + K$ with $\mathbf{u} \in H$ and $\mathbf{v} \in K$. Then

$$c\mathbf{w} = c(\mathbf{u} + \mathbf{v}) = (c\mathbf{u}) + (c\mathbf{v})$$

  since $c\mathbf{u} \in H$ and $c\mathbf{v} \in K$ (because they are subspaces of V), then $c\mathbf{w} \in H + K$.

b. H is a subspace of $H + K$ because any vector of $\mathbf{u} \in H$ can be written as

$$\mathbf{w} = \mathbf{u} + \mathbf{0} \in H + K$$

consequently H is a subset of $H + K$. Since H is a subspace of V, H is also a subspace of $H + K$. Similarly, any vector $\mathbf{v} \in K$ can be written as

$$\mathbf{w} = \mathbf{0} + \mathbf{v} \in H + K$$

and K is a subset of $H + K$ and since K is a subspace of V, then it is also a subspace of $H + K$.

Lay, 4.2.1 (3rd ed.)

María Postigo Fliquete, Dec. 7th, 2014

Determine if $\mathbf{w}=\begin{pmatrix}1\\3\\-4\end{pmatrix}$ is in Nul$\{A\}$, with $A=\begin{pmatrix}3&-5&-3\\6&-2&0\\-8&4&1\end{pmatrix}$.

Solution:

$$\begin{pmatrix}3&-5&-3\\6&-2&0\\-8&4&1\end{pmatrix}\begin{pmatrix}1\\3\\-4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

So $\mathbf{w}$ is in Nul$\{A\}$.

Lay, 4.2.2 (3rd ed.)

María Postigo Fliquete, Dec. 7th, 2014

Determine if $\mathbf{w}=\begin{pmatrix}5\\-3\\2\end{pmatrix}$ is in Nul$\{A\}$, with $A=\begin{pmatrix}5&21&19\\13&23&2\\8&14&1\end{pmatrix}$.

Solution:

$$\begin{pmatrix}5&21&19\\13&23&2\\8&14&1\end{pmatrix}\begin{pmatrix}5\\-3\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

So $\mathbf{w}$ is in Nul$\{A\}$.

Lay, 4.2.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Find an explicit description of the null space of A by listing the vectors that span it.

$$A=\begin{pmatrix}1&2&4&0\\0&1&3&-2\end{pmatrix}$$

Solution: The null space of A is defined as those vectors such that

$$A\mathbf{x}=\mathbf{0}$$

If we construct the augmented matrix of this equation system we get

$$\left(\begin{array}{cccc|c}1&2&4&0&0\\0&1&3&-2&0\end{array}\right)\sim\left(\begin{array}{cccc|c}1&0&-2&4&0\\0&1&3&-2&0\end{array}\right)$$

So all points satisfying $A\mathbf{x}=\mathbf{0}$ are of the form

$$\left.\begin{array}{r}x_1=2x_3-4x_4\\x_2=-3x_3+2x_4\end{array}\right\}\Rightarrow\mathbf{x}=x_3(2,-3,1,0)+x_4(-4,2,0,1)$$

So a basis of Nul$\{A\}$ is given by

$$\text{Basis}\{\text{Nul}\{A\}\}=\{(2,-3,1,0),(-4,2,0,1)\}$$

Lay, 4.2.4

Ignacio Sanchez Lopez, Jan. 15th, 2015

Find an explicit description of the null space of A by listing the vectors that span it.

$$A = \begin{pmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Solution: The null space of A is defined as those vectors such that

$$A\mathbf{x} = \mathbf{0}$$

If we construct the augmented matrix of this equation system we get

$$\left(\begin{array}{cccc|c} 1 & -6 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

So all points satisfying $A\mathbf{x} = \mathbf{0}$ are of the form

$$\left. x_1 = 6x_2 \ \right\} \Rightarrow \mathbf{x} = \begin{pmatrix} 6x_2 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So a basis of $\text{Nul}\{A\}$ is given by

$$\text{Basis}\{\text{Nul}\{A\}\} = \{(6,1,0,0),(0,0,1,0),(0,0,0,1)\}$$

Lay, 4.2.5 (3rd ed.)

María Postigo Fliquete, Dec. 7th, 2014

Find an explicit description of $\text{Nul}\{A\}$, by listing vectors that span the null space. $A = \begin{pmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Solution: The Null space is composed by the set of solutions of the system $A\mathbf{x} = \mathbf{0}$

$$\begin{pmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + 4x_4 \\ x_3 - 9x_4 \\ x_5 \end{pmatrix}$$

$$\begin{pmatrix} x_1 - 2x_2 + 4x_4 \\ x_3 - 9x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

That is

$$x_1 = -4x_4 + 2x_2$$

$$x_3 = 9x_4$$

$$x_5 = 0$$

or equivalently

$$\text{Nul}\{A\} \ni \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 - 4x_4 \\ x_2 \\ 9x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{pmatrix}$$

So the vectors are $\begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}-4\\0\\9\\1\\0\end{pmatrix}$ constitute a basis of the Null space of A.

Lay, 4.2.6

Ignacio Sanchez Lopez, Jan. 15th, 2015

Find an explicit description of the null space of A by listing the vectors that span it.

$$A = \begin{pmatrix}1 & 5 & -4 & -3 & 1\\0 & 1 & -2 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{pmatrix}$$

Solution: The null space of A is defined as those vectors such that

$$A\mathbf{x} = \mathbf{0}$$

If we construct the augmented matrix of this equation system we get

$$\left(\begin{array}{ccccc|c}1 & 5 & -4 & -3 & 1 & 0\\0 & 1 & -2 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0\end{array}\right) \sim \left(\begin{array}{ccccc|c}1 & 0 & 6 & -8 & 1 & 0\\0 & 1 & -2 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0\end{array}\right)$$

So all points satisfying $A\mathbf{x} = \mathbf{0}$ are of the form

$$\left.\begin{array}{c}x_1 = -6x_3 + 8x_4 - x_5\\x_2 = 2x_3 - x_4\end{array}\right\} \Rightarrow \mathbf{x} = x_3\begin{pmatrix}-6\\2\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}8\\-1\\0\\1\\0\end{pmatrix} + x_5\begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix}$$

So a basis of Nul$\{A\}$ is given by

$$\text{Basis}\{\text{Nul}\{A\}\} = \{(-6, 2, 1, 0, 0), (8, -1, 0, 1, 0)(-1, 0, 0, 0, 1)\}$$

Lay, 4.2.9

Carlos Oscar Sorzano, Aug. 31st, 2013

For the set below, either find an appropriate theorem to show that W is a vector space or find a specific example to show the contrary.

$$W = \left\{\begin{pmatrix}p\\q\\r\\s\end{pmatrix} | p - 3q = 4s, 2p = s + 5r\right\}$$

Solution: We can rewrite the two conditions for the vectors in W as

$$\begin{pmatrix}1 & -3 & 0 & -4\\2 & 0 & -5 & -1\end{pmatrix}\begin{pmatrix}p\\q\\r\\s\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

So, W is nothing more than the null space of the matrix $A = \begin{pmatrix} 1 & -3 & 0 & -4 \\ 2 & 0 & -5 & -1 \end{pmatrix}$ and consequently it is a vector subspace of $\mathbb{R}^4$. Since any vector subspace is a vector space, then W is a vector space.

Lay, 4.2.10

Ana Peña Gil, Jan. 19th 2014

For the set below, either find an appropriate theorem to show that W is a vector space or find a specific example to show the contrary.

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} | 3a + b = c, a + b + 2c = 2d \right\}$$

Solution: We can rewrite the two conditions for the vectors in W as

$$\begin{pmatrix} 3 & 1 & -1 & 0 \\ 1 & 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, W is nothing more than the null space of the matrix $A = \begin{pmatrix} 3 & 1 & -1 & 0 \\ 1 & 1 & 2 & -2 \end{pmatrix}$ and consequently it is a vector subspace of $\mathbb{R}^4$. Since any vector subspace is a vector space, then W is a vector space.

Lay, 4.2.11

Carlos Oscar Sorzano, Aug. 31st, 2013

For the set below, either find an appropriate theorem to show that W is a vector space or find a specific example to show the contrary.

$$W = \left\{ \begin{pmatrix} s - 2t \\ 3 + 3s \\ 3s + t \\ 2s \end{pmatrix} | \forall s, t \in \mathbb{R} \right\}$$

Solution: W is not a subspace because $\mathbb{R}^4 \ni \mathbf{0} \notin W$. To show why, consider the vector equation

$$\begin{pmatrix} s - 2t \\ 3 + 3s \\ 3s + t \\ 2s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The equation for the last component implies

$$2s = 0 \Rightarrow s = 0$$

But for the second component

$$3 + 3s = 0 \Rightarrow s = -1$$

Since s cannot take the values 0 and -1 at the same time, we conclude that there are no values of s and t such that $\mathbf{0} \in W$, and consequently, the set W cannot be a vector space.

Lay, 4.2.13 (3rd ed.)

María Postigo Fliquete, Dec. 8th, 2014

Use an appropriate theorem to show that the given set, W , is a vector space, or find a specific example to the contrary.

$$W = \left\{ \begin{pmatrix} c - 6d \\ d \\ c \end{pmatrix} \forall c, d \in \mathbb{R} \right\}$$

Solution:

$$\begin{pmatrix} c - 6d \\ d \\ c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + d \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}$$

That is $W = \text{Col}\{A\}$ for $A = \begin{pmatrix} 1 & -6 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, so W is a vector space.

Lay, 4.2.26

Carlos Oscar Sorzano, Dec. 16th, 2014

Let A be a $m \times n$ matrix. Mark each statement True or False. Justify each answer.

1. The null space of A is a vector space.
2. The column space of A is in $\mathbb{R}^m$.
3. $\text{Col}\{A\}$ is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.
4. $\text{Nul}\{A\}$ is the kernel of the mapping $\mathbf{x} \to A\mathbf{x}$.
5. The range of a linear transformation is a vector space.
6. The set of all solutions of a homogeneous linear differential equation is the kernel of a linear transformation.

Solution:

1. True, the null space of a matrix is a vector subspace (see Theorem 4.2.2) and any vector subspace is a vector space.
2. True, the column space of A is the vector subspace of $\mathbb{R}^m$ formed by all the vectors that can be obtained as linear combinations of the columns of A (which are of size m).
3. False, see previous answer for the definition of $\text{Col}\{A\}$.
4. True, the kernel of the proposed transformation are all those vectors in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$, but this is the defintion of $\text{Nul}\{A\}$.

5. True. To show that the range is a vector space, let us first prove that it is a vector subspace. Let us denote as T to the linear transformation.

   - $\mathbf{0} \in \text{Range}\{T\}$. We know that for any linear transformation $T(\mathbf{0}) = \mathbf{0}$.
   - $\forall \mathbf{y}_1, \mathbf{y}_2 \in \text{Range}\{T\} \Rightarrow \mathbf{y}_1 + \mathbf{y}_2 \in \text{Range}\{T\}$. Let us denote as $\mathbf{x}_1$ and $\mathbf{x}_2$ two vectors in the input space such that $T(\mathbf{x}_1) = \mathbf{y}_1$ and $T(\mathbf{x}_2) = \mathbf{y}_2$. Since the transformation is linear we know that $T(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{y}_1 + \mathbf{y}_2$. Consequently, $\mathbf{y}_1 + \mathbf{y}_2 \in \text{Range}\{T\}$.
   - $\forall \mathbf{y} \in \text{Range}\{T\}, \forall c \in \mathbb{R} \Rightarrow c\mathbf{y} \in \text{Range}\{T\}$. Let us denote as $\mathbf{x}$ a vector such that $T(\mathbf{x}) = \mathbf{y}$. Since T is linear we have $T(c\mathbf{x}) = c\mathbf{y}$ and, consequently, $c\mathbf{y} \in \text{Range}\{T\}$.

   So, $\text{Range}\{T\}$ is a vector subspace of the output vector space, but any vector subspace is a vector space.

6. True, a homogeneous linear differential equation is one of the form

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + ... + f_1(x)y' + f_0(x)y = 0$$

   Let us define the transformation

$$T(y) = f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + ... + f_1(x)y' + f_0(x)y$$

   Let us show that T is a linear transformation

$$\begin{aligned}
T(y_1 + y_2) &= f_n(x)(y_1 + y_2)^{(n)} + f_{n-1}(x)(y_1 + y_2)^{(n-1)} + ... + f_1(x)(y_1 + y_2)' + f_0(x)(y_1 + y_2) \\
&= f_n(x)(y_1^{(n)} + y_2^{(n)}) + f_{n-1}(x)(y_1^{(n-1)} + y_2^{(n-1)}) + ... + f_1(x)(y_1' + y_2') + f_0(x)(y_1 + y_2) \\
&= \left[f_n(x)y_1^{(n)} + f_{n-1}(x)y_1^{(n-1)} + ... + f_1(x)y_1' + f_0(x)y_1\right] + \\
&\quad \left[f_n(x)y_2^{(n)} + f_{n-1}(x)y_2^{(n-1)} + ... + f_1(x)y_2' + f_0(x)y_2\right] \\
&= T(y_1) + T(y_2) \\
T(cy) &= f_n(x)(cy)^{(n)} + f_{n-1}(x)(cy)^{(n-1)} + ... + f_1(x)(cy)' + f_0(x)(cy) \\
&= f_n(x)cy^{(n)} + f_{n-1}(x)cy^{(n-1)} + ... + f_1(x)cy' + f_0(x)cy \\
&= c(f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + ... + f_1(x)y' + f_0(x)y) \\
&= cT(y)
\end{aligned}$$

   The kernel of this linear transformation is

$$\text{Ker}\{T\} = \{y | T(y) = 0\}$$

   that is, the set of all solutions of the homogeneous linear differential equation

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + ... + f_1(x)y' + f_0(x)y = 0$$

Lay, 4.2.30

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. [Hint: typical elements of the range have the form $T(\mathbf{u})$ and $T(\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in V$.]
Solution: We need to show that the range of T meets the three requirements to be a subspace

- $\mathbf{0}_W \in \text{Range}\{T\}$
  We know that for any linear transformation $T(\mathbf{0}_V) = \mathbf{0}_W$, so $\mathbf{0}_W$ is in the range of T.

- Given any two vectors $T(\mathbf{u}), T(\mathbf{v}) \in \text{Range}\{T\}$, $\in T(\mathbf{u}) + T(\mathbf{v})\text{Range}\{T\}$
  Since T is a linear transformation

  $$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

  So $T(\mathbf{u}) + T(\mathbf{v})$ is also in the range of T.

- Given any vector $T(\mathbf{u}) \in \text{Range}\{T\}$ and $c \in \mathbb{R}$, $cT(\mathbf{u}) \in \text{Range}\{T\}$
  Again, exploiting the fact that T is linear

  $$T(c\mathbf{u}) = cT(\mathbf{u})$$

  So $cT(\mathbf{u})$ is in the range of T

Since Range$\{T\}$ meets all properties, Range$\{T\}$ is a subspace of W.

Lay, 4.2.31

Carlos Oscar Sorzano, Aug. 31st, 2013

Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(p(t)) = (p(0), p(1))$. For instance, if $p(t) = 3 + 5t + 7t^2$, then $T(p(t)) = (3, 15)$.

a. Show that T is a linear transformation. [Hint: for arbitrary polynomials $p(t)$ and $q(t)$ in $\mathbb{P}_2$, compute $T(p(t) + q(t))$ and $T(cp(t))$].

b. Find a polynomial $p(t)$ in $\mathbb{P}_2$ that spans the kernel of T, and describe the range of T.

Solution:

a. Let $p(t) = a_p + b_pt + c_pt^2$ and $q(t) = a_q + b_qt + c_qt^2$. We have

$$\begin{array}{rcl}
T(p(t)+q(t)) & = & T((a_p + b_pt + c_pt^2) + (a_q + b_qt + c_qt^2)) \\
 & = & T((a_p + a_q) + (b_p + b_q)t + (c_p + c_q)t^2) \\
 & = & (a_p + a_q, a_p + a_q + b_p + b_q + c_p + c_q) \\
 & = & (a_p, a_p + b_p + c_p) + (a_q, a_q + b_q + c_q) \\
 & = & T(p(t)) + T(q(t))
\end{array}$$

$$\begin{array}{rcl}
T(cp(t)) & = & T(c(a_p + b_pt + c_pt^2)) \\
 & = & T(ca_p + cb_pt + cc_pt^2) \\
 & = & (ca_p, ca_p + cb_p + cc_p) \\
 & = & c(a_p, a_p + b_p + c_p) \\
 & = & cT(p(t))
\end{array}$$

Since T meets the two conditions to be a linear transformation, it is a linear transformation.

b. The kernel of T is formed by all those polynomials such that

$$T(p(t)) \quad = \quad (a_p, a_p + b_p + c_p) = (0, 0)$$

for that we need

$$a_p = 0$$
$$a_p + b_p + c_p = 0 \Rightarrow c_p = -b_p$$

That is, all polynomials in the kernel of T are of the form $p(t) = b_p t - b_p t^2$, in particular

$$\text{Ker}\{T\} = \text{Span}\{t - t^2\}$$

The range of T is formed by all those vectors of the form

$$\text{Range}\{T\} = \{(a_p, a_p + b_p + c_p) \forall a_p, b_p, c_p \in \mathbb{R}\} = \mathbb{R}^2$$

Lay, 4.2.33 (3rd ed.)

María Postigo Fliquete, Dec. 8th, 2014

Let $M_{2\times 2}$ be the vector space of all 2x2 matrices, and define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

a. Show that T is a linear transformation.

b. Let B be any element of $M_{2\times 2}$ such that $B^T{=}B$. Find an A in $M_{2\times 2}$ such that $T(A) = B$.

c. Show that the range of T is the set of B in $M_{2\times 2}$ with the property that $B^T = B$.

d. Describe the kernel of T.

Solution:

a. To show that a transformation is linear we need to show that

$$\begin{aligned} T(A+B) &= (A+B) + (A+B)^T = A + B + A^T + B^T \\ &= (A + A^T) + (B + B^T) = T(A) + T(B) \\ T(cA) &= (cA) + (cA)^T = cA + cA^T = c(A + A^T) = cT(A) \end{aligned}$$

b. Let $A = \frac{1}{2}B$, then

$$T(A) = \left(\frac{1}{2}B\right) + \left(\frac{1}{2}B\right)^T = \frac{1}{2}B + \frac{1}{2}B^T = \frac{1}{2}B + \frac{1}{2}B = B$$

c. Consider any matrix B such that $B = T(A)$. Then

$$B^T = (T(A))^T = (A + A^T)^T = A^T + A = T(A) = B$$

d. The kernel of T is composed by all matrices such that

$$T(A) = 0 = A + A^T$$

This requires that $a_{ii} = 0$ and $a_{ij} = -a_{ji}$. Any matrix fulfilling this condition is in the kernel of T.

Lay, 4.3.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine whether the set $B = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis or not for $\mathbb{R}^3$. If it is not, determine if it is linearly independent.

Solution: B has three linearly independent vectors because if we form the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ the unique solution of the equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Since B has 3 linearly independent vectors, it spans $\mathbb{R}^3$ and it is, therefore, a basis for $\mathbb{R}^3$.

Lay, 4.3.2

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine whether the set $B = \{(1,1,0),(0,0,0),(0,1,1)\}$ is a basis or not for $\mathbb{R}^3$. If it is not, determine if it is linearly independent.

Solution: The set B is not linearly independent because it contains the vector $(0,0,0)$. So it cannot be a basis for $\mathbb{R}^3$.

Lay, 4.3.3

Ana Peña Gil, Jan. 19th 2014

Determine whether the set $B = \{(1,0,-3),(3,1,-4),(-2,-1,1)\}$ is a basis or not for $\mathbb{R}^3$. If it is not, determine if it is linearly independent.

Solution: B has three linearly dependent vectors because if we form the matrix

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & -4 & 1 \end{pmatrix}$$

its determinant $\det\{A\} = 0$, which means that A is formed by linearly dependent vectors.

Since B has 3 linearly dependent vectors, it does not span $\mathbb{R}^3$ and it is not, therefore, a basis for $\mathbb{R}^3$.

Lay, 4.3.6

Ana Peña Gil, Jan. 19th 2014

Determine whether the set $B = \{(1,2,-4),(-4,3,6)\}$ is a basis or not for $\mathbb{R}^3$. If it is not, determine if it is linearly independent.

Solution: B does not span $\mathbb{R}^3$ because it has only two vectors. It needs exactly three independent vectors to span $\mathbb{R}^3$. And therefore, it cannot be a basis for $\mathbb{R}^3$.

The two vectors of the set B are linearly independent because none of them is a multiple of the other.

Lay, 4.3.7

Andrea Santos Cortés, Nov. 16th, 2014

Determine whether the set $B = \{(-2,3,0),(6,-1,5)\}$ is a basis or not for $\mathbb{R}^3$. If it is not, determine if it is linearly independent.
Solution: B does not span $\mathbb{R}^3$ because it has only two vectors. It needs exactly three linearly independent vectors to span $\mathbb{R}^3$. And therefore, it cannot be a basis for $\mathbb{R}^3$. The two vectors of the set B are linearly independent because none of them is a multiple of the other

Lay, 4.3.8

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine whether the set $B = \{(1,-2,3),(0,3,-1),(2,-1,5),(0,0,-1)\}$ is a basis or not for $\mathbb{R}^3$. If it is not, determine if it is linearly independent.
Solution: The set B cannot be linearly independent because in $\mathbb{R}^3$ there can be at most 3 linearly independent vectors.

Lay, 4.3.9

María Postigo Fliquete, Dec. 8th, 2014

Find bases for the null space of the matrix $\begin{pmatrix}1 & 0 & -3 & 2\\ 0 & 1 & -5 & 4\\ 3 & -2 & 1 & -2\end{pmatrix}$

Solution:

$$\begin{pmatrix}1 & 0 & -3 & 2\\ 0 & 1 & -5 & 4\\ 3 & -2 & 1 & -2\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -3 & 2\\ 0 & 1 & -5 & 4\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & -3 & 2\\ 0 & 1 & -5 & 4\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}x_1-3x_3+2x_4\\ x_2-5x_3+4x_4\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$x_1 = 3x_3 - 2x_4$$

$$x_2 = 5x_3 - 4x_4$$

Any vector in the null space is of the form

$$\begin{pmatrix}3x_3-2x_4\\ 5x_3-4x_4\\ x_3\\ x_4\end{pmatrix} = x_3\begin{pmatrix}3\\5\\1\\0\end{pmatrix} + x_4\begin{pmatrix}-2\\-4\\0\\1\end{pmatrix}$$

So a basis for Nul$\{A\}$ is formed by the vectors $\begin{pmatrix}3\\5\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\-4\\0\\1\end{pmatrix}$.

Lay, 4.3.12

Carlos Oscar Sorzano, Aug. 31st, 2013

Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = -3x$.
Solution: All these vectors are of the form

$$\mathbf{r} = \begin{pmatrix}x\\ -3x\end{pmatrix} = x\begin{pmatrix}1\\ -3\end{pmatrix}$$

So, a basis for these vectors is

$$B = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}$$

Lay, 4.3.13 (3rd ed.)

María Postigo Fliquete, Dec. 8th, 2014

Assume that $A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}$ is row equivalent to $B = \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
Find bases for Nul$\{A\}$ is and Col$\{A\}$.
Solution:

$$\begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 6x_3 + 5x_4 \\ 2x_2 + 5x_3 + 3x_4 \\ 0 \end{pmatrix}$$

That is

$$x_1 = -6x_3 - 5x_4$$
$$x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4$$

So any vector in Nul$\{A\}$ is of the form

$$\begin{pmatrix} -6x_3 - 5x_4 \\ -\frac{5}{2}x_3 - \frac{3}{2}x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$$

A basis for Nul$\{A\}$ is formed by the set

$$\left\{ \begin{pmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix} \right\}$$

The basis for Col$\{A\}$ are the colums of A corresponding to the pivot columns (columns 1 and 2) of B:

$$\left\{ \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ 8 \end{pmatrix} \right\}$$

Lay, 4.3.14

Ana Sanmartín, Dec. 15th, 2014

Assume that A is row equivalent to B. Find bases for Nul$\{A\}$ and Col$\{A\}$.

$$A = \begin{pmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{pmatrix} \sim B = \begin{pmatrix} 1 & 2 & 0 & -2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: If B is the reduced echelon form of A, then, the pivot columns of B form a linearly independent set. Because the dependence relationships among columns are not affected by row operations, the corresponding pivot columns of A are also linearly independent and, consequently a basis of $\text{Col}\{A\}$:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \\ 9 \\ 9 \end{pmatrix} \right\}$$

For the basis of $\text{Nul}\{A\}$ we make use of the equations implied by $B\mathbf{x} = \mathbf{0}$:

$$\begin{array}{rcr} x_1 + 2x_2 - 2x_4 + 5x_5 = 0 & & x_1 = -2x_2 + 2x_4 \\ 3x_3 - 6x_4 + 3x_5 = 0 & \Rightarrow & x_3 = 2x_4 \\ -7x_5 = 0 & & x_5 = 0 \end{array}$$

$$\text{Nul}\{A\} \ni \mathbf{x} = \begin{pmatrix} -2x_2 + 2x_4 \\ x_2 \\ 2x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

So a basis for $\text{Nul}\{A\}$ is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Lay, 4.3.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be a linearly independent set of $\mathbf{R}^n$. Explain why B must be basis for $\mathbb{R}^n$.

Solution: To be a basis for $\mathbb{R}^n$ a set needs to be linearly independent (B is so by hypothesis) and span $\mathbb{R}^n$. Let's check this latter requirement. Let us form the matrix $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & ... & \mathbf{v}_n \end{pmatrix}$. B spans $\mathbb{R}^n$ if for any vector $\mathbf{b} \in \mathbb{R}^n$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution. Since the columns of A are linearly independent, by the invertible theorem matrix (Theorem 5.1, Chapter 3), we know that the matrix equation above has a solution, and consequently, the columns of A (that is B) spans $\mathbb{R}^n$.

Lay, 4.3.25

Carlos Oscar Sorzano, Jan. 20th, 2013

Let $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (0, 1, 1)$ and $\mathbf{v}_3 = (0, 1, 0)$, and let H be the set of vectors of $\mathbb{R}^3$ whose second and third entries are equal. Then every vector in H has a unique expansion as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ because $(s, t, t) = s(1, 0, 1) + (t - s)(0, 1, 1) + s(0, 1, 0)$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of H? Why or why not?

Solution: $\mathbf{v}_1$ and $\mathbf{v}_3$ do not belong to H (because they do not have the same values in the second and third position), so they cannot participate of any basis of H.

Lay, 4.3.31

Carlos Oscar Sorzano, Aug. 31st, 2013

Let V and W be vector spaces, and $T : V \to W$ a linear transformation between the two. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ be a subset of V. Show that if S is linearly dependent in V, then the set of images $T(S) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_p)\}$ is linearly dependent in W. This fact shows that if a linear transformation maps S onto a linearly independent set of vectors, then S is also linearly independent (because it cannot be linearly dependent).
Solution: If S is linearly dependent, then there exist coefficients c_1, c_2, ..., c_p not all of them zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = \mathbf{0}_V$$

Apply the lienar transformation T to both sides yields

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + ... + c_pT(\mathbf{v}_p) = \mathbf{0}_W$$

that is, there exist coefficients c_1, c_2, ..., c_p not all of them zero such that the linear combination of the transformed vectors is $\mathbf{0}_W$. This means that $S' = T(S)$ is a set of linearly dependent vectors.

Lay, 4.3.32

Carlos Oscar Sorzano, Aug. 31st, 2013

Let V and W be vector spaces, and $T : V \to W$ a linear transformation between the two. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ be a subset of V. Suppose T is a one-to-one transformation, so that an equation $T(\mathbf{u}) = T(\mathbf{v})$ always implies $\mathbf{u} = \mathbf{v}$. Show that if the set of images $T(S) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_p)\}$ is linearly dependent, then $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ is also linearly dependent. This fact shows that a one-to-one linear transformation maps a linearly independent set onto a linearly independent set (because in this case the set of images cannot be linearly dependent).
Solution: If $T(S)$ is linearly dependent, then there exist coefficients c_1, c_2, ..., c_p not all of them zero such that

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + ... + c_pT(\mathbf{v}_p) = \mathbf{0}_W$$

If T is one-to-one, we infer that it must also be

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = \mathbf{0}_V$$

that is, there exist coefficients c_1, c_2, ..., c_p not all of them zero such that the linear combination of the transformed vectors is $\mathbf{0}_V$. This means that S is a set of linearly dependent vectors.

Lay, 4.3.33

Carlos Oscar Sorzano, Aug. 31st, 2013

Consider the polynomials $p_1(t) = 1 + t^2$ and $p_2(t) = 1 - t^2$. Is $\{p_1(t), p_2(t)\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?
Solution: We may define the linear transformation $T : \mathbb{P}_3 \to \mathbb{R}^4$ such that $T(a + bt + ct^2 + dt^3) = (a, b, c, d)$. It can be easily verified that T is a linear transformation.

The polynomials $p_1(t)$ and $p_2(t)$ are transformed to

$$T(p_1(t)) = (1,0,1,0)$$
$$T(p_2(t)) = (1,0,-1,0)$$

which is clearly a linear independent set in $\mathbb{R}^4$ and by Exercise Lay 4.3.31, this implies that $\{p_1(t), p_2(t)\}$ is a linearly independent set in $\mathbb{P}_3$.

Lay, 4.4.1

Andrea Santos Cortés, Nov. 17th, 2014

Given the coordinate $[\mathbf{x}]_B = (5,3)$ and the basis $B = \{(3,-5),(-4,6)\}$, find the vector $\mathbf{x}$.
Solution: The coordinates of $\mathbf{x}$ in the basis B specify the linear combination of the vectors in the basis B to find $\mathbf{x}$

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = 5\begin{pmatrix}3\\-5\end{pmatrix} + 3\begin{pmatrix}-4\\6\end{pmatrix} = \begin{pmatrix}3\\-7\end{pmatrix}$$

Lay, 4.4.2

Andrea Santos Cortés, Nov. 17th, 2014

Given the coordinate $[\mathbf{x}]_B = (-2,5)$ and the basis $B = \{(3,2),(-4,1)\}$, find the vector $\mathbf{x}$.
Solution: The coordinates of $\mathbf{x}$ in the basis B specify the linear combination of the vectors in the basis B to find $\mathbf{x}$

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = -2\begin{pmatrix}3\\2\end{pmatrix} + 5\begin{pmatrix}-4\\1\end{pmatrix} = \begin{pmatrix}-26\\1\end{pmatrix}$$

Lay, 4.4.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Given the coordinate $[\mathbf{x}]_B = (1,0,-2)$ and the basis $B = \{(1,-2,3),(5,0,-2),(4,-3,0)\}$, find the vector $\mathbf{x}$.
Solution: The coordinates of $\mathbf{x}$ in the basis B specify the linear combination of the vectors in the basis B to find $\mathbf{x}$

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 = 1\begin{pmatrix}1\\-2\\3\end{pmatrix} + 0\begin{pmatrix}5\\0\\-2\end{pmatrix} - 2\begin{pmatrix}4\\-3\\0\end{pmatrix} = \begin{pmatrix}-7\\4\\0\end{pmatrix}$$

Lay, 4.4.4

Andrea Santos Cortés, Nov. 17th, 2014

Given the coordinate $[\mathbf{x}]_B = (-3,2,-1)$ and the basis $B = \{(-2,2,0),(3,0,2),(4,-1,3)\}$, find the vector $\mathbf{x}$.
Solution: The coordinates of $\mathbf{x}$ in the basis B specify the linear combination of the vectors in the basis B to find $\mathbf{x}$

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 = -3\begin{pmatrix}-2\\2\\0\end{pmatrix} + 2\begin{pmatrix}3\\0\\2\end{pmatrix} - 1\begin{pmatrix}4\\-1\\3\end{pmatrix} = \begin{pmatrix}8\\-5\\1\end{pmatrix}$$

Lay, 4.4.5

Andrea Santos Cortés, Nov. 17th, 2014

Find the coordinates of $\mathbf{x} = (-1, 1)$ relative to the basis $B = \{(1, -2), (3, -5)\}$
Solution: The coordinates of $\mathbf{x}$ in the basis B specify the linear combination of the vectors in the basis B to find $\mathbf{x}$. We need to find the weights such that

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

We can solve this problem through the augmented matrix

$$\left(\begin{array}{rr|r} 1 & 3 & -1 \\ -2 & -5 & 1 \end{array} \right) \sim \left(\begin{array}{rr|r} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array} \right)$$

The coordinates are $[\mathbf{x}]_B = (5, -2)$.

Lay, 4.4.8

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the coordinates of $\mathbf{x} = (0, 0, -2)$ relative to the basis $B = \{(1, 1, 3), (2, 0, 8), (1, -1, 3)\}$
Solution: The coordinates of $\mathbf{x}$ in the basis B specify the linear combination of the vectors in the basis B to find $\mathbf{x}$. We need to find the weights such that

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 = x_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

We can solve this problem through the augmented matrix

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 8 & 3 & -2 \end{array} \right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

The coordinates are $[\mathbf{x}]_B = (1, -1, 1)$.

Lay, 4.4.9

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the change-of-coordinates matrix from the basis $B = \{(1, -3), (2, -5)\}$ to the standard basis of $\mathbb{R}^2$.
Solution: The matrix sought is the one whose columns are the vectors in the basis B

$$P_{E \leftarrow B} = \begin{pmatrix} 1 & 2 \\ -3 & -5 \end{pmatrix}$$

Lay, 4.4.11 (3rd ed.)

María Postigo Fliquete, Dec. 8th, 2014

Use an inverse matrix to find $[\mathbf{x}]_B$ for the given $\mathbf{x}$ and B.
$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right\}$

$\mathbf{x} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

Solution:

$$\mathbf{x} = P_B[\mathbf{x}]_B$$

$$P_B = \begin{pmatrix} 3 & -4 \\ -5 & 6 \end{pmatrix}$$

We calculate the inverse matrix

$$\left(\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -5 & 6 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -9/3 & -2 \\ 0 & 1 & -5/2 & -3/2 \end{array}\right)$$

$$P_B^{-1} = \begin{pmatrix} -3/9 & -2 \\ -5/2 & -3/2 \end{pmatrix}$$

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x}$$

$$[\mathbf{x}]_B = \begin{pmatrix} -3/9 & -2 \\ -5/2 & -3/2 \end{pmatrix} \begin{pmatrix} 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Lay, 4.4.13

Carlos Oscar Sorzano, Aug. 31st, 2013

The set $B = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to B.

Solution: Consider the standard basis of $\mathbb{P}_2$ ($\{1, t, t^2\}$). The change-of-coordinate matrix from B to the standard basis is the one whose columns are the expression of each one of the elements in the basis B in the standard basis

$$P_{E\leftarrow B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

We use this matrix to convert B-coordinates into E-coordinates

$$[\mathbf{x}]_E = P_{E\leftarrow B}[\mathbf{x}]_B$$

Conversely, we may invert this equation to find the B-coordinates of the polynomial $p(t)$.

$$[\mathbf{x}]_B = P_{E\leftarrow B}^{-1}[\mathbf{x}]_E = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix}$$

Lay, 4.4.14 (3rd ed.)

María Postigo Fliquete, Dec. 8th, 2014

The set $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t) = 3 + t + 6t^2$ relative to $\mathcal{B}$.

Solution:

$$p(t) = 3 + t + 6t^2 \Rightarrow [p(t)]_E = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

$$\mathcal{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\} \Rightarrow P_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

The following equation relates the coordinates in the standard basis and the $\mathcal{B}$ basis

$$[p(t)]_E = P_B[p(t)]_B$$

We solve the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow [p(t)]_B = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$$

Lay, 4.4.17

Carlos Oscar Sorzano, Aug. 31st, 2013

The vectors $\mathbf{v}_1 = (1, -3)$, $\mathbf{v}_2 = (2, -8)$ and $\mathbf{v}_3 = (-3, 7)$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\mathbf{x} = (1, 1)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: We need to find x_1, x_2, x_3 such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{x}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -5 & 5 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

So any linear combination of the form

$$\begin{array}{c} x_1 = 5 + 5x_3 \\ x_2 = -2 - x_3 \end{array} \Rightarrow \begin{pmatrix} 5 + 5x_3 \\ -2 - x_3 \\ x_3 \end{pmatrix}$$

is a representation of $\mathbf{x}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Lay, 4.4.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Let S be a finite set of in a vector space V with the property that every $\mathbf{x}$ in V has a unique representation as a linear combination of the elements of S. Show that S is a basis of V.

Solution: The fact that every $\mathbf{x}$ in V has a representation as a linear combination of elements of S means that S spans V. The fact that this representation is unique implies that the set S is linearly independent. These are the two conditions to become a basis and, therefore, the set S is a basis of V.

Lay, 4.4.20

Carlos Oscar Sorzano, Feb. 15th, 2014

Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly dependent set spanning a vector space V. Show that each $\mathbf{w}$ in V can be expressed in more than one way as a linear

combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. [*Hint*: Let $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$ be an arbitrary vector of V. Use the linear dependence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ to produce another representation of $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.]
Solution: Without loss of generality, let us assume that $\mathbf{v}_4$ is a linear combination of the rest of the vectors

$$\mathbf{v}_4 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$$

Then, we can reexpress $\mathbf{w}$ as

$$\begin{aligned}\mathbf{w} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 \\ &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \\ &= (k_1 + k_4a_1)\mathbf{v}_1 + (k_2 + k_4a_2)\mathbf{v}_2 + (k_3 + k_4a_3)\mathbf{v}_3\end{aligned}$$

Lay, 4.4.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that the coordinate mapping is *onto* $\mathbb{R}^n$. That is, given any $\mathbf{y} \in \mathbb{R}^n$, with entries $y_1, y_2, ..., y_n$, produce a $\mathbf{u} \in V$ such that $[\mathbf{u}]_B = y$.
Solution: Assume that the basis B is formed by the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_n$. Then the coordinates of the vector

$$\mathbf{u} = y_1\mathbf{b}_1 + y_2\mathbf{b}_2 + ... + y_n\mathbf{b}_n$$

are $\mathbf{y}$.

Lay, 4.4.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that the subset $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_p\}$ in V is linearly independent if and only if the set of coordinate vectors $\{[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, ..., [\mathbf{u}_p]_B\}$ is linearly independent in $\mathbb{R}^n$. *Hint*: Since the coordinate mapping is one-to-one, the following equations have the same solution $c_1, c_2, ..., c_p$:

$$\begin{gathered}c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_p\mathbf{u}_p = \mathbf{0}_V \\ c_1[\mathbf{u}_1]_B + c_2[\mathbf{u}_2]_B + ... + c_p[\mathbf{u}_p]_B = \mathbf{0}\end{gathered}$$

Solution: Consider the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_p\mathbf{u}_p = \mathbf{0}_V$$

and its solution coefficients $c_1, c_2, ..., c_p$. Since the coordinate mapping is one-to-one, the set of solutions is the same as the one for the equation

$$[c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + ... + c_p\mathbf{u}_p]_B = [\mathbf{0}_V]_B$$

and because the coordinate mapping is a linear transformation, we have

$$c_1[\mathbf{u}_1]_B + c_2[\mathbf{u}_2]_B + ... + c_p[\mathbf{u}_p]_B = \mathbf{0}$$

But, by hypothesis, the set $\{[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, ..., [\mathbf{u}_p]_B\}$ is linearly independent. That means that the only solution of the equation is $c_1 = c_2 = ... = c_p = 0$ and so is the solution of the first equation in the sequel.

Lay, 4.5.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Find a basis for the subspace below and state its dimension

$$S = \left\{ \begin{pmatrix} s-2t \\ s+t \\ 3t \end{pmatrix} \quad \forall s,t \in \mathbb{R} \right\}$$

Solution: We may write the set as

$$S = \left\{ s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \quad \forall s,t \in \mathbb{R} \right\}$$

Thus, a basis is given by the vectors

$$\text{Basis}\{S\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\}$$

Since the basis has two vectors, the dimension of S is 2.

Lay, 4.5.2

Marta Monsalve Buendía, Dic. 4th, 2014

Find a basis for the subspace below and state its dimension

$$S = \left\{ \begin{pmatrix} 2a \\ -4b \\ -2a \end{pmatrix} \quad \forall a,b \in \mathbb{R} \right\}$$

Solution: We may write the set as

$$S = \left\{ a \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} \quad \forall a,b \in \mathbb{R} \right\}$$

Thus, a basis is given by the vectors

$$\text{Basis}\{S\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} \right\}$$

Since the basis has two vectors, the dimension of S is 2.

Lay, 4.5.3

Marta Monsalve Buendía, Dic. 4th, 2014

Find a basis for the subspace below and state its dimension

$$S = \left\{ \begin{pmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{pmatrix} \quad \forall a,b,c \in \mathbb{R} \right\}$$

Solution: We may write the set as

$$S = \left\{ a \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix} + c \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix} \quad \forall a,b,c \in \mathbb{R} \right\}$$

Thus, a basis is given by the vectors

$$\text{Basis}\{S\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix} \right\}$$

Since the basis has three vectors, the dimension of S is 3.

Lay, 4.5.4

Ana Peña Gil, Jan. 19th 2014

Find a basis for the subspace below and state its dimension

$$S = \left\{ \begin{pmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{pmatrix} \quad \forall s, t \in \mathbb{R} \right\}$$

Solution: We may write the set as

$$S = \left\{ p \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} + q \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad \forall p, q \in \mathbb{R} \right\}$$

Thus, a basis is given by the vectors

$$\text{Basis}\{S\} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Since the basis has two vectors, the dimension of S is 2.

Lay, 4.5.5

Ana Peña Gil, Jan. 19th 2014

Find a basis for the subspace below and state its dimension

$$S = \left\{ \begin{pmatrix} p-2q \\ 2p+5r \\ -2q+2r \\ -3p+6r \end{pmatrix} \quad \forall p, q, r \in \mathbb{R} \right\}$$

Solution: We may write the set as

$$S = \left\{ p \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix} + q \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 5 \\ 2 \\ 6 \end{pmatrix} \quad \forall p, q, r \in \mathbb{R} \right\}$$

Thus, a basis is given by the vectors

$$\text{Basis}\{S\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 2 \\ 6 \end{pmatrix} \right\}$$

Since the basis has three vectors, the dimension of S is 3.

Lay, 4.5.6

Marta Monsalve Buendía, Dic. 4th, 2014

Find a basis for the subspace below and state its dimension

$$S = \left\{ \begin{pmatrix} 3a - c \\ -b - 3c \\ -7a + 6b + 5c \\ -3a + c \end{pmatrix} \quad \forall a, b, c \in \mathbb{R} \right\}$$

Solution: We may write the set as

$$S = \left\{ a \begin{pmatrix} 3 \\ 0 \\ -7 \\ -3 \end{pmatrix} + b \begin{pmatrix} 0 \\ -1 \\ 6 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ -3 \\ 5 \\ 1 \end{pmatrix} \quad \forall a, b, c \in \mathbb{R} \right\}$$

Thus, a basis is given by the vectors

$$\text{Basis}\{S\} = \left\{ \begin{pmatrix} 3 \\ 0 \\ -7 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 5 \\ 1 \end{pmatrix} \right\}$$

Since the basis has three vectors, the dimension of S is 3.

Lay, 4.5.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine the dimensions of Nul$\{A\}$ and Col$\{A\}$ for the matrix

$$A \begin{pmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: The basis of the null space is formed by those non-pivot columns, in the case of A, the third and fifth columns. So, the dimension of Nul$\{A\}$ is 2. The basis of the column space is formed by the pivot columns, in this case, the first, second and fourth columns. So, the dimension of Col$\{A\}$ is 3.

Lay, 4.5.15 (3rd ed.)

María Postigo Fliquete, Dec. 8th, 2014

Determine the dimensions of Nul$\{A\}$ and Col$\{A\}$

$$A = \begin{pmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 41 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

The given matrix has two pivot colums so $\dim\{\text{Col}\{A\}\} = 2$ and two non-pivot columns so $\dim\{\text{Nul}\{A\}\} = 2$.

Lay, 4.5.17 (3rd ed.)

Ignacio Sanchez Lopez, Jan. 17th, 2015

Determine the dimensions of Nul$\{A\}$ and Col$\{A\}$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The given matrix has three pivot colums so $\dim\{\text{Col}\{A\}\} = 3$ and zero non-pivot columns so $\dim\{\text{Nul}\{A\}\} = 0$.

Lay, 4.5.20

Carlos Oscar Sorzano, Jan. 20th, 2014

Mark each statement as true or false. Justify each answer.

1. $\mathbb{R}^2$ is a two-dimensional subspace of $\mathbb{R}^3$
2. The number of variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the dimension of Nul$\{A\}$.
3. A vector space is infinite-dimensional if it is spanned by an infinite set.
4. If $\dim\{V\} = n$ and if S spans V, then S is a basis of V.
5. The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

Solution:

1. False, $\mathbb{R}^2$ is not a subpace of $\mathbb{R}^3$. Although it is true that $\mathbb{R}^2$ is isomorph to a two-dimensional subspace of $\mathbb{R}^3$.

2. False, the number of variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the number of columns of A. The dimension of $\mathrm{Nul}\{A\}$ is equal to the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

3. False, the infinite set must be a basis of A, that is, there cannot be a proper subset of S that also spans the same vector space.

4. False, for this to be true we also need that S has n vectors.

5. True, $\mathbb{R}^3$ is said to be an improper subspace of itself.

Lay, 4.5.21

Carlos Oscar Sorzano, Aug. 31st, 2013

The first four Hermite polynomials are 1, $2t$, $-2 + 4t^2$ and $-12t + 8t^3$. These polynomials arise naturally in the study of certain important differential equations in mathematical physics. Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

Solution: Consider the standard basis of $\mathbb{P}_3$:

$$E = \{1, t, t^2, t^3\}$$

In order to know whether the four Hermite polynomials are linearly independent or not we resort to the following augmented matrix whose columns are the expression of the Hermite polynomials in the standard basis of $\mathbb{P}_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & -12 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

So the four Hermite polynomials are linearly independent. Since they are 4 and the dimension of $\mathbb{P}_3$ is also 4, then by Theorem 9.4 of Chapter 5, the four Hermite polynomials are a basis of $\mathbb{P}_3$.

Lay, 4.5.22

Carlos Oscar Sorzano, Jan. 20th, 2013

The first four Laguerre polynomials are 1, $1-t$, $2-4t+t^2$ and $6-18t+9t^2-t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

Solution: Consider the standard basis of $\mathbb{P}_3$:

$$E = \{1, t, t^2, t^3\}$$

In order to know whether the four Laguerre polynomials are linearly independent or not we resort to the following augmented matrix whose columns are the expression of the Laguerre polynomials in the standard basis of $\mathbb{P}_3$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 6 & 0 \\ 0 & -1 & -4 & -18 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

So the four Laguerre polynomials are linearly independent. Since they are 4 and the dimension of $\mathbb{P}_3$ is also 4, then by Theorem 9.4 of Chapter 5, the four Laguerre polynomials are a basis of $\mathbb{P}_3$.

Lay, 4.5.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Let S be a subset of an n-dimensional vector space V, and suppose S contains fewer than n vectors. Explain why S cannot span V.

Solution: Note that $n \geq 1$ because S cannot have fewer than 0 vectors. If S spans V, then there exists a subset $S' \subseteq S$ that is a basis of V. S' must have fewer than n vectors (because S has fewer than n vectors), but by Theorem 9.2 of Chapter 5, this is impossible because all bases of V have n vectors.

Lay, 4.5.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

Solution: Let B_H be a basis of H. Since H is an n-dimensional vector space, B_H must have n vectors. But by Theorem 9.4 of Chapter 5, any set of n linearly independent vectors of V is a basis of V. So B_H is also a basis for V. Since both spaces have the same basis, both spaces are the same.

Lay, 4.5.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Explain why the space $\mathbb{P}$ of all polynomials is an infinite-dimensional space.

Solution: Let us assume that $\mathbb{P}$ is finite-dimensional and that its dimension is n. Consider the set of polynomials of degree n ($\mathbb{P}_n$). This obviously a subset of $\mathbb{P}$ so the dimension of $\mathbb{P}$ is larger than the dimension of $\mathbb{P}_n$

$$\mathbb{P}_n \subset \mathbb{P} \Rightarrow \dim\{\mathbb{P}_n\} < \dim\{\mathbb{P}\}$$

The dimension of $\mathbb{P}_n$ is $n+1$, but $n+1 > n$ so this is a contradiction with our hypothesis, and $\mathbb{P}$ is an infinite-dimensional space.

Lay, 4.5.28

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that the space $C(\mathbb{R})$ of all continuous functions defined on the real line is an infinite-dimensional space.

Solution: In the previous exercise we showed that the space of polynomials $\mathbb{P}$ is infinite-dimensional. Since all polynomials are continuous functions on the real line we have $\mathbb{P} \subseteq C(\mathbb{R})$. Consequently, $C(\mathbb{R})$ must be infinite-dimensional since its dimension cannot be smaller than the dimension of $\mathbb{P}$.

Lay, 4.5.31

Carlos Oscar Sorzano, Aug. 31st, 2013

Consider finite-dimensionals spaces V and W, and a linear transformation $T : V \to W$. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Then $T(H)$ is a subspace of W, by Exercise 35 in Section 4.2. Prove that $\dim\{T(H)\} \leq \dim\{H\}$.

Solution: Let $k = \dim\{T(H)\}$. If $k = 0$, then $k < \dim\{H\}$. If $k > 0$, then $T(H)$ must have a basis formed by k vectors in $T(H)$ of the form $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$,

..., $T(\mathbf{v}_k)$. Since the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_k)\}$ is linearly independent and T is a linear transformation, then by Exercise 4.3.31 the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is also linearly independent in H and consequently $\dim\{H\} \geq k$.

Lay, 4.5.32

Carlos Oscar Sorzano, Aug. 31st, 2013

Consider finite-dimensionals spaces V and W, and a linear transformation $T : V \to W$. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Suppose, additionally, that T is a one-to-one mapping. Prove that $\dim\{T(H)\} = \dim\{H\}$. If T happens to be a one-to-one mapping of V onto W, then $\dim\{V\} = \dim\{W\}$. Isomorphic finite-dimensional vector spaces have the same dimension.

Solution: If T is a one-to-one mapping, it means that it maps linearly independent sets from V to W and viceversa. If the dimension of H is k, then any basis of $H \subseteq V$ has k linearly independent vectors, that are mapped by T onto k linearly independent vectors of W. So they are also a basis of $T(H)$, and consequently $\dim\{T(H)\} = k$.

Lay, 4.6.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Assume

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

Without calculations list $\text{Rank}\{A\}$ and $\dim\{\text{Nul}\{A\}\}$. Then find bases for $\text{Col}\{A\}$, $\text{Row}\{A\}$, and $\text{Nul}\{A\}$.

Solution: The rank of A is the dimension of the column space of A that is given by the number of pivot columns of A. This is the same as the number of pivot columns of B, that is, 2 (first and second columns). The dimension of the null space of A, thanks to the rank theorem, can be calculated as the number of columns of A minus its rank, in this case, $4 - 2 = 2$.

The basis of $\text{Col}\{A\}$ is given by the pivot columns of A, that are the same as the pivot columns of B:

$$\text{Basis}\{\text{Col}\{A\}\} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix} \right\}$$

The space spanned by the rows of B is the same as the space spanned by the rows of A. So a basis for the row space of A is

$$\text{Basis}\{\text{Row}\{A\}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 5 \\ -6 \end{pmatrix} \right\}$$

For the null space of A we write the equations implied by the two rows of B

$$\begin{array}{r} x_1 = x_3 - x_4 \\ -2x_2 = -5x_3 + 6x_4 \end{array} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ \frac{5}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

So the basis of Nul$\{A\}\}$ is given by

$$\text{Basis}\{\text{Row}\{A\}\} = \left\{ \begin{pmatrix} 1 \\ \frac{5}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Lay, 4.6.3 (3rd ed.)

María Postigo Fliquete, Dec. 8th, 2014

Assume that the matrix A is row equivalent to B. Without calculations, list Rank$\{A\}$ and dim$\{$Nul$\{A\}\}$. Then find bases for Col$\{A\}$, Row$\{A\}$, and Nul$\{A\}$.

$$A = \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: The matrix has 3 pivot colums so Rank$\{A\} = \dim\{\text{Col}\{A\}\} = 3$ and 2 non pivot colums so $\dim\{\text{Nul}\{A\}\} = 2$. The basis of Col$\{A\}$ is formed by the columns of A corresponding to the pivot columns of B:

$$\left\{ \begin{pmatrix} 2 \\ -2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ -3 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \\ -4 \end{pmatrix} \right\}$$

The basis of Nul$\{A\}$ comes from the equations implied by B

$$\begin{matrix} 2x_1 - 3x_2 + 6x_3 + 2x_4 + 5x_5 = 0 \\ 3x_3 - x_4 + x_5 = 0 \\ x_4 + 3x_5 = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} x_1 = \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ x_3 = -\frac{4}{3}x_5 \\ x_4 = -3x_5 \end{matrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} \frac{9}{2} \\ 0 \\ -\frac{4}{3} \\ -3 \\ 1 \end{pmatrix}$$

So the basis of Nul$\{A\}$ is

$$\left\{ \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{9}{2} \\ 0 \\ -\frac{4}{3} \\ -3 \\ 1 \end{pmatrix} \right\}$$

Finally the basis of Row$\{A\}$ comes from the rows of B

$$\left\{ \begin{pmatrix} 2 \\ -3 \\ 6 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}$$

Lay, 4.6.5

Carlos Oscar Sorzano, June, 6th 2014

If a 4×7 matrix A has rank 3, find dim$\{$Nul$\{A\}\}$, dim$\{$Row$\{A\}\}$, and rank of A^T.

Solution: According to the rank theorem, the rank of A plus dim$\{$Nul$\{A\}\}$ must be the number of columns of A. Since the rank of A is 3, dim$\{$Nul$\{A\}\}$ must be 4. On another side, the rank is defined as the dimension of the row space of A, so dim$\{$Row$\{A\}\} = 3$. Finally

$$\text{rank}\{A^T\} = \dim\{\text{Row}\{A^T\}\} = \dim\{\text{Col}\{A\}\} = \text{rank}\{A\} = 3$$

Lay, 4.6.13

Carlos Oscar Sorzano, Aug. 31st, 2013

If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A? Explain your answers.

Solution: In both cases the rank can be 5 at maximum, because for any $m \times n$ matrix the rank meets (Rank Theorem)

$$\dim\{\text{Row}\{A\}\} = \dim\{\text{Col}\{A\}\} = \text{Rank}\{A\}$$

In the first case, the rank cannot be larger than 5 because there are only 5 columns. In the second case, the rank cannot be larger than 5 because there are only 5 rows.

Lay, 4.6.15

Carlos Oscar Sorzano, Aug. 31st, 2013

If A is a 3×7 matrix, what is the smallest possible dimension of Nul$\{A\}$?

Solution: By the Rank Theorem we know that for a $m \times n$ matrix

$$\dim\{\text{Nul}\{A\}\} + \text{Rank}\{A\} = n$$

The rank cannot be larger than 3, so the dimension of Nul$\{A\}$ cannot be smaller than 4, i.e., dim$\{$Nul$\{A\}\} \geq 4$.

Lay, 4.6.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose the solutions of a homogeneous system of 5 linear equations in 6 unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equation? Explain.

Solution: The fact that all homogeneous solutions are multiples of one nonzero solution implies that the null space is 1-dimensional. By the Rank Theorem, the rank of A (the system matrix) is 5 (so that $5 + 1 = 6$). So,

$$\text{Rank}\{A\} = \dim\{\text{Col}\{A\}\} = 5$$

Since the matrix A is 5×6, its column space must be a subspace of $\mathbb{R}^5$. On the other side, since its dimension is 5, then

$$\text{Col}\{A\} = \mathbb{R}^5$$

and consequently, for every $\mathbf{b} \in \mathbb{R}^5$ there is a solution of the equation $A\mathbf{x} = \mathbf{b}$.

Lay, 4.6.20

Carlos Oscar Sorzano, Jan. 20th, 2014

Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution, with two free variables. Is it possible to change some constants on the equations' right sides to make the new system inconsistent? Explain.

Solution: If the equation system has a solution with two free variables, it means that the system matrix has rank 6. By changing the right side we cannot make the system inconsistent since for this we would need at least one of the equations to have only 0 coefficients in the left side (which is not possible because the rank of the matrix is 6).

Lay, 4.6.26

Carlos Oscar Sorzano, Aug. 31st, 2013

In statistical theory, a common requirement is that a matrix be of *full rank*. That is, the rank should be as large as possible. Explain why an $m \times n$ matrix with more rows than columns has full rank if and only if its columns are linearly independent.

Solution: The if there are more rows than columns then $m > n$ and the rank can be at most n. The rank is n iff the dimension of the column space is n. But since there are only n columns in the matrix, this can only be achieved if they are linearly independent.

Lay, 4.6.28

Carlos Oscar Sorzano, Aug. 31st, 2013

Justify the following equalities:

a. $\dim\{\text{Row}\{A\}\} + \dim\{\text{Nul}\{A\}\} = n$

b. $\dim\{\text{Col}\{A\}\} + \dim\{\text{Nul}\{A^T\}\} = m$

Solution:

a. By the Rank Theorem we know that

$$\dim\{\text{Col}\{A\}\} + \dim\{\text{Nul}\{A\}\} = n$$

Since $\text{Rank}\{A\} = \dim\{\text{Col}\{A\}\} = \dim\{\text{Row}\{A\}\}$ we have immediately the proposed equality.

b. If we apply the Rank Theorem to A^T we get

$$\dim\{\text{Col}\{A^T\}\} + \dim\{\text{Nul}\{A^T\}\} = m$$

But $\dim\{\text{Col}\{A^T\}\} = \dim\{\text{Row}\{A\}\} = \dim\{\text{Col}\{A\}\}$, and again we have the proposed equality.

Lay, 4.6.29

Carlos Oscar Sorzano, Aug. 31st, 2013

Use Exercise 28 to explain why the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$ if and only if the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solution: If the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the dimension of its null space is 0 ($\dim\{\text{Nul}\{A^T\}\} = 0$) and by the proposition b of Exercise 4.6.28, $\dim\{\text{Col}\{A\}\} = m$. This means that the columns of A span $\mathbb{R}^m$ and by the Invertible Matrix theorem, there is a solution of the equation $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^m$.

This reasoning could be reversed in all its steps to show that if there is a solution of the equation $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^m$, then the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Lay, 4.6.33

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be any 2×3 matrix such that $\text{Rank}\{A\} = 1$, let $\mathbf{u}$ be the first column of A, and suppose that $\mathbf{u} \neq 0$. Explain why there is a vector $\mathbf{v} \in \mathbb{R}^3$ such that $A = \mathbf{u}\mathbf{v}^T$. How could this construction be modified if the first column of A were zero.

Solution: If $\text{Rank}\{A\} = 1$, then the dimension of the column space of A is 1, meaning that all columns are multiples of a single vector. Without loss of generality, we may assume that the first column of A is the basis of the column space. Then

$$\mathcal{M}_{2\times 3} \ni A = \begin{pmatrix} \mathbf{u} & a\mathbf{u} & b\mathbf{u} \end{pmatrix}$$

The vector $\mathbf{v}$ proposed by the problem is

$$\mathbf{v} = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$$

It can be easily verified that $A = \mathbf{u}\mathbf{v}^T$.

If the first column of A is zero, then the new vector $\mathbf{v}$ would be of the form

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ a \end{pmatrix}$$

and now the basis of the column space of A is given by its second column and not by its first column.

Lay, 4.7.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.

a. Find the change-of-coordinates matrix from B to C.

b. Find $[\mathbf{x}]_C$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$

Solution:

a. The change-of-coordinates matrix is

$$P_{C\leftarrow B} = \begin{pmatrix} [\mathbf{b}_1]_C & [\mathbf{b}_2]_C \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix}$$

b. We note that $[\mathbf{x}]_B = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, then

$$[\mathbf{x}]_C = P_{C\leftarrow B}[\mathbf{x}]_B = \begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Lay, 4.7.9

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space $\mathbb{R}^2$ with $\mathbf{b}_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$, $\mathbf{c}_1 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, and $\mathbf{c}_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. Find the change-of-coordinates matrices from B to C and from C to B

Solution: The change-of-coordinates matrices from B and C to the standard basis of $\mathbb{R}^2$ are given by

$$P_{E\leftarrow B} = \begin{pmatrix} [\mathbf{b}_1]_E & [\mathbf{b}_2]_E \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ 5 & -1 \end{pmatrix}$$
$$P_{E\leftarrow C} = \begin{pmatrix} [\mathbf{c}_1]_E & [\mathbf{c}_2]_E \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -5 & 2 \end{pmatrix}$$

Now we note that for each one of the basis we have

$$[\mathbf{x}]_E = P_{E\leftarrow B}[\mathbf{x}]_B = P_{E\leftarrow C}[\mathbf{x}]_C \Rightarrow [\mathbf{x}]_C = P_{E\leftarrow C}^{-1} P_{E\leftarrow B}[\mathbf{x}]_B$$

In this particular case

$$P_{C\leftarrow B} = P_{E\leftarrow C}^{-1} P_{E\leftarrow B} = \begin{pmatrix} 1 & -2 \\ -5 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 7 & -3 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}$$

In the other direction

$$P_{B\leftarrow C} = P_{C\leftarrow B}^{-1} = \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$$

Lay, 4.7.13

Carlos Oscar Sorzano, Dec. 16th, 2014

In $\mathbb{P}_2$ find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find the $\mathcal{B}$-coordinate vector for $-1 + 2t$.

Solution: The matrix sought is the one whose columns are the coordinates in the $\mathcal{C}$ coordinate system of the vectors in the basis $\mathcal{B}$

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{pmatrix}$$

This matrix can be used to convert coordinates in $\mathcal{B}$ into coordinates in $\mathcal{C}$

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

Conversely

$$[\mathbf{x}]_{\mathcal{B}} = P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{C}}$$

In the case of the example of this problem

$$[\mathbf{x}]_{\mathcal{B}} = P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

Lay, 4.Suppl.4

Carlos Oscar Sorzano, Jan. 19th 2015

Explain what is wrong with the following discussion: Let $f(t) = 3 + t$ and $g(t) = 3t + t^2$. Note that $g(t) = tf(t)$. The set $\{f, g\}$ is linearly dependent because g is a multiple of f.

Solution: g is not a multiple of f. To be a multiple it should be $g = kf$. Actually, g and f are linearly independent because their ratio is not constant

$$\frac{g}{f} = t$$

Lay, 4.Suppl.9

Carlos Oscar Sorzano, Jan. 19th 2015

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1. What is the dimension of Range$\{T\}$ if T is a one-to-one mapping? Explain.
2. What is the dimension of Ker$\{T\}$ if T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$? Explain.

Solution:

1. If T is a one-to-one mapping, it means that there are vectors in $\mathbb{R}^m$ that do not come from any vector in $\mathbb{R}^n$. So the dimension of Range$\{T\}$ would be at most $m - 1$. For being linear, we also know that Range$\{T\}$ can be at most n. So we have

$$\dim\{\text{Range}\{T\}\} \leq \min(n, m)$$

2. If T is surjective, it means that all vectors in $\mathbb{R}^m$ come at least from 1 vector in $\mathbb{R}^n$. Since T is linear and the dimension of Range$\{T\}$ can be at most n, it means that
$$\dim\{\text{Range}\{T\}\} = m \leq n$$
The difference between n and m must be the dimension of the kernel of T.
$$\dim\{\text{Ker}\{T\}\} = n - m$$
Note that this dimension can be 0 if $n = m$.

Lay, 4 Suppl. 12

Show from parts (a) and (b) that rank(AB) cannot exceed the rank of A and the rank of B. (In general, the rank of a product of matrices cannot exceed the rank of any factor in the product.)

- Show that if B is $n \times p$, then $\text{rank}(AB) \leq \text{rank}(A)$. [*Hint*: Explain why every vector in the column space of AB is in the column space of A.]
- Show that if B is $n \times p$, then $\text{rank}(AB) \leq \text{rank}(B)$. [*Hint*: Use part (a) to study $\text{rank}((AB)^T)$.]

Solution:

- Let $\mathbf{b}_i$ be one of the columns of the matrix B. The product $A\mathbf{b}_i$ is a linear combination of the columns of A with the weights given by the elements of the vector $\mathbf{b}_i$ and, consequently, it belongs to the column space of the matrix A. For the same reason, all columns of the product AB also belong to the column space of the matrix A. The rank of a matrix is the dimension of the column space of that matrix. Since all the columns of AB are in the column space of A, the rank of AB cannot be larger than the rank of A.
- $(AB)^T = B^T A^T$. Applying the same reasoning as above the rank of $(AB)^T$ cannot be larger than the rank of A^T. On the other side, we know that the rank of a matrix is equal to the rank of its transpose. Consequently
$$\text{Rank}(AB) = \text{Rank}((AB)^T) \leq \text{Rank}(A^T) = \text{Rank}(A)$$

Lay, 4 Suppl. 13

Show that if P is an invertible $m \times m$ matrix, then $\text{Rank}(PA) = \text{Rank}(A)$. [*Hint*: Apply 4.Suppl. 12 to PA and $P^{-1}(PA)$.]
Solution: Applying 4.Suppl. 12 we have that
$$\text{Rank}(A) = \text{Rank}(P^{-1}(PA)) \leq \text{Rank}(PA) \leq \text{Rank}(A)$$
That is
$$\text{Rank}(A) \leq \text{Rank}(PA) \leq \text{Rank}(A) \Rightarrow \text{Rank}(PA) = \text{Rank}(A)$$

5 Chapter 5

Lay, 5.1.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Is $\lambda = 2$ an eigenvalue of $\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$? Why or why not?

Solution: To check whether $\lambda = 2$ is an eigenvalue or not, we test whether it is a solution of the equation

$$\left|\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} - \lambda I\right| = 0 \Rightarrow \left|\begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0 \Rightarrow \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0 \Rightarrow 0 = 0$$

Since we have got an identity ($0 = 0$), $\lambda = 2$ is a solution of the eigenvalue problem and it is an eigenvalue of the proposed matrix.

Lay, 5.1.2 *Yolanda Manrique Marcos, Dec. 17th, 2013*

Is $\lambda = -3$ an eigenvalue of $\begin{vmatrix} -1 & 4 \\ 6 & 9 \end{vmatrix}$? Why or why not?

Solution: To check whether $\lambda = -3$ is an eigenvalue or not, we test whether it is a solution of the equation

$$\left|\begin{pmatrix} -1 & 4 \\ 6 & 9 \end{pmatrix} - \lambda I\right| = 0 \Leftrightarrow$$
$$\left|\begin{pmatrix} -1 & 4 \\ 6 & 9 \end{pmatrix} - (-3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0 \Leftrightarrow$$
$$\begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix} = 0 \Leftrightarrow$$
$$0 = 0$$

Since we have got an identity ($0 = 0$), $\lambda = -3$ is a solution of the eigenvalue problem and it is an eigenvalue of the proposed matrix.

Lay, 5.1.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 1 & -1 \\ 6 & -4 \end{pmatrix}$? If so, find the eigenvalue.

Solution: To check whether $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector or not, we test whether it has the property

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ \begin{pmatrix} 1 & -1 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} &= \begin{pmatrix} -2 \\ -6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{aligned}$$

So, $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector and its associated eigenvalue is -2.

Lay, 5.1.4

Ana Peña Gil, Jan. 19th 2014

Is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}$? If so, find its eigenvalue.

Solution: To check whether $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector or not, we test whether it has the property

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ \begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} &= \begin{pmatrix} -3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}$$

So, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector and its associated eigenvalue is 3.

Lay, 5.1.5

Marta Monsalve Buendía, Dic. 6th, 2014

Is $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{pmatrix}$? If so, find the eigenvalue.

Solution: To check whether $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector or not, we test whether it has the property

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ \begin{pmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} &= \begin{pmatrix} -15 \\ 10 \\ -5 \end{pmatrix} = -5\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

So, $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector and its associated eigenvalue is -5.

Lay, 5.1.9

Carlos Oscar Sorzano, Aug. 31st, 2013

Find a basis for the eigenspace corresponding to each of the eigenvalues of $A = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$, $\lambda = 1, 3$

Solution: We need to find the set of vectors such that for each eigenvalue they meet

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$$

$\underline{\lambda = 1}$

$$\begin{gathered} \left(\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{v} = \mathbf{0} \\ \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}\mathbf{v} = \mathbf{0} \end{gathered}$$

The general solution of this homogeneous equation system is $\mathbf{v} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$. This is the eigenspace associated to the eigenvalue $\lambda = 1$ and one of its basis is $\{(0, 1)\}$.

$\underline{\lambda = 3}$

$$\left(\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{v} = \mathbf{0}$$
$$\begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}\mathbf{v} = \mathbf{0}$$

The general solution of this homogeneous equation system is $\mathbf{v} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$. This is the eigenspace associated to the eigenvalue $\lambda = 3$ and one of its basis is $\{(1,1)\}$.

Lay, 5.1.17

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the eigenvalues of $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{pmatrix}$.

Solution: We solve for λ the equation

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & -2-\lambda \end{vmatrix} = -\lambda(3-\lambda)(-2-\lambda) = 0$$

whose roots are $\lambda = 0$, $\lambda = 3$, and $\lambda = -2$.

Lay, 5.1.18

Marta Monsalve Buendía, Dic. 6th, 2014

Find the eigenvalues of $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 3 \end{pmatrix}$.

Solution: We solve for λ the equation

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (5-\lambda)(-\lambda)(3-\lambda) = 0$$

whose roots are $\lambda = 0$, $\lambda = 5$, and $\lambda = 3$.

Lay, 5.1.19

Carlos Oscar Sorzano, Aug. 31st, 2013

For $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

Solution: The determinant of A is zero because it has duplicated rows. On the other hand, the determinant is the product of the matrix eigenvalues, so at least one of the eigenvalues of A must be zero.

Lay, 5.1.20

Carlos Oscar Sorzano, Aug. 31st, 2013

Without calculation, find one eigenvalue and two linearly independent vectors of $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. Justify your answer.

Solution: The determinant of A is zero because it has duplicated rows. On the other hand, the determinant is the product of the matrix eigenvalues, so at least one of the eigenvalues of A must be zero. The eigenspace associated to the eigenvalue $\lambda = 0$ is the set of vectors satisfying $A\mathbf{x} = \mathbf{0}$. Since the three rows are the same, we can eliminate the last two by subtracting the first one. To find the eigenvectors we note that if we subtract the first column to the second column we get a null vector (the corresponding eigenvector is $(-1, 1, 0)$). Similarly, if we subtract the first column to the third column, we again get a null vector (a second eigenvector is $(-1, 0, 1)$).

Lay, 5.1.21

Ana Sanmartin, Jan. 16th, 2015

Mark each statement as True or False.

- If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.
- A matrix A is not invertible if and only if 0 is an eigenvalue of A.
- A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a non trivial solution.
- Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.
- To find the eigenvalues of A, reduce A to echelon form.

Solution:

- FALSE, this is true as long as the vector $\mathbf{x}$ is not the zero vector.
- TRUE, we need to prove it in both ways such that:
  - If A is not invertible, then 0 is an eigenvalue. Proof: If we assume that A is not invertible, then $A\mathbf{x} = \mathbf{0}$ does not have only the trivial solution. So by the equation $A\mathbf{x} = \mathbf{0} = 0\mathbf{x}$ we can obtain that by definition, 0 is an eigenvalue.
  - If 0 is an eigenvalue, then A is not invertible. Proof: if A were invertible, by the invertible matrix theorem, there would only be the trivial solution to the problem $A\mathbf{x} = \mathbf{0}$, that is $\mathbf{x} = \mathbf{0}$. But the eigenvector associated to the eigenvalue 0 is a non-zero vector that fulfills $A\mathbf{x} = \mathbf{0}$. Since the solution to the problem $A\mathbf{x} = \mathbf{0}$ is not unique, A is not invertible.
- TRUE, we may rearrange $(A - cI)\mathbf{x} = \mathbf{0}$ as $A\mathbf{x} - c\mathbf{x} = \mathbf{0}$ or, what is the same, $A\mathbf{x} = c\mathbf{x}$. Since the first problem, $(A - cI)\mathbf{x} = \mathbf{0}$ has a non-trivial solution, so does the problem $A\mathbf{x} = c\mathbf{x}$. But this is the definition of $\mathbf{x}$ being an eigenvector associated to the eigenvalue c.

- TRUE, to find an eigenvector of A we need to look for the eigenvalues of A, and then solve an equation system. But for cheking is a given vector is in fact an eigenvector, we just need to see if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$.
- FALSE, by row reducing we make the matrix easier to participate in equation systems, but this operation changes the eigenvectors and eigenvalues.

Lay, 5.1.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why a $n \times n$ matrix can have at most n distinct eigenvalues.
Solution: The eigenvalue problem

$$|A - \lambda I| = 0$$

implies finding the roots of a polynomial of degree n ($|A - \lambda I|$). Since a polynomial of degree n can have at most n distinct roots, then A can have at most n distinct eigenvalues.

Lay, 5.1.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [*Hint*: suppose a non-zero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]
Solution: Suppose

$$A\mathbf{x} = \lambda\mathbf{x}$$

Let's multiply on both sides by A^{-1}

$$\begin{gathered} \mathbf{x} = \lambda A^{-1}\mathbf{x} \\ \lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x} \end{gathered}$$

So λ^{-1} is an eigenvalue of A^{-1} and $\mathbf{x}$ is its associated eigenvector. It is noteworthy to see that $\mathbf{x}$ is an eigenvector of A and of A^{-1}.

Lay, 5.1.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.
Solution: Suppose λ is an eigenvalue of A

$$A\mathbf{x} = \lambda\mathbf{x}$$

Let's multiply on both sides by A

$$\begin{gathered} A^2\mathbf{x} = \lambda A\mathbf{x} \\ \mathbf{0} = \lambda(\lambda\mathbf{x}) \\ \mathbf{0} = \lambda^2\mathbf{x} \end{gathered}$$

But $\mathbf{x}$ is non-zero (by the definition of eigenvector). Then, $\lambda^2 = 0$ and this implies that the only eigenvalue of A is 0.

Lay, 5.1.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [*Hint*: Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]
Solution: Let us calculate the transpose of the matrix $A - \lambda I$

$$(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$$

Now, by Theorem 6c of Section 2.2, $(A - \lambda I)^T$ is not invertible if and only if $A - \lambda I$ is not invertible. If λ is one of the eigenvalues, then $A - \lambda I$ is not invertible. So $(A - \lambda I)^T = A^T - \lambda I$ is not invertible neither, and λ is one of the eigenvalues of A^T.

Lay, 5.1.29

Carlos Oscar Sorzano, Feb. 15th, 2014

Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is a eigenvalue of A. [*Hint*: Find an eigenvector.]
Solution: We will show that the vector $\mathbf{v} = (1, 1, ..., 1)^T$ is an eigenvector of A. The product $A\mathbf{v}$ is equal to

$$A\mathbf{v} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^{n} a_{1j} \\ \sum\limits_{j=1}^{n} a_{2j} \\ ... \\ \sum\limits_{j=1}^{n} a_{nj} \end{pmatrix} = \begin{pmatrix} s \\ s \\ ... \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix}$$

Lay, 5.2.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the characteristic equation and the real eigenvalues of the matrix $A = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$.
Solution: The characteristic equation is $|A - \lambda I| = 0$. In this particular case

$$\begin{array}{c} \left| \begin{array}{cc} 2-\lambda & 7 \\ 7 & 2-\lambda \end{array} \right| = 0 \\ (2-\lambda)^2 - 49 = 0 \\ 4 + \lambda^2 - 4\lambda - 49 = 0 \\ \lambda^2 - 4\lambda - 45 = 0 \\ \lambda = \frac{4 \pm \sqrt{16 + 4 \cdot 45}}{2} = \frac{4 \pm 14}{2} = \left\{ \begin{array}{c} 9 \\ -5 \end{array} \right. \end{array}$$

The two real eigenvalues are $\lambda = 9$ and $\lambda = -5$.

Lay, 5.2.2

Andrea Santos Cortés, Nov. 25th, 2014

Find the characteristic equation and the real eigenvalues of the matrix $A = \begin{pmatrix} -4 & -1 \\ 6 & 1 \end{pmatrix}$.
Solution: The characteristic equation is $|A - \lambda I| = 0$. In this particular case

$$\begin{vmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{vmatrix} = 0$$
$$(-4-\lambda)\cdot(1-\lambda)+6=0$$
$$-4+4\lambda-\lambda+\lambda^2+6=0$$
$$\lambda^2+3\lambda 2=0$$
$$\lambda = \frac{-3\pm\sqrt{9-4\cdot 2}}{2} = \frac{-3\pm 1}{2} = \begin{cases} -1 \\ -2 \end{cases}$$

The two real eigenvalues are $\lambda=-1$ and $\lambda=-2$.

Lay, 5.2.3

Andrea Santos Cortés, Nov. 25th, 2014

Find the characteristic equation and the real eigenvalues of the matrix $A = \begin{pmatrix} -4 & 2 \\ 6 & 7 \end{pmatrix}$.

Solution: The characteristic equation is $|A-\lambda I|=0$. In this particular case

$$\begin{vmatrix} -4-\lambda & 2 \\ 6 & 7-\lambda \end{vmatrix} = 0$$
$$(-4-\lambda)\cdot(7-\lambda)-12=0$$
$$-28+4\lambda-7\lambda+\lambda^2-12=0$$
$$\lambda^2-3\lambda-40=0$$
$$\lambda = \frac{3\pm\sqrt{9-4\cdot -40}}{2} = \frac{-3\pm 13}{2} = \begin{cases} 8 \\ -5 \end{cases}$$

The two real eigenvalues are $\lambda=8$ and $\lambda=-5$.

Lay, 5.2.4

Andrea Santos Cortés, Nov. 25th, 2014

Find the characteristic equation and the real eigenvalues of the matrix $A = \begin{pmatrix} 8 & 2 \\ 3 & 3 \end{pmatrix}$.

Solution: The characteristic equation is $|A-\lambda I|=0$. In this particular case

$$\begin{vmatrix} 8-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = 0$$
$$(8-\lambda)\cdot(3-\lambda)-6=0$$
$$24-8\lambda-3\lambda+\lambda^2-6=0$$
$$\lambda^2-11\lambda+18=0$$
$$\lambda = \frac{11\pm\sqrt{121-4\cdot 18}}{2} = \frac{11\pm 7}{2} = \begin{cases} 9 \\ 2 \end{cases}$$

The two real eigenvalues are $\lambda=9$ and $\lambda=2$.

Lay, 5.2.5

Andrea Santos Cortés, Nov. 25th, 2014

Find the characteristic equation and the real eigenvalues of the matrix $A = \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$.

Solution: The characteristic equation is $|A-\lambda I|=0$. In this particular case

$$\begin{vmatrix} 8-\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)^2 - 16 = 0$$
$$64 + \lambda^2 - 16\lambda - 16 = 0$$
$$\lambda^2 - 16\lambda + 48 = 0$$
$$\lambda = \tfrac{16 \pm \sqrt{256 - 4\cdot 48}}{2} = \tfrac{16\pm 8}{2} = \left\{ \begin{array}{r} 4 \\ 12 \end{array} \right.$$

The two real eigenvalues are $\lambda = 4$ and $\lambda = 12$.

Lay, 5.2.9

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the characteristic equation and the real eigenvalues of the matrix $A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{pmatrix}$.

Solution: The characteristic equation is $|A - \lambda I| = 0$. In this particular case

$$\begin{vmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

We calculate this determinant by expanding the factors and cofactors of the second column. Disregarding the cofactors multiplied by a zero value, we have

$$(4-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$
$$(4-\lambda)((4-\lambda)(2-\lambda)+1) = 0$$

Now, we factorize the term $(4-\lambda)(2-\lambda)+1$

$$(4-\lambda)(2-\lambda)+1 = (8+\lambda^2-6\lambda)+1 = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2$$

So the characteristic equation is

$$(4-\lambda)(\lambda-3)^2 = 0$$

whose solutions are $\lambda = 4$ and $\lambda = 3$ (with multiplicity 2).

Lay, 5.2.13

Ana Sanmartin, Jan. 16th, 2015

Find the characteristic polynomial using either a cofactor expansion or the special formula of 3×3 determinants of the matrix $A = \begin{pmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{pmatrix}$.

Solution: The characteristic polynomial is $|A - \lambda I| = 0$. In this particular case:

$$\begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 9-\lambda & 0 \\ 5 & 8 & 3-\lambda \end{vmatrix} = 0$$

We calculate this determinant by expanding the factors and cofactors of the third column. Disregarding the cofactors multiplied by a zero value we have

$$(3-\lambda)\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)((6-\lambda)(9-\lambda)-4) = 0$$

Now we factorize the term $(6-\lambda)(9-\lambda)-4$

$$(6-\lambda)(9-\lambda)-4=(54-15\lambda+\lambda^2)-4=50-15\lambda+\lambda^2=(\lambda-10)(\lambda-5)$$

So the characteristic equation is

$$(3-\lambda)(\lambda-10)(\lambda-5)=0$$

whose solutions are $\lambda=3$, $\lambda=10$ and $\lambda=5$.

Lay, 5.2.18

Carlos Oscar Sorzano, Aug. 31st, 2013

It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace of $\lambda=4$ is two dimensional.

$$A=\begin{pmatrix}4&2&3&3\\0&2&h&3\\0&0&4&14\\0&0&0&2\end{pmatrix}$$

Solution: Let's calculate the eigenspace associated to the eigenvalue $\lambda=4$. For doing so we solve the homogenous equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$ making use of the augmented matrix below

$$\left(\begin{array}{cccc|c}0&2&3&3&0\\0&-2&h&3&0\\0&0&0&14&0\\0&0&0&-2&0\end{array}\right)\sim\left(\begin{array}{cccc|c}0&1&\frac{3}{2}&0&0\\0&0&h+3&0&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right)$$

Note that we have not made the elements $a_{13}=0$ and $a_{23}=1$ because for doing that we would need to perform the row operations

$$\begin{array}{c}\mathbf{r}_1\leftarrow\mathbf{r}_1-\frac{\frac{3}{2}}{h+3}\mathbf{r}_2\\ \mathbf{r}_2\leftarrow\frac{1}{h+3}\mathbf{r}_2\end{array}$$

which are not allowed if $h=-3$. If $h\neq-3$, then the eigenspace is formed by all the vectors of the form $\{x_1,0,0,0\}$ whose dimension is 1. If $h=-3$, then the eigenspace is formed by all the vectors of the form $\{x_1,-\frac{3}{2}x_3,x_3,0\}$ whose dimension is 2.

Lay, 5.2.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $n\times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1,\lambda_2,...,\lambda_n$, repeated according to multiplicities, so that

$$\det\{A-\lambda I\}=(\lambda_1-\lambda)(\lambda_2-\lambda)...(\lambda_n-\lambda)$$

Explain why $\det\{A\}$ is the product of the n eigenvalues of A. (This result is true for any square matrix when complex eigenvalues are considered.)

Solution: Since the equation above is true for any value of λ, we simply take $\lambda=0$ to obtain

$$\det\{A\}=\lambda_1\lambda_2...\lambda_n$$

Lay, 5.2.20

Carlos Oscar Sorzano, Aug. 31st, 2013

Use a property of determinants to show that A and A^T have the same characteristic polynomial.

Solution: The characteristic polynomial of A is given by

$$\det\{A-\lambda I\}$$

For any matrix X, we know that $\det\{X\}=\det\{X^T\}$, then

$$\det\{A-\lambda I\}=\det\{(A-\lambda I)^T\}=\det\{A^T-\lambda I^T\}=\det\{A^T-\lambda I\}$$

But this latter expression is the characteristic polynomial of A^T.

Lay, 5.2.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if $A=QR$ with Q invertible, then A is similar to $A_1=RQ$.

Solution: We remind that the matrices A and B are similar if there exists an invertible matrix P such that $B=P^{-1}AP$ (with $A,B,P\in\mathcal{M}_{n\times n}$). This means that we need to find an invertible matrix P such that

$$\begin{array}{c} A_1=P^{-1}AP \\ RQ=P^{-1}QRP \end{array}$$

If we make $P=Q$, since Q is invertible, we have $P^{-1}=Q^{-1}$ and

$$RQ=Q^{-1}QRQ=RQ$$

So, we have proven that A and A_1 are similar.

Lay, 5.2.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if A and B are similar, then $\det\{A\}=\det\{B\}$.

Solution: If A and B are similar, then there exists an invertible matrix P such that

$$B=P^{-1}AP$$

Applying the determinant on both sides we have

$$\begin{array}{rcl} \det\{B\} &=& \det\{P^{-1}AP\}=\det\{P^{-1}\}\det\{A\}\det\{P\} \\ &=& \frac{1}{\det\{P\}}\det\{A\}\det\{P\}=\det\{A\} \end{array}$$

Lay, 5.3.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A=PDP^{-1}$ with $P=\begin{pmatrix}1&2\\2&3\end{pmatrix}$ and $D=\begin{pmatrix}1&0\\0&3\end{pmatrix}$. Calculate A^4.

Solution: If $A=PDP^{-1}$, then

$$\begin{aligned}
A^4 &= PD^4P^{-1} \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1^4 & 0 \\ 0 & 3^4 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 321 & -160 \\ 480 & -239 \end{pmatrix}
\end{aligned}$$

Lay, 5.3.3

Carlos Oscar Sorzano, Dec. 16th, 2014

Calculate A^k with

$$A = \begin{pmatrix} a & 0 \\ 2(a-b) & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Solution: Since $A = PDP^{-1}$ we can calculate A^k as

$$A^k = PD^kP^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 2(a^k - b^k) & b^k \end{pmatrix}$$

Lay, 5.3.17

Carlos Oscar Sorzano, June, 6th 2014

Diagonalize the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$.

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 2 & 2-\lambda & 0 \\ 2 & 2 & 2-\lambda \end{pmatrix} = (2-\lambda)^3$$

whose only root is $\lambda = 2$ with multiplicity 3. We now analyze the eigenvectors of the matrix $A - 2I$

$$(A - 2I)\mathbf{x} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

It can be easily seen that the corresponding eigenspace has $x_1 = x_2 = 0$, while x_3 is free. That is, $\mathbf{x}$ is of the form $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}$. Consequently, the dimension of this eigenspace is 1. But to be diagonalizable we need that the sum of the dimensions of all eigenspaces is equal to the number of columns of the matrix A (see Theorem 5.3.7), so we conclude that the matrix A is not diagonalizable.

Lay, 5.3.23

Carlos Oscar Sorzano, Aug. 31st, 2013

A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

Solution: According to Theorem 6.3.7, if the sum of the dimensions of the different eigenspaces is equal to the number of columns of A, then A is diagonalizable. This is the case of the matrix A of the problem for which $3 + 2 = 5$, and consequently A is diagonalizable.

Lay, 5.3.24

Carlos Oscar Sorzano, Aug. 31st, 2013

A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

Solution: According to Theorem 5.3.7b, a matrix is diagonalizable if and only if the sum of the dimensions of all the eigenspaces is equal to the number of rows and columns of the matrix A. Since in this case $1 + 1 = 2 \neq 3$, A is not diagonalizable.

Lay, 5.3.26

Ana Sanmartin, Jan. 16th, 2015

A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

Solution: The matrix A will be diagonalizable if $\mathbf{v}_1, \mathbf{v}_2$ are two independent eigenvectors for the first eigenvalue, $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are three independent eigenvectors for the second eigenvalue, and $\mathbf{p}_1, \mathbf{p}_2$ are two independent eigenvectors for the third eigenvalue. We have to diagonalize A, so we will write A such as $A = PDP^{-1}$. Let us define

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}$$

Then, A is diagonalizable if it can be written as

$$A = P \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix} P^{-1}$$

If A is diagonalizable, then P must be invertible, and for that $\mathbf{p}_1$ and $\mathbf{p}_2$ must be linearly independent. If $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly dependent, then A is not diagonalizable. In other words, A may be non-diagonalizable if the dimension of the 3rd eigenspace is not 2.

Lay, 5.3.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if A is both diagonalizable and invertible, then so is A^{-1}

Solution: If A is diagonalizable, then there exist an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

If A is invertible, then

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$$

So, D is also invertible and we see that A^{-1} is also diagonalizable.

Lay, 5.3.28

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if A has n linearly independent eigenvectors, then so does A^T. [*Hint*: Use the Diagonalization Theorem.]
Solution: The Diagonalization Theorem states that A is diagonalizable if and only if it has n linearly independent eigenvectors, and that in that case, A can be expressed as

$$A = PDP^{-1},$$

where the columns of P are the n linearly independent eigenvectors. Taking the transpose in both sides, we have

$$A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D^T P^T$$

So, A^T is also diagonalizable and, by the Diagonalization Theorem again, it must have n linearly independent eigenvectors.

Lay, 5.3.29

Carlos Oscar Sorzano, Aug. 31st, 2013

The diagonalization of a matrix is not unique. Given the following diagonalization of the matrix A

$$\begin{gathered} A = PDP^{-1} \\ \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} \end{gathered}$$

Now consider a new factorization of the form $A = P_1 D_1 P_1^{-1}$ with $D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$. Find the matrix P_1.
Solution: P_1 is simply the reorganization of the columns in P such that each eigenvector is in the same column as its corresponding eigenvalue in D_1

$$P_1 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

Lay, 5.3.31

Carlos Oscar Sorzano, Aug. 31st, 2013

Construct a 2×2 matrix that is invertible but not diagonalizable.
Solution: The matrix A below is such a matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Its inverse is

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

But it is not diagonalizable. Let's see why. Let's calculate the eigenspace associated to $\lambda = 1$.

$$\begin{gathered} (A - I)\mathbf{v} = \mathbf{0} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0} \end{gathered}$$

whose set of solutions is formed by all vectors of the form $\mathbf{v} = (x_1, 0)$ and its basis is $\{(1,0)\}$. Since the dimension of the eigenspace is 1 and there are 2 columns in A, by Theorem 6.3.7, the matrix is not diagonalizable.

Lay, 5.3.32

Carlos Oscar Sorzano, Aug. 31st, 2013

Construct a 2×2 matrix that is diagonalizable but not invertible.

Solution: Consider $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Now let us construct the matrix

$$A = PDP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

It is obviously diagonalizable by construction, but it is not invertible because one of its eigenvalues is 0, and D is not invertible.

Lay, 5.4.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$\begin{aligned} T(\mathbf{b}_1) &= 3\mathbf{d}_1 - 5\mathbf{d}_2 \\ T(\mathbf{b}_2) &= -\mathbf{d}_1 + 6\mathbf{d}_2 \\ T(\mathbf{b}_3) &= 4\mathbf{d}_2 \end{aligned}$$

Find the matrix of T relative to $\mathcal{B}$ and $\mathcal{D}$

Solution: The matrix sought is

$$M = \begin{pmatrix} [T(\mathbf{b}_1)]_\mathcal{D} & [T(\mathbf{b}_2)]_\mathcal{D} & [T(\mathbf{b}_3)]_\mathcal{D} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{pmatrix}$$

We can apply it as

$$[T(\mathbf{x})]_\mathcal{D} = M[\mathbf{x}]_\mathcal{B}$$

Lay, 5.4.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V, and let $T : \mathbb{R}^3 \to V$ be a linear transformation with the property that

$$T(x_1, x_2, x_3) = (2x_3 - x_2)\mathbf{b}_1 - (2x_2)\mathbf{b}_2 + (x_1 + 3x_3)\mathbf{b}_3$$

a. Compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$ and $T(\mathbf{e}_3)$.

b. Compute $[T(\mathbf{e}_1)]_\mathcal{B}$, $[T(\mathbf{e}_2)]_\mathcal{B}$ and $[T(\mathbf{e}_3)]_\mathcal{B}$.

c. Find the matrix for T relative to $\mathcal{E}$ and $\mathcal{B}$

Solution:

a. Applying the transformation T to the three standard vectors we get

$$\begin{array}{l} T(\mathbf{e}_1) = T(1,0,0) = \mathbf{b}_3 \\ T(\mathbf{e}_2) = T(0,1,0) = -\mathbf{b}_1 - 2\mathbf{b}_2 \\ T(\mathbf{e}_3) = T(0,0,1) = 2\mathbf{b}_1 + 3\mathbf{b}_3 \end{array}$$

b. Let's calculate now the coordinates of the different transformed vectors in $\mathcal{B}$

$$\begin{array}{l} [T(\mathbf{e}_1)]_{\mathcal{B}} = (0,0,1) \\ [T(\mathbf{e}_2)]_{\mathcal{B}} = (-1,-2,0) \\ [T(\mathbf{e}_3)]_{\mathcal{B}} = (2,0,3) \end{array}$$

c. The matrix sought is the one whose columns are the vectors in part b.

$$M = \begin{pmatrix} 0 & -1 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Lay, 5.4.5

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the transformation that maps a polynomial $p(t)$ into the polynomial $(t+3)p(t)$.

a. Find the image of $p(t) = 3 - 2t + t^2$

b. Show that T is a linear transformation

c. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$.

Solution:

a. $T(3 - 2t + t^2) = (t+3)(3 - 2t + t^2) = 9 - 3t + t^2 + t^3$

b. We need to show that $T(p_1(t)+p_2(t)) = T(p_1(t))+T(p_2(t))$ and $T(c(p(t)) = cT(p(t))$

- $T(p_1(t) + p_2(t)) = T(p_1(t)) + T(p_2(t))$

$$\begin{array}{rcl} T(p_1(t) + p_2(t)) & = & (t+3)(p_1(t) + p_2(t)) = (t+3)p_1(t) + (t+3)p_2(t) \\ & = & T(p_1(t)) + T(p_2(t)) \end{array}$$

- $T(c(p(t)) = cT(p(t))$

$$T(c(p(t)) \quad = \quad (t+3)(cp(t)) = c(t+3)p(t) = c((t+3)p(t)) = cT(p(t))$$

c. We need to calculate the transformation of each of the elements in the basis $\{1, t, t^2\}$

$$\begin{array}{rcl} T(1) & = & (t+3)1 = t+3 \\ T(t) & = & (t+3)t = t^2 + 3t \\ T(t^2) & = & (t+3)t^2 = t^3 + 3t^2 \end{array}$$

The matrix sought is

$$M = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Lay, 5.4.9

Carlos Oscar Sorzano, Feb. 15th, 2014

Define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by $T(p(t)) = (p(-1), p(0), p(1))^T$.

1. Find the image under T of $p(t) = 5 + 3t$.
2. Show that T is a linear transformation.
3. Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis of $\mathbb{R}^3$.

Solution:

1. $T(5 + 3t) = (5 + 3 \cdot (-1), 5 + 3 \cdot 0, 5 + 3 \cdot (1))^T = (2, 5, 8)^T$
2. To show that T is a linear transformation we will show that

$$\begin{aligned} T(k_1p_1(t) + k_2p_2(t)) &= k_1T(p_1(t)) + k_2T(p_2(t)) \\ T(k_1p_1(t) + k_2p_2(t)) &= ((k_1p_1 + k_2p_2)(-1), (k_1p_1 + k_2p_2)(0), (k_1p_1 + k_2p_2)(1))^T \\ &= (k_1p_1(-1) + k_2p_2(-1), k_1p_1(0) + k_2p_2(0), k_1p_1(1) + k_2p_2(1))^T \\ &= (k_1p_1(-1), k_1p_1(0), k_1p_1(1))^T + (k_2p_2(-1), k_2p_2(0), k_2p_2(1))^T \\ &= k_1(p_1(-1), p_1(0), p_1(1))^T + k_2(p_2(-1), p_2(0), p_2(1))^T \\ &= k_1T(p_1(t)) + k_2T(p_2(t)) \end{aligned}$$

3. Let us refer to $\{1, t, t^2\}$ as $\{p_0(t), p_1(t), p_2(t)\}$. The required matrix is given by

$$M = \begin{pmatrix} T(p_0(t)) & T(p_1(t)) & T(p_2(t)) \end{pmatrix} = \begin{pmatrix} p_0(-1) & p_1(-1) & p_2(-1) \\ p_0(0) & p_1(0) & p_2(0) \\ p_0(1) & p_1(1) & p_2(1) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Lay, 5.4.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property $[T]_\mathcal{B}$ is diagonal.

Solution: Let's diagonalize A

$$A = PDP^{-1} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} =$$
$$\begin{pmatrix} -0.3162 & -0.7071 \\ -0.9487 & -0.7071 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -0.3162 & -0.7071 \\ -0.9487 & -0.7071 \end{pmatrix}^{-1}$$

If we construct the basis $\mathcal{B} = \{(-0.3162, -0.9487), (-0.7071, -0.7071)\}$ is a basis in which the matrix of T relative to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

Lay, 5.4.18

Carlos Oscar Sorzano, Aug. 31st, 2013

Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 3×3 matrix with eigenvalues 5, 5 and -2. Does there exist a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that the $\mathcal{B}$-matrix of T is a diagonal matrix? Discuss.
Solution: It depends on whether A is diagonalizable or not. Since A does not have all its eigen values distinct, the condition is (see Theorem 5.3.7) that the dimension of the eigenspace associated to eigenvalue 5 is 2, and that the dimension of the eigenspace associated to eigenvalue -2 is 1.

Lay, 5.4.21

Carlos Oscar Sorzano, Dec. 16th, 2014

Show that if B is similar to A and C is similar to A, then B is similar to C.
Solution: If B and C are similar to A is because there exist two invertible matrices P_B and P_C such that

$$B = P_B^{-1} A P_B$$

$$C = P_C^{-1} A P_C$$

From the second equation we see that

$$A = P_C C P_C^{-1}$$

Substituting into the first one

$$B = P_B^{-1} P_C C P_C^{-1} P_B = (P_C^{-1} P_B)^{-1} C (P_C^{-1} P_B)$$

From which it can be seen that B is similar to C because there exists an invertible matrix $P = P_C^{-1} P_B$ such that B can be written as

$$B = P^{-1} C P$$

Lay, 5.4.22

Carlos Oscar Sorzano, Aug. 31st, 2013

If A is diagonalizable and B is similar to A, then B is also diagonalizable.
Solution: If A is diagonalizable, there exist an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

If B is similar to A, then there exists an invertible matrix Q such that

$$B = QAQ^{-1}$$

Combining both results we have

$$B = Q(PDP^{-1})Q^{-1} = (QP)D(P^{-1}Q^{-1})$$

So B is also diagonalizable since it can be expressed as

$$B = P'D(P')^{-1}$$

being $P' = QP$ an invertible matrix and D a diagonal matrix.

Lay, 5.4.23

Carlos Oscar Sorzano, Aug. 31st, 2013

If $B = P^{-1}AP$ and $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding also to an eigenvalue λ.
Solution: Let's check whether the statement proposed by the problem is true or not. If it is true, it means that

$$B(P^{-1}\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$$

According to the problem we have that $B = P^{-1}AP$, so

$$B(P^{-1}\mathbf{x}) = (P^{-1}AP)(P^{-1}\mathbf{x}) = P^{-1}A\mathbf{x}$$

But by hypothesis $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ, that is $A\mathbf{x} = \lambda\mathbf{x}$. Consequently,

$$P^{-1}A\mathbf{x} = P^{-1}(\lambda\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$$

Finally, we have proven that, as stated by the problem,

$$B(P^{-1}\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$$

Lay, 5.4.25

Carlos Oscar Sorzano, Aug. 31st, 2013

The trace of a square matrix A is the sum of the diagonal entries in A and is denoted as $\mathrm{tr}\{A\}$. It can be verified that $\mathrm{tr}\{FG\} = \mathrm{tr}\{GF\}$ for any two $n \times n$ matrices F and G. Show that if A and B are similar, then $\mathrm{tr}\{A\} = \mathrm{tr}\{B\}$.
Solution: If A and B are similar, then there exists an invertible matrix P such that

$$B = PAP^{-1}$$

Taking the trace of both sides

$$\begin{array}{rcll}
\mathrm{tr}\{B\} & = & \mathrm{tr}\{PAP^{-1}\} & \\
& = & \mathrm{tr}\{P(AP^{-1})\} & \text{[by trace property]} \\
& = & \mathrm{tr}\{(AP^{-1})P\} & \\
& = & \mathrm{tr}\{A(P^{-1}P)\} & \\
& = & \mathrm{tr}\{A\} &
\end{array}$$

Lay, 5.4.26

Carlos Oscar Sorzano, Aug. 31st, 2013

It can be shown that the trace of a matrix equals the sum of its eigenvalues. Verify this statement for the case when A is diagonalizable.
Solution: If A is diagonalizable, then $A = PDP^{-1}$, that is D is similar to A. Then, by Exercise 5.4.25

$$\mathrm{tr}\{A\} = \mathrm{tr}\{D\} = \sum_{i=1}^{n} \lambda_i$$

Lay, 5.4.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Let V be $\mathbb{R}^n$ with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$; let W be $\mathbb{R}^n$ with the standard basis, denoted here by $\mathcal{E}$; and consider the identity transformation $I : V \to W$, $I(\mathbf{x}) = \mathbf{x}$. Find the matrix for I relative to $\mathcal{B}$ and $\mathcal{E}$. What was this matrix called in the context of coordinate systems (Section 4.4)?
Solution: The transformation matrix is given by

$$\begin{array}{rcl} M & = & \begin{pmatrix} [I(\mathbf{b}_1)]_{\mathcal{E}} & [I(\mathbf{b}_2)]_{\mathcal{E}} & ... & [I(\mathbf{b}_n)]_{\mathcal{E}} \end{pmatrix} \\ & = & \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & ... & \mathbf{b}_n \end{pmatrix} \end{array}$$

This was the change of coordinates matrix in Section 4.4, denoted as $P_{\mathcal{E}\leftarrow\mathcal{B}}$.

Lay, 5.5.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Let the matrix $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ act on $\mathbb{C}^2$. Find the eigenvalues and a basis for each of the eigenspace in $\mathbb{C}^2$.
Solution: The eigenvalues are the solutions of the characteristic equation

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = 0$$

To solve this equation we multiply the two monomials to get

$$|A - \lambda I| = \lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = 2 \pm i$$

The two eigenvalues are complex, and the characteristic polynomial can be factorized as

$$|A - \lambda I| = (\lambda - (2+i))(\lambda - (2-i))$$

Let's find now a basis for each one of the eigenspaces.

$\underline{\lambda = 2 + i}$
We need to solve the homogeneous equation system $(A - \lambda I)\mathbf{v} = \mathbf{0}$

$$\begin{pmatrix} 1-(2+i) & -2 \\ 1 & 3-(2+i) \end{pmatrix} \mathbf{v} = \mathbf{0}$$

We use the augmented matrix below

$$\left(\begin{array}{cc|c} -1-i & -2 & 0 \\ 1 & 1-i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1-i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

All vectors in this eigenspace are of the form $\mathbf{v} = ((-1+i)x_2, x_2) \quad x_2 \in \mathbb{R}$. One of its bases is $\{(-1+i, 1)\}$

$\underline{\lambda = 2 - i}$
We need to solve the homogeneous equation system $(A - \lambda I)\mathbf{v} = \mathbf{0}$

$$\begin{pmatrix} 1-(2-i) & -2 \\ 1 & 3-(2-i) \end{pmatrix}\mathbf{v} = \mathbf{0}$$

We use the augmented matrix below

$$\left(\begin{array}{cc|c} -1+i & -2 & 0 \\ 1 & 1+i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1+i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

All vectors in this eigenspace are of the form $\mathbf{v} = ((-1-i)x_2, x_2) \quad x_2 \in \mathbb{R}$. One of its bases is $\{(-1-i, 1)\}$

In fact this is a general result, if λ and λ^* are two complex conjugate eigenvalues, then their corresponding bases are also related by a complex conjugate operation.

Lay, 5.5.3

Ignacio Sánchez López, Dec. 15th, 2014

Let the matrix $A = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$ act on $\mathbb{C}^2$. Find the eigenvalues and a basis for each of the eigenspace in $\mathbb{C}^2$.

Solution: The eigenvalues are the solutions of the characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 5 \\ -2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 10 = (\lambda - (2+3i))(\lambda - (2-3i)) = 0$$

The two eigenvalues are complex. Let's find now a basis for each one of the eigenspaces.

$\underline{\lambda = 2+3i}$

We need to solve the homogeneous equation system $(A - \lambda I)\mathbf{v} = \mathbf{0}$

$$\begin{pmatrix} 1-(2+3i) & 5 \\ -2 & 3-(2+3i) \end{pmatrix}\mathbf{v} = \mathbf{0}$$

We use the augmented matrix below

$$\left(\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -\frac{1}{2}+\frac{3}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

All vectors in this eigenspace are of the form $\mathbf{v} = \left(\left(\frac{1}{2} - \frac{3}{2}i\right)x_2, x_2\right) \quad x_2 \in \mathbb{C}$. One of its bases is $\{(1-3i, 2)\}$

$\underline{\lambda = 2-3i}$

For the second eigenvalue, we do not need to perform the same analysis because the two eigenvalues are complex conjugate (because they come from a real-valued matrix). Consequently, the two eigenspaces are also complex conjugates. All vectors in this eigenspace are of the form $\mathbf{v} = \left(\left(\frac{1}{2} + \frac{3}{2}i\right)x_2, x_2\right) \quad x_2 \in \mathbb{R}$. One of its bases is $\{(1+3i, 2)\}$.

Lay, 5.5.7

Carlos Oscar Sorzano, Aug. 31st, 2013

Let the matrix $A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(\mathbf{x}) = A\mathbf{x}$. T is the composition of a scaling and a rotation. Give the scaling factor and the rotation angle.

Solution: The eigenvalues of A are

$$|A - \lambda I| = 0$$
$$\begin{vmatrix} \sqrt{3}-\lambda & -1 \\ 1 & \sqrt{3}-\lambda \end{vmatrix} = (\sqrt{3}-\lambda)^2 + 1 = 0$$
$$\sqrt{3}-\lambda = \pm i$$
$$\lambda = \sqrt{3} \pm i = 2e^{\pm i30^\circ}$$

The scaling factor is 2 and the rotation angle 30° or -30°(in fact looking only at the eigenvalues we cannot determine the sign of the rotation). However We note that

$$A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{pmatrix}$$

So, the rotation angle is 30°.

Lay, 5.5.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Let the matrix $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$. Find an invertible matrix P and a matrix C of the form $\begin{pmatrix} b & -a \\ a & b \end{pmatrix}$ such that $A = PCP^{-1}$.

Solution: As in Exercise 5.5.1, the eigenvalues of A are $\lambda = 2 \pm i$. A basis of the eigenvalue $\lambda = 2 - i$ is $\{(-1-i), 1\}$. According to Theorem 5.5.9, $\lambda = a - bi$ and $\mathbf{v}$ is a basis of its eigenspace, we find the P and C matrices as

$$P = \begin{pmatrix} \mathrm{Re}\{\mathbf{v}\} & \mathrm{Im}\{\mathbf{v}\} \end{pmatrix}$$
$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

In this case,

$$P = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

It can be easily verified that

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$$

Lay, 5.5.21

Carlos Oscar Sorzano, Feb. 15th, 2014

Let $A = \begin{pmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{pmatrix}$, whose eigenvalues are $0.8 \pm 0.6i$. The equation to determine the eigenvectors associated to the eigenvalue $0.8 - 0.6i$ are

$$A - (0.8 - 0.6i)\mathbf{v} = \mathbf{0} \Leftrightarrow \begin{array}{rcrcl} (-0.3 + 0.6i)x_1 & - & 0.6x_2 & = & 0 \\ 0.75x_1 & + & (0.3 + 0.6i)x_2 & = & 0 \end{array}$$

Solve the first equation above for x_2 and from that produce the eigenvector $\mathbf{v} = (2, -1 + 2i)$. Show that $\mathbf{v}$ is a (complex) multiple of $\mathbf{v}_1 = (-2 - 4i, 5)$.

Solution:

$$x_2 = \frac{-0.3+0.6i}{0.6}x_1 = \left(-\frac{1}{2}+i\right)x_1$$

The corresponding eigenvector is, therefore,

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \left(-\frac{1}{2}+i\right)x_1 \end{pmatrix}$$

For $x_1 = 2$ we get

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1+2i \end{pmatrix}$$

Let k be such that $\mathbf{v} = k\mathbf{v}_1$. That means

$$\begin{pmatrix} 2 \\ -1+2i \end{pmatrix} = \begin{pmatrix} (-2-4i)k \\ 5k \end{pmatrix}$$

From where

$$\begin{array}{l} 2 = (-2-4i)k \\ -1+2i = 5k \end{array}$$

and

$$k = \frac{-1+2i}{5} = -\frac{1}{5}+\frac{2}{5}i$$

We still need to prove that

$$2 = (-2-4i)k$$

But this is actually the case because $(-2-4i)\left(-\frac{1}{5}+\frac{2}{5}i\right) = 2$. So $\mathbf{v}$ is a multiple of $\mathbf{v}_1$ and the proportionality constant is $-\frac{1}{5}+\frac{2}{5}i$.

Lay, 5.5.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $n\times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q = (\mathbf{x}^*)^T A\mathbf{x}$. Show that q is a real number.
Solution: We need to show that $q^* = q$.

$$\begin{array}{rcll} q^* & = & ((\mathbf{x}^*)^T A\mathbf{x})^* & [(AB)^* = A^*B^*; (\mathbf{x}^*)^T = (\mathbf{x}^T)^*)] \\ & = & \mathbf{x}^T A^* \mathbf{x}^* & [\text{by hypothesis A is real}] \\ & = & \mathbf{x}^T A \mathbf{x}^* & [q^T = q; (ABC)^T = C^T B^T A^T] \\ & = & (\mathbf{x}^*)^T A^T \mathbf{x} & [\text{by hypothesis } A^T = A] \\ & = & (\mathbf{x}^*)^T A\mathbf{x} & \\ & = & q & \end{array}$$

Lay, 5.5.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $n\times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}\in\mathbb{C}^n$, then, in fact, λ is real and the real part of $\mathbf{x}$ is an eigenvector of A. [*Hint*: Compute $(\mathbf{x}^*)^T A\mathbf{x}$ and use Exercise 5.5.23. Also, examine the real part of $A\mathbf{x}$.]
Solution: Let us calculate $q = (\mathbf{x}^*)^T A\mathbf{x}$

$$q = (\mathbf{x}^*)^T A\mathbf{x} = (\mathbf{x}^*)^T (A\mathbf{x}) = (\mathbf{x}^*)^T (\lambda\mathbf{x}) = \lambda\|\mathbf{x}\|^2$$

Since $\|\mathbf{x}\|^2$ is a real number and q is a real number, then λ is a real number.

Let us calculate the real part on both sides of the equation $A\mathbf{x} = \lambda\mathbf{x}$

$$\begin{aligned} \text{Real}\{A\mathbf{x}\} &= \text{Real}\{\lambda\mathbf{x}\} \quad \text{[A and } \lambda \text{ are real]} \\ A\text{Real}\{\mathbf{x}\} &= \lambda\text{Real}\{\mathbf{x}\} \end{aligned}$$

So Real$\{\mathbf{x}\}$ is an eigenvector of A.

Lay, 5.5.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be a real $n \times n$ matrix, and let $\mathbf{x} \in \mathbb{C}^n$. Show that Real$\{A\mathbf{x}\} = A$Real$\{\mathbf{x}\}$ and Imag$\{A\mathbf{x}\} = A$Imag$\{\mathbf{x}\}$.

Solution: Consider

$$\mathbf{x} = \text{Real}\{\mathbf{x}\} + i\text{Imag}\{\mathbf{x}\}$$

Multiplying on both sides by A on the left

$$\begin{aligned} A\mathbf{x} &= A(\text{Real}\{\mathbf{x}\} + i\text{Imag}\{\mathbf{x}\}) \\ &= A\text{Real}\{\mathbf{x}\} + iA\text{Imag}\{\mathbf{x}\} \end{aligned}$$

Now simply by taking the real and imaginary parts of $A\mathbf{x}$ and taking into account that A is a real matrix, we get the properties proposed:

$$\begin{aligned} \text{Real}\{A\mathbf{x}\} &= A\text{Real}\{\mathbf{x}\} \\ \text{Imag}\{A\mathbf{x}\} &= A\text{Imag}\{\mathbf{x}\} \end{aligned}$$

Lay, 5.5.26

Carlos Oscar Sorzano, Jan. 20th, 2014

Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector in $\mathbf{v} \in \mathbb{C}^2$.

1. Show that AReal$\{\mathbf{v}\} = a$Real$\{\mathbf{v}\}+b$Imag$\{\mathbf{v}\}$ and AImag$\{\mathbf{v}\} = -b$Real$\{\mathbf{v}\}+ a$Imag$\{\mathbf{v}\}$ (*Hint:* Write $\mathbf{v} = $ Real$\{\mathbf{v}\} + i$Imag$\{\mathbf{v}\}$ and compute $A\mathbf{v}$.)
2. Verify that if A is diagonalized as $A = PCP^{-1}$, then $AP = PC$.

Solution:

1. Let us calculate $A\mathbf{v}$

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ &= (a - bi)\text{Real}\{\mathbf{v}\} + i\text{Imag}\{\mathbf{v}\} \\ &= a\text{Real}\{\mathbf{v}\} + ia\text{Imag}\{\mathbf{v}\} - bi\text{Real}\{\mathbf{v}\} + b\text{Imag}\{\mathbf{v}\} \\ &= (a\text{Real}\{\mathbf{v}\} + b\text{Imag}\{\mathbf{v}\}) + i(a\text{Imag}\{\mathbf{v}\} - b\text{Real}\{\mathbf{v}\}) \end{aligned} \tag{8}$$

2. It suffices to multiply by P on the right to get

$$AP = (PCP^{-1})P = PC(P^{-1}P) = PC \tag{9}$$

Lay, 5.Suppl.2

Carlos Oscar Sorzano, Jan. 19th 2015

Show that if $\mathbf{x}$ is an eigenvector of the product AB and $B\mathbf{x} \neq \mathbf{0}$, then $B\mathbf{x}$ is an eigenvector of BA.

Solution: Let us calculate

$$BA(B\mathbf{x}) = B(AB)\mathbf{x}$$

Since $\mathbf{x}$ is an eigenvector of AB it must be

$$AB\mathbf{x} = \lambda\mathbf{x}$$

Consequently

$$BA(B\mathbf{x}) = B\lambda\mathbf{x} = \lambda(B\mathbf{x})$$

that means that $B\mathbf{x}$ is eigenvector of BA and its eigenvalue is λ.

Strang, 6.2.28

Carlos Oscar Sorzano, Jan. 19th 2015

Heisenberg's uncertainty principle. In quantum mechanics the position matrix P and the momentum matrix Q do not commute. In fact $QP - PQ = I$. Then, we cannot have $P\mathbf{x} = \mathbf{0}$ and at the same time $Q\mathbf{x} = \mathbf{0}$ (unless $\mathbf{x} = \mathbf{0}$).If we know the position exactly, we could not also know the momentum exactly. This is Heisenberg's uncertainty principle

$$\frac{\|P\mathbf{x}\|}{\|\mathbf{x}\|}\frac{\|Q\mathbf{x}\|}{\|\mathbf{x}\|} \geq \frac{1}{2}$$

Assume that $P = P^T$ and $Q = -Q^T$. Then

$$\mathbf{x}^T\mathbf{x} = \mathbf{x}^T(QP - PQ)\mathbf{x} = \mathbf{x}^T(QP)\mathbf{x} - \mathbf{x}^T(PQ)\mathbf{x}$$

Use Schwarz inequality to show why

$$\mathbf{x}^T\mathbf{x} \leq 2\|P\mathbf{x}\|\|Q\mathbf{x}\|$$

or what is the same

$$\frac{\|P\mathbf{x}\|}{\|\mathbf{x}\|}\frac{\|Q\mathbf{x}\|}{\|\mathbf{x}\|} \geq \frac{1}{2}$$

Solution: Let us rewrite

$$\mathbf{x}^T(QP)\mathbf{x} = (\mathbf{x}^TQ)(P\mathbf{x}) = (Q^T\mathbf{x})^T(P\mathbf{x}) = (-Q\mathbf{x})^T(P\mathbf{x}) = -\langle Q\mathbf{x}, P\mathbf{x}\rangle$$

$$-\mathbf{x}^T(PQ)\mathbf{x} = -(\mathbf{x}^TP)(Q\mathbf{x}) = -(P^T\mathbf{x})^T(Q\mathbf{x}) = -(P\mathbf{x})^T(Q\mathbf{x}) = -\langle P\mathbf{x}, Q\mathbf{x}\rangle$$

Then

$$\|\mathbf{x}\|^2 = \mathbf{x}^T(QP)\mathbf{x} - \mathbf{x}^T(PQ)\mathbf{x} = -2\langle Q\mathbf{x}, P\mathbf{x}\rangle$$

from where

$$\langle Q\mathbf{x}, P\mathbf{x}\rangle = -\frac{1}{2}\|\mathbf{x}\|^2$$

Taking absolute values on both sides

$$|\langle Q\mathbf{x}, P\mathbf{x}\rangle| = \frac{1}{2}\|\mathbf{x}\|^2$$

Schwarz inequality states that

$$\frac{1}{2}\|\mathbf{x}\|^2 = |\langle Q\mathbf{x}, P\mathbf{x}\rangle| \leq \|Q\mathbf{x}\|\|P\mathbf{x}\|$$

or what is the same

$$\frac{\|P\mathbf{x}\|}{\|\mathbf{x}\|}\frac{\|Q\mathbf{x}\|}{\|\mathbf{x}\|} \geq \frac{1}{2}$$

6 Chapter 6

Lay, 6.1.15

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{a} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$. Determine if both vectors are orthgonal.

Solution: $\mathbf{a}$ is orthogonal to $\mathbf{b}$ if their inner product is 0

$$\mathbf{a} \cdot \mathbf{b} = 0$$
$$8 \cdot (-2) + (-5) \cdot (-3) = -16 - 15 = -31 \neq 0$$

So, they are not orthogonal.

Lay, 6.1.16

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u} = \begin{pmatrix} 12 \\ 3 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$. Determine if both vectors are orthgonal.

Solution: $\mathbf{u}$ is orthogonal to $\mathbf{v}$ if their inner product is 0

$$\mathbf{u} \cdot \mathbf{v} = 0$$
$$12 \cdot (2) + (3) \cdot (-3) + (-5) \cdot (3) = 24 - 9 - 15 = 0$$

So, they are orthogonal.

Lay, 6.1.22

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u} = (u_1, u_2, u_3)$. Explain why $\mathbf{u} \cdot \mathbf{u} \geq 0$. When is $\mathbf{u} \cdot \mathbf{u} = 0$?

Solution:

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2$$

Any real number squared is non-negative, and the sum of non-negative numbers is also non-negative. So, $\mathbf{u} \cdot \mathbf{u} \geq 0$. $\mathbf{u} \cdot \mathbf{u} = 0$ if all its terms are 0, that is, $u_1 = u_2 = u_3 = 0$, or what is the same, if $\mathbf{u} = \mathbf{0}$.

Lay, 6.1.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Verify the *Parallelogram Law* for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2=2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2$$

Solution: We know that $\|\mathbf{x}\|^2=\mathbf{x}\cdot\mathbf{x}$. In $\mathbb{R}^n$ the most standard inner product is defined as

$$\mathbf{x}\cdot\mathbf{y}=\mathbf{x}^T\mathbf{y}$$

Then,

$$\begin{array}{rcl}\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 &=& (\mathbf{u}+\mathbf{v})^T(\mathbf{u}+\mathbf{v})+(\mathbf{u}-\mathbf{v})^T(\mathbf{u}-\mathbf{v})\\ &=& (\|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\mathbf{u}\cdot\mathbf{v})+(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\mathbf{u}\cdot\mathbf{v})\\ &=& 2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2\end{array}$$

Lay, 6.1.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}=\begin{pmatrix}5\\-6\\7\end{pmatrix}$, and let W the set of all $\mathbf{x}\in\mathbb{R}^3$ such that $\mathbf{u}\cdot\mathbf{x}=0$. What theorem of Chapter 4 can be used to show that W is a subspace of $\mathbb{R}^3$? Describe W in geometric language.

Solution: We may use Theorem 4.2.2 in which it is stated that the null space of an $m\times n$ matrix is a subspace of $\mathbb{R}^n$. We simply need to use the matrix $A=\mathbf{u}^T$. Its null space is formed by all those vectors such that

$$A\mathbf{x}=\mathbf{u}^T\mathbf{x}=\mathbf{u}\cdot\mathbf{x}=0$$

Geometrically, W is formed by the plane through the origin and perpendicular to the vector $\mathbf{u}=\begin{pmatrix}5\\-6\\7\end{pmatrix}$.

Lay, 6.1.27

Carlos Oscar Sorzano, Feb. 15th, 2014

Suppose a vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to $\mathbf{u}+\mathbf{v}$.

Solution: If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{y}\cdot\mathbf{u}=0$ and $\mathbf{y}\cdot\mathbf{v}=0$. We now calculate the dot product between $\mathbf{y}$ and $\mathbf{u}+\mathbf{v}$

$$\mathbf{y}\cdot(\mathbf{u}+\mathbf{v})=\mathbf{y}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v}=0+0=0$$

Lay, 6.1.28

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in Span$\{\mathbf{u},\mathbf{v}\}$. [*Hint*: An arbitrary $\mathbf{w}$ in Span$\{\mathbf{u},\mathbf{v}\}$ has the form $\mathbf{w}=c_1\mathbf{u}+c_2\mathbf{v}$.]

Solution: Let us calculate the inner product between $\mathbf{y}$ and $\mathbf{w}$

$$\begin{array}{rcl}\mathbf{y}\cdot\mathbf{w} &=& \mathbf{y}\cdot(c_1\mathbf{u}+c_2\mathbf{v})\\ &=& c_1\mathbf{y}\cdot\mathbf{u}+c_2\mathbf{y}\cdot\mathbf{v}\\ &=& c_1\cdot 0+c_2\cdot 0\\ &=& 0\end{array}$$

where we have made used of the fact that $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, that is, $\mathbf{y} \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{v} = 0$.

Lay, 6.1.30

Carlos Oscar Sorzano, Aug. 31st, 2013

Let W be a subspace of $\mathbb{R}^n$, and let $W^\perp$ be the set of all vectors ortohogonal to W. Show that $W^\perp$ is a subspace of $\mathbb{R}^n$ using the following steps:

a. Take $\mathbf{z} \in W^\perp$ and let $\mathbf{u}$ represent any vector in W. Then, $\mathbf{z} \cdot \mathbf{u} = 0$. Take any scalar c and show that $c\mathbf{z}$ is orthogonal to $\mathbf{u}$. (Since $\mathbf{u}$ is any arbitrary vector in W, this will show that $c\mathbf{z}$ is in $W^\perp$.)

b. Take $\mathbf{z}_1, \mathbf{z}_2 \in W^\perp$, and let $\mathbf{u}$ be any vector in W. Show that $\mathbf{z}_1 + \mathbf{z}_2$ is orthogonal to $\mathbf{u}$. What can you conclude about $\mathbf{z}_1 + \mathbf{z}_2$? Why?

c. Finish the proof that $W^\perp$ is a subspace of $\mathbb{R}^n$

Solution:

a. Let us calculate $(c\mathbf{z}) \cdot \mathbf{u}$

$$(c\mathbf{z}) \cdot \mathbf{u} = c(\mathbf{z} \cdot \mathbf{u}) = c \cdot 0 = 0$$

So $c\mathbf{z}$ is orthogonal to any vector $\mathbf{u}$ in W, and consequently $c\mathbf{z}$ belongs to $W^\perp$.

b. Let us calculate $(\mathbf{z}_1 + \mathbf{z}_2) \cdot \mathbf{u}$

$$(\mathbf{z}_1 + \mathbf{z}_2) \cdot \mathbf{u} = \mathbf{z}_1 \cdot \mathbf{u} + \mathbf{z}_2 \cdot \mathbf{u} = 0 + 0 = 0$$

So $\mathbf{z}_1 + \mathbf{z}_2$ is orthogonal to any vector $\mathbf{u}$ in W, and consequently $\mathbf{z}_1 + \mathbf{z}_2$ belongs to $W^\perp$.

c. We still need to show that $\mathbf{0} \in W^\perp$

$$\mathbf{0} \cdot \mathbf{u} = 0$$

So $\mathbf{0} \in W^\perp$.

Lay, 6.2.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}_1 = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$. Is the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ orthogonal?

Solution: To check whether S is orthogonal, we calculate all possible inner products to check if they are 0 or not

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 6$$
$$\mathbf{u}_1 \cdot \mathbf{u}_3 = -40$$
$$\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$$

Only $\mathbf{u}_2$ is orthogonal to $\mathbf{u}_3$. The rest of vectors are not orthogonal to each other, and consequently, the set S is not orthogonal.

Lay, 6.2.2

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$. Is the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ orthogonal?

Solution: To check whether S is orthogonal, we calculate all possible inner products to check if they are 0 or not

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 \cdot 0 + (-2) \cdot 1 + 1 \cdot 2 = 0 + (-2) + 2 = 0$$
$$\mathbf{u}_1 \cdot \mathbf{u}_3 = 1 \cdot (-5) + (-2) \cdot (-2) + 1 \cdot 1 = (-5) + 4 + 1 = 0$$
$$\mathbf{u}_2 \cdot \mathbf{u}_3 = 0 \cdot (-5) + 1 \cdot (-2) + 2 \cdot 1 = 0 + (-2) + 2 = 0$$

All vectors are orthogonal to each other, and consequently, the set S is orthogonal.

Lay, 6.2.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -7 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -6 \\ -3 \\ 9 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$. Is the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ orthogonal?

Solution: To check whether S is orthogonal, we calculate all possible inner products to check if they are 0 or not

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 2 \cdot (-6) + (-7) \cdot (-3) + (-1) \cdot 9 = 0$$
$$\mathbf{u}_1 \cdot \mathbf{u}_3 = 2 \cdot 3 + (-7) \cdot 1 + (-1) \cdot (-1) = 0$$
$$\mathbf{u}_2 \cdot \mathbf{u}_3 = (-6) \cdot 3 + (-3) \cdot 1 + 9 \cdot (-1) = -30$$

$\mathbf{u}_2$ is not orthogonal to $\mathbf{u}_3$. Therefore, S is not orthogonal.

Lay, 6.2.4

Marta Monsalve Buendía,Dic. 13th, 2014

Let $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$. Is the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ orthogonal?

Solution: To check whether S is orthogonal, we calculate all possible inner products to check if they are 0 or not

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$$
$$\mathbf{u}_1 \cdot \mathbf{u}_3 = -5$$
$$\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$$

$\mathbf{u}_1$ is not orthogonal to $\mathbf{u}_3$. The rest of vectors are orthogonal to each other, and consequently, the set S is not orthogonal.

Lay, 6.2.5

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}_1 = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 4 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 3 \\ 8 \\ 7 \\ 0 \end{pmatrix}$. Is the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ orthogonal?

Solution: To check whether S is orthogonal, we calculate all possible inner products to check if they are 0 or not

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= 0 \\ \mathbf{u}_1 \cdot \mathbf{u}_3 &= 2 \end{aligned}$$

$\mathbf{u}_1$ is not orthogonal to $\mathbf{u}_3$. Therefore, S is not orthogonal.

Lay, 6.2.6

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}_1 = \begin{pmatrix} 5 \\ -4 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -4 \\ 1 \\ -3 \\ 8 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 3 \\ 3 \\ 5 \\ -1 \end{pmatrix}$. Is the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ orthogonal?

Solution: To check whether S is orthogonal, we calculate all possible inner products to check if they are 0 or not

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= 5 \cdot (-4) + (-4) \cdot 1 + 0 \cdot (-3) + 3 \cdot 8 = -20 - 4 + 24 = 0 \\ \mathbf{u}_1 \cdot \mathbf{u}_3 &= 5 \cdot 3 + (-4) \cdot 3 + 0 \cdot 5 + 3 \cdot (-1) = 15 - 12 - 3 = 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_3 &= (-4) \cdot 3 + 1 \cdot 3 + (-3) \cdot 5 + 8 \cdot (-1) = -12 + 3 - 15 - 8 = -32 \end{aligned}$$

$\mathbf{u}_2$ is not orthogonal to $\mathbf{u}_3$. Consequently, the set S is not orthogonal.

Lay, 6.2.9

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$. Show that the set $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$. Let $\mathbf{x} = \begin{pmatrix} 8 \\ -4 \\ -3 \end{pmatrix}$. Express $\mathbf{x}$ as a linear combination of the $\mathbf{u}_i$'s.

Solution: To check whether $\mathcal{B}$ is orthogonal, we calculate all possible inner products to check if they are 0 or not

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= 0 \\ \mathbf{u}_1 \cdot \mathbf{u}_3 &= 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_3 &= 0 \end{aligned}$$

So, the set $\mathcal{B}$ is orthogonal. To express $\mathbf{x}$ as a linear combination of the $\mathbf{u}_i$'s we construct the matrix $A = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$ and solve the equation system $A[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & -2 \end{pmatrix} [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 8 \\ -4 \\ -3 \end{pmatrix} \Rightarrow [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ -4 \\ -3 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -1.5 \\ 2 \end{pmatrix}$$

Lay, 6.2.10

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$. Show that the set $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$. Let $\mathbf{x} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$. Express $\mathbf{x}$ as a linear combination of the $\mathbf{u}_i$'s.

Solution: To check whether $\mathcal{B}$ is orthogonal, we calculate all possible inner products to check if they are 0 or not

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 = 0 \\ \mathbf{u}_1 \cdot \mathbf{u}_3 = 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_3 = 0 \end{aligned}$$

So, the set $\mathcal{B}$ is orthogonal. To express $\mathbf{x}$ as a linear combination of the $\mathbf{u}_i$'s we construct the matrix $A = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$ and solve the equation system $A[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$

$$\begin{pmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix} [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} \Rightarrow [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

Lay, 6.2.15

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{y} = (3, 1)$ and $\mathbf{u} = (8, 6)$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

Solution: Let's compute the first the projection of $\mathbf{y}$ onto $\mathbf{u}$.

$$\mathbf{y}_u = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{30}{100}\begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 2.4 \\ 1.8 \end{pmatrix}$$

The distance asked by the problem is the distance between $\mathbf{y}$ and $\mathbf{y}_u$:

$$d = \|\mathbf{y} - \mathbf{y}_u\| = \left\| \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2.4 \\ 1.8 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0.6 \\ -0.8 \end{pmatrix} \right\| = 1$$

Lay, 6.2.16

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{y} = (-3, 9)$ and $\mathbf{u} = (1, 2)$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

Solution: Let's compute the first the projection of $\mathbf{y}$ onto $\mathbf{u}$.

$$\mathbf{y}_u = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{15}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

The distance asked by the problem is the distance between $\mathbf{y}$ and $\mathbf{y}_u$:

$$d = \|\mathbf{y} - \mathbf{y}_u\| = \left\| \begin{pmatrix} -3 \\ 9 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -6 \\ 3 \end{pmatrix} \right\| = \sqrt{(-6)^2 + 3^2} = \sqrt{45}$$

Lay, 6.2.25

Carlos Oscar Sorzano, Aug. 31st, 2013

Prove the following theorem:
Let $U \in \mathcal{M}_{n\times n}$ be an orthonormal matrix and $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0 \Leftrightarrow \mathbf{x} \cdot \mathbf{y} = 0$

Solution: Let's prove first point b:

$$(U\mathbf{x}) \cdot (U\mathbf{y}) \quad = \quad (U\mathbf{x})^T(U\mathbf{y}) = \mathbf{x}^T U^T U\mathbf{y} = \mathbf{x}^{\mathbf{y}} = \mathbf{x} \cdot \mathbf{y}$$

where we have made used that for any orthonormal matrix $U^TU = I$.

Let's prove now point a:

$$\|U\mathbf{x}\|^2 \quad = \quad (U\mathbf{x}) \cdot (U\mathbf{x}) = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$$

Taking the square root

$$\|U\mathbf{x}\| \quad = \quad \|\mathbf{x}\|$$

Finally, point c. From point b we know that $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$. So, it is obvious that $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ iff $\mathbf{x} \cdot \mathbf{y} = 0$.

Lay, 6.2.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose W is a subspace of $\mathbb{R}^n$ spanned by n non-zero orthogonal vectors. Explain $W = \mathbb{R}^n$.

Solution: A set of orthogonal vectors is always linearly independent (see Theorem 6.2.4). We also know that any set of n linearly independent vectors is a basis of $\mathbb{R}^n$ (see Theorem 4.5.12). So, the same set spans W and $\mathbb{R}^n$, so both sets are equal.

Lay, 6.2.27

Carlos Oscar Sorzano, Feb. 15th, 2014

Let U be a square matrix with orthonormal columns. Explain why U is invertible.

Solution: If U has orthonormal columns, they are linearly independent (Theorem 6.2.4) and its determinant must be different from 0 (Theorem 6.2.6; since $U^TU = I \Rightarrow |U^TU| = |I| = 1 = |U^T||U| = |U|^2 \Rightarrow |U| = \pm 1$). Consequently, the matrix U is invertible (Theorem 5.2.3).

Lay, 6.2.29

Carlos Oscar Sorzano, Aug. 31st, 2013

Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix.

Solution: Let's calculate $(UV)^{-1}$

$$
\begin{aligned}
(UV)^{-1} &= V^{-1}U^{-1} && \text{Properties of matrix inverse; U,V are invertible} \\
&= V^TU^T && \text{U and V are orthogonal matrices} \\
&= (UV)^T && \text{Properties of matrix transpose}
\end{aligned}
$$

So, UV is invertible and its inverse is $(UV)^T$.

Lay, 6.3.1

Carlos Oscar Sorzano, Aug. 31st, 2013

You may assume that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthogonal basis of $\mathbb{R}^4$. Let $\mathbf{u}_1 = (0, 1, -4, -1)$, $\mathbf{u}_2 = (3, 5, 1, 1)$, $\mathbf{u}_3 = (1, 0, 1, -4)$, $\mathbf{u}_4 = (5, -3, -1, 1)$. Let $\mathbf{x} = (10, -8, 2, 0)$. Write $\mathbf{x}$ as the sum of two vectors, one in Span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and the other in Span$\{\mathbf{u}_4\}$.

Solution: We project $\mathbf{x}$ onto Span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

$$
\begin{aligned}
\mathbf{x}_{123} &= \frac{\mathbf{x}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{x}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 \\
&= \frac{-16}{18}\begin{pmatrix} 0 \\ 1 \\ -4 \\ -1 \end{pmatrix} + \frac{-8}{36}\begin{pmatrix} 3 \\ 5 \\ 1 \\ 1 \end{pmatrix} + \frac{12}{18}\begin{pmatrix} 1 \\ 0 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 4 \\ -2 \end{pmatrix} \\
\mathbf{x}_4 &= \frac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 = \frac{72}{36}\begin{pmatrix} 5 \\ -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ -2 \\ 2 \end{pmatrix}
\end{aligned}
$$

It can be easily verified that $\mathbf{x} = \mathbf{x}_{123} + \mathbf{x}_4$.

Lay, 6.3.2

Carlos Oscar Sorzano, Aug. 31st, 2013

You may assume that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthogonal basis of $\mathbb{R}^4$. Let $\mathbf{u}_1 = (1, 2, 1, 1)$, $\mathbf{u}_2 = (-2, 1, -1, 1)$, $\mathbf{u}_3 = (1, 1, -2, -1)$, $\mathbf{u}_4 = (-1, 1, 1, -2)$. Let $\mathbf{v} = (4, 5, -3, 3)$. Write $\mathbf{v}$ as the sum of two vectors, one in Span$\{\mathbf{u}_1\}$ and the other in Span$\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

Solution: We project $\mathbf{v}$ onto Span$\{\mathbf{u}_1\}$ and then onto Span$\{\mathbf{u}_{234}\}$:

$$
\begin{aligned}
\mathbf{v}_1 &= \frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \frac{14}{7}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 2 \end{pmatrix} \\
\mathbf{v}_{234} &= \frac{\mathbf{v}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{v}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 + \frac{\mathbf{v}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 \\
&= \frac{3}{7}\begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \frac{12}{7}\begin{pmatrix} 1 \\ 1 \\ -2 \\ -1 \end{pmatrix} + \frac{-8}{7}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}
\end{aligned}
$$

It can be easily verified that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_{234}$.

Lay, 6.3.5

Ignacio Sánchez López ,Dic. 15th, 2014

Verify that $\mathbf{u}_1$,$\mathbf{u}_2$ is an ortogonal set, and then find the ortogonal proyection of y onto Span$\{\mathbf{u}_1,\mathbf{u}_2\}$. $y = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}$, $\mathbf{u}_1 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$, and $\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Solution: To check if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an ortogonal set we compute $\mathbf{u}_1 \cdot \mathbf{u}_2$ to seee if its inner product is 0 or not.

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 3 \cdot 1 + (-1) \cdot (-1) + 2 \cdot (-2)$$

$\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2$. Then we calculate the ortogonal proyection of $\mathbf{y}$ onto $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\mathbf{x}_W = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \frac{-17}{14}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + \frac{9}{6}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -15 \\ -2 \\ 38 \end{pmatrix}$$

Lay, 6.3.7

Carlos Oscar Sorzano, Aug. 31st, 2013

Let W be the space spanned by $\mathbf{u}_1 = (1, 3, -2)$ and $\mathbf{u}_2 = (5, 1, 4)$ and let $\mathbf{y} = (1, 3, 5)$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

Solution: We project $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$

$$\begin{aligned} \mathbf{x}_W &= \tfrac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \tfrac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\ &= \tfrac{0}{14}\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \tfrac{28}{42}\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{pmatrix} \end{aligned}$$

To find the vector perpendicular to W, we simply calculate

$$\mathbf{x}_{W^\perp} = \mathbf{x} - \mathbf{x}_W = \begin{pmatrix} -\frac{7}{3} \\ \frac{7}{3} \\ \frac{7}{3} \end{pmatrix}$$

By construction, we have $\mathbf{x} = \mathbf{x}_W + \mathbf{x}_{W^\perp}$.

Lay, 6.3.15

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{y} = (5, -9, 5)$, $\mathbf{u}_1 = (-3, -5, 1)$ and $\mathbf{u}_2 = (-3, 2, 1)$. Find the distance from $\mathbf{y} = (5, -9, 5)$ to the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

Solution: We project $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$

$$\begin{aligned} \mathbf{y}_W &= \tfrac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \tfrac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\ &= \tfrac{35}{35}\begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} + \tfrac{-28}{14}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ 1 \end{pmatrix} \end{aligned}$$

To find the vector perpendicular to W, we simply calculate

$$\mathbf{y}_{W^\perp} = \mathbf{y} - \mathbf{y}_W = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$$

The required distance is simply the norm of this vector that is $\sqrt{40}$.

Lay, 6.3.22

Carlos Oscar Sorzano, June, 6th 2014

Matk each statement as true or false. Justify your answer.

1. If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

2. In the Orthogonal Decomposition Theorem (see Theorem 6.3.8), each term of the formula

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$$

is itself a projection of $\mathbf{y}$ onto a subspace of W.

3. If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$ where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

4. The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \text{Proj}_W\{\mathbf{y}\}$.

5. If an $n \times p$ matrix U has orthonormal columns, then $UU^T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Solution:

1. True. $\mathbf{0}$ is the only vector that belongs simultaneously to W and $W^\perp$.

2. True. The term $\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1$ is the orthogonal projection of $\mathbf{y}$ onto the vector space spanned by $\mathbf{u}_1$. This latter space is a subspace of W (the space spanned by the vectors $\mathbf{v}_1$, ..., $\mathbf{v}_p$).

3. True, because it is an orthogonal decomposition ($\mathbf{z}_1 \perp \mathbf{v}_2$) and according to the Orthogonal Decomposition Theorem(see Theorem 6.3.8), this decomposition is unique. Consequently, $\mathbf{v}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

4. False. The best approximation is $\text{Proj}_W\{\mathbf{y}\}$. The proposed vector, $\mathbf{y} - \text{Proj}_W\{\mathbf{y}\}$, is called the residual: the part of $\mathbf{y}$ that cannot be explained by W.

5. False. If U has orthonormal columns, then $UU^T\mathbf{x}$ is the projection of the vector $\mathbf{x}$ onto the subspace spanned by the columns of U. Unless $\mathbf{x}$ is already in W, in general, $\mathbf{x}$ is different from its projection onto W. Consequently, in general, $UU^T\mathbf{x} \neq \mathbf{x}$.

Lay, 6.3.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $m \times n$ matrix. Prove that every vector in $\mathbf{x} \in \mathbb{R}^n$ can be written in the form $\mathbf{x} = \mathbf{p} + \mathbf{u}$, where $\mathbf{p}$ is in $\text{Row}\{A\}$ and $\mathbf{u}$ is in $\text{Nul}\{A\}$. Also, show that if the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then there is a unique $\mathbf{p}$ in $\text{Row}\{A\}$ such that $A\mathbf{p} = \mathbf{b}$.
Solution: First, we'll show that $\text{Row}\{A\}$ and $\text{Nul}\{A\}$ are orthogonal subspaces. Let $\mathbf{a}_i$ ($i = 1, 2, ..., m$) be the rows of matrix A. Any vector $\mathbf{u}$ in $\text{Nul}\{A\}$ is such that

$$A\mathbf{u} = \mathbf{0}$$

In particular, we may consider the multiplication of the i-th row of A and $\mathbf{u}$

$$\mathbf{a}_i^T\mathbf{u} = 0 \Rightarrow \mathbf{a}_i \cdot \mathbf{u} = 0 \Rightarrow \mathbf{a}_i \perp \mathbf{u}$$

Any vector $\mathbf{p}$ in Row$\{A\}$ can be written as a linear combination of the rows of A

$$\mathbf{p} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + ... + c_m\mathbf{a}_m$$

Let's calculate the inner product between $\mathbf{p}$ and $\mathbf{u}$

$$\mathbf{p} \cdot \mathbf{u} = c_1\mathbf{a}_1 \cdot \mathbf{u} + c_2\mathbf{a}_2 \cdot \mathbf{u} + ... + c_m\mathbf{a}_m \cdot \mathbf{u} = 0$$

So $\mathbf{p} \perp \mathbf{u}$ for any $\mathbf{p}$ in Row$\{A\}$ and any $\mathbf{u}$ in Nul$\{A\}$.

Since, both spaces are orthogonal to each other we may orthogonally project $\mathbf{x} \in \mathbb{R}^n$ onto Row$\{A\}$ (obtaining $\mathbf{p}$) and onto Nul$\{A\}$ (obtaining $\mathbf{u}$). By the Orthogonal Decomposition theorem we know that $\mathbf{x}$ can be uniquely decomposed as a vector in Row$\{A\}$ and a vector in $(\text{Row}\{A\})^\perp = \text{Nul}\{A\}$. This proves that $\mathbf{x} = \mathbf{p} + \mathbf{u}$.

For the second part of the problem, let us presume that there are two distinct solutions $\mathbf{p}_1$ and $\mathbf{p}_2$ in Row$\{A\}$ such that

$$A\mathbf{p}_1 = \mathbf{b}$$
$$A\mathbf{p}_2 = \mathbf{b}$$

Subtracting both equations we have

$$A(\mathbf{p}_1 - \mathbf{p}_2) = \mathbf{0}$$

That means that $\mathbf{p}_1 - \mathbf{p}_2$ is in Nul$\{A\}$. But at the same time it is in Row$\{A\}$ (because it is the linear combination of two vectors in Row$\{A\}$ and Row$\{A\}$ is a vector space). The only vector that belongs both to Nul$\{A\}$ and Row$\{A\}$ is the zero vector so

$$\mathbf{p}_1 - \mathbf{p}_2 = \mathbf{0} \Rightarrow \mathbf{p}_1 = \mathbf{p}_2$$

which is a contradiction to our hypothesis that both solutions were distinct, and therefore, there is a single solution $\mathbf{p}$ in Row$\{A\}$ of the problem $A\mathbf{p} = \mathbf{b}$.

Lay, 6.3.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_p\}$, and let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_q\}$ be an ortohogonal basis for $W^\perp$.

a. Explain why $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_p, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_q\}$ is an orthogonal set.

b. Explain why the set in part (a) spans $\mathbb{R}^n$.

c. Show that $\dim\{W\} + \dim\{W^\perp\} = n$

Solution:

a. Since both sets $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_p\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_q\}$ are orthogonal bases, all products $\mathbf{w}_i \cdot \mathbf{w}_j = 0 = \mathbf{v}_i \cdot \mathbf{v}_j$ (for $i \neq j$). We still need to show that the products $\mathbf{w}_i \cdot \mathbf{v}_j = 0$ for any $i \in \{1, 2, ..., p\}$ and $j \in \{1, 2, ..., q\}$. But this is true since $\mathbf{w}_i \in W$ and $\mathbf{v}_j \in W^\perp$.

b. Since W and $W^\perp$ are perpendicular spaces contained in $\mathbb{R}^n$ by the Orthogonal Decomposition Theorem (Theorem 6.3.8) we have that any vector can be decomposed as a sum of a vector in W and a vector in $W^\perp$. But any vector in W can be expressed as a linear combination of the $\mathbf{w}_i$ vectors and any vector in $W^\perp$ can be expressed as a linear combination of the $\mathbf{v}_i$ vectors. So the combined set $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_p, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_q\}$ can generate both parts of the orthogonal decomposition, and consequently, can generate any vector in $\mathbb{R}^n$. In fact, since all the vectors in the set are orthogonal, the set is a basis of $\mathbb{R}^n$.

c. We know that $\dim\{W\} = p$ and $\dim\{W^\perp\} = q$. We need to show that $p+q = n$. But this is true since the set in part (a) has $p+q$ vectors, and we have stated in part (b) that these $p+q$ vectors is a basis for $\mathbb{R}^n$, so it must have exactly n vectors.

Lay, 6.4.7

Carlos Oscar Sorzano, Aug. 31st, 2013

The set $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2\} = \{(2,-5,1),(4,-1,2)\}$ is a basis for a subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W. Then, normalize it to have an orthonormal basis.

Solution: In Gram-Schmidt process, the first vector is any of the vectors in the basis, let's say

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$

The second vector is calculated as any other vector in the basis minus its projection onto the already explained subspace

$$\begin{array}{rcl} \mathbf{v}_2 & = & \mathbf{x}_2 - \mathrm{Proj}_{\mathrm{Span}\{\mathbf{v}_1\}}\{\mathbf{x}_2\} \\ & = & \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\ & = & \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} - \frac{15}{30}\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \end{array}$$

The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis of W. To produce an orthonormal basis, we have to normalize each vector

$$\begin{array}{rcl} \mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} & = & \frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \\ \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} & = & \frac{1}{\sqrt{\frac{27}{2}}}\begin{pmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = \sqrt{\frac{2}{27}}\begin{pmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \end{array}$$

Lay, 6.4.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{pmatrix}$ and $Q = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{5}{6} \\ -\frac{3}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{3}{6} \end{pmatrix}$. The columns of Q were obtained by applying the Gram-Schmidt process to the columns of A. Find an upper triangular matrix R such that $A = QR$.
Solution: Since Q is an orthogonal matrix, its inverse is its transpose $Q^TQ = I$. Then, we simply multiply the decomposition $A = QR$ by Q^T on the left to obtain

$$\begin{gathered} A = QR \\ Q^TA = R \\ R = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{3}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{6} & \frac{3}{6} \end{pmatrix} \begin{pmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 12 \\ 0 & 6 \end{pmatrix} \end{gathered}$$

Lay, 6.4.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose that $A = QR$ where Q is $m \times n$ and R is $n \times n$. Show that if the columns of A are linearly independent, then R must be invertible. [*Hint*: Study the equation $R\mathbf{x} = \mathbf{0}$ and use the fact that $A = QR$.]
Solution: Since Q is orthogonal and meets that $Q^TQ = I$ we have

$$\begin{gathered} A = QR \\ Q^TA = R \end{gathered}$$

Then, the equation $R\mathbf{x} = \mathbf{0}$ becomes

$$Q^TA\mathbf{x} = \mathbf{0}$$

Multiplying both sides by Q we have

$$A\mathbf{x} = Q\mathbf{0} = \mathbf{0}$$

Since the columns of A are linearly independent, the only solution of this problem is $\mathbf{x} = \mathbf{0}$ (see Equation 1.7.3) and, consequently, the only solution of $R\mathbf{x} = \mathbf{0}$ is also $\mathbf{x} = \mathbf{0}$. But this implies, by the Invertible Matrix Theorem, that R is invertible.

Lay, 6.4.20

Carlos Oscar Sorzano, Feb. 15th, 2014

Suppose $A = QR$ where R is an invertible matrix. Show that A and Q have the same column space. [*Hint*: Given $\mathbf{y}$ in Col$\{A\}$, show that $\mathbf{y} = Q\mathbf{x}$ for some $\mathbf{x}$. Also, given $\mathbf{y} \in$ Col$\{Q\}$, show that $\mathbf{y} = A\mathbf{x}$.]
Solution: Consider an arbitrary $\mathbf{y}$ in Col$\{A\}$, that is, there exists a vector $\mathbf{b}$ such that

$$\mathbf{y} = A\mathbf{b} = (QR)\mathbf{b} = Q(R\mathbf{b}) = Q\mathbf{x}$$

where $\mathbf{x} = R\mathbf{b}$. That is $\mathbf{y}$ belongs to the column space of Q.

Alternatively, consider any $\mathbf{y}$ in Col$\{Q\}$, that is, there exists a vector $\mathbf{b}$ such that

$$\mathbf{y} = Q\mathbf{b}$$

Remind that $A = QR$. Since R is invertible, we can write $AR^{-1} = Q$. Consequently, we can rewrite the equation above as

$$\mathbf{y} = (AR^{-1})\mathbf{b} = A(R^{-1}\mathbf{b}) = A\mathbf{x}$$

where $\mathbf{x} = R^{-1}\mathbf{b}$. That is, $\mathbf{y}$ belongs to the column space of A.

Lay, 6.4.22

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\mathbf{x}) = \mathrm{Proj}_W\{\mathbf{x}\}$. Show that T is a linear transformation.

Solution: Since $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_p\}$ is an orthogonal basis of W, the projection onto W can be calculated as

$$T(\mathbf{x}) = \mathrm{Proj}_W\{\mathbf{x}\} = \tfrac{\mathbf{x}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + \tfrac{\mathbf{x}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p$$

To show that T is linear let us show that $T(c\mathbf{x}) = cT(\mathbf{x})$

$$\begin{array}{rcl} T(c\mathbf{x}) & = & \frac{(c\mathbf{x})\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + \frac{(c\mathbf{x})\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \\ & = & c\frac{\mathbf{x}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + c\frac{\mathbf{x}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \\ & = & c\left(\frac{\mathbf{x}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + \frac{\mathbf{x}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p\right) \\ & = & cT(\mathbf{x}) \end{array}$$

and that $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$

$$\begin{array}{rcl} T(\mathbf{x}_1 + \mathbf{x}_2) & = & \frac{(\mathbf{x}_1+\mathbf{x}_2)\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + \frac{(\mathbf{x}_1+\mathbf{x}_2)\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \\ & = & \frac{\mathbf{x}_1\cdot\mathbf{u}_1+\mathbf{x}_2\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + \frac{\mathbf{x}_1\cdot\mathbf{u}_p+\mathbf{x}_2\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \\ & = & \left(\frac{\mathbf{x}_1\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + \frac{\mathbf{x}_1\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p\right) + \left(\frac{\mathbf{x}_2\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + ... + \frac{\mathbf{x}_2\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p\right) \\ & = & T(\mathbf{x}_1) + T(\mathbf{x}_2) \end{array}$$

Lay, 6.5.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ by constructing the normal equations for $\hat{\mathbf{x}}$ and solving for it with $A = \begin{pmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$.

Solution: The normal equations are given by

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

In this particular case

$$\begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix}\hat{\mathbf{x}} = \begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix}\hat{\mathbf{x}} = \begin{pmatrix} -4 \\ 11 \end{pmatrix}$$

$$\hat{\mathbf{x}} = \begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix}^{-1}\begin{pmatrix} -4 \\ 11 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Note that

$$A\hat{\mathbf{x}} = \begin{pmatrix}1\\0\\3\end{pmatrix} \neq \begin{pmatrix}4\\1\\2\end{pmatrix} = \mathbf{b}$$

The error vector is

$$\epsilon = \mathbf{b} - \hat{\mathbf{b}} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$$

and its norm

$$\sigma_\epsilon^2 = \|\epsilon\| = \sqrt{11}$$

Lay, 6.5.2

Ignacio Sánchez López, Dec. 15th, 2014

Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ by constructing the normal equations for $\hat{\mathbf{x}}$ and solving for it with $A = \begin{pmatrix}2 & 1\\-2 & 0\\2 & 3\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}-5\\8\\1\end{pmatrix}$.

Solution: The normal equations are given by

$$A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$$

In this particular case

$$\begin{pmatrix}2 & -2 & 2\\1 & 0 & 3\end{pmatrix}\begin{pmatrix}2 & 1\\-2 & 0\\2 & 3\end{pmatrix}\hat{\mathbf{x}} = \begin{pmatrix}2 & -2 & 2\\1 & 0 & 3\end{pmatrix}\begin{pmatrix}-5\\8\\1\end{pmatrix}$$

$$\begin{pmatrix}12 & 8\\8 & 10\end{pmatrix}\hat{\mathbf{x}} = \begin{pmatrix}-24\\8\end{pmatrix}$$

$$\hat{\mathbf{x}} = \begin{pmatrix}12 & 8\\8 & 10\end{pmatrix}^{-1}\begin{pmatrix}-24\\8\end{pmatrix} = \begin{pmatrix}38/7\\36/7\end{pmatrix}$$

Lay, 6.5.9

Carlos Oscar Sorzano, Dec. 16th, 2014

Find the orthogonal projection of $\mathbf{b} = (4, -2, -3)^T$ onto the column space of $A = \begin{pmatrix}1 & 5\\3 & 1\\-2 & 4\end{pmatrix}$. Then, using this result find a least squares solution of the problem $A\mathbf{x} = \mathbf{b}$.

Solution: It can be seen that the two columns of A are orthogonal to each other, so to project $\mathbf{b}$ onto the column space of A we simply calculate

$$\begin{aligned}\hat{\mathbf{b}} &= \frac{\mathbf{b}\cdot\mathbf{a}_1}{\|\mathbf{a}_1\|^2}\mathbf{a}_1 + \frac{\mathbf{b}\cdot\mathbf{a}_2}{\|\mathbf{a}_2\|^2}\mathbf{a}_2\\ &= \frac{4\cdot1-2\cdot3-3\cdot(-2)}{1^2+3^2+(-2)^2}\begin{pmatrix}1\\3\\-2\end{pmatrix} + \frac{4\cdot5-2\cdot1-3\cdot4}{5^2+1^2+4^2}\begin{pmatrix}5\\1\\4\end{pmatrix}\\ &= \frac{4}{14}\begin{pmatrix}1\\3\\-2\end{pmatrix} + \frac{6}{42}\begin{pmatrix}5\\1\\4\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}\end{aligned}$$

To solve the least-squares problem we need to solve the problem

$$A\hat{\mathbf{x}} = \hat{\mathbf{b}}$$

whose augmented matrix is

$$\left(\begin{array}{rr|r} 1 & 5 & 1 \\ 3 & 1 & 1 \\ -2 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{rr|r} 1 & 0 & \frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 0 \end{array}\right)$$

So, the least-squares solution is

$$\hat{\mathbf{x}} = \frac{1}{7}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Lay, 6.5.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $m \times n$ matrix. Use the steps below to show that a vector $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$ if and only if $A^T A\mathbf{x} = \mathbf{0}$. This will show that $\text{Nul}\{A\} = \text{Nul}\{A^T A\}$.

a. Show that if $A\mathbf{x} = \mathbf{0}$, then $A^T A\mathbf{x} = \mathbf{0}$

b. Suppose $A^T A\mathbf{x} = \mathbf{0}$. Explain why $\mathbf{x}^T A^T A\mathbf{x} = 0$, and use this to show that $A\mathbf{x} = \mathbf{0}$

Solution:

a. Let us assume that

$$A\mathbf{x} = \mathbf{0}$$

Multiplying on the left by A^T, we get

$$A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}$$

b. Let us assume that

$$A^T A\mathbf{x} = \mathbf{0}$$

Multiplying both sides by $\mathbf{x}^T$, we get

$$\mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0$$

But this means that the norm of $A\mathbf{x}$ is null because

$$\mathbf{x}^T A^T A\mathbf{x} = \|A\mathbf{x}\|^2 = 0$$

So $A\mathbf{x} = \mathbf{0}$.

Lay, 6.5.20

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $m \times n$ matrix such that $A^T A$ is invertible. Show that the columns of A are linerly independent. [*Careful*: You may not assume that A is invertible; it may not even be square.]

Solution: By the Invertible Matrix Theorem (see Section 2.9), if $A^T A$ is invertible, then $\text{Nul}\{A^T A\} = \mathbf{0}$, that means that the only solution of the problem

$$A^T A\mathbf{x} = \{\mathbf{0}\}$$

is the vector $\mathbf{x} = \mathbf{0}$. In Exercise 6.5.19 we showed that $\text{Nul}\{A^T A\} = \text{Nul}\{A\}$, so the only vector in $\text{Nul}\{A\} = \{\mathbf{0}\}$, that is the only solution of the problem

$$A\mathbf{x} = \mathbf{0}$$

is $\mathbf{x} = \mathbf{0}$, and consequently, the columns of A are linearly independent.

Lay, 6.5.21

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be an $m \times n$ matrix whose columns are linearly independent. [*Careful*: A need not be square.]

a. Use Exercise 6.5.19 to show that $A^T A$ is an invertible matrix.

b. Explain why A must have at least as many rows as columns.

c. Determine the rank of A.

Solution:

1. If the columns of A are linearly independent, then the only solution of the problem

   $$A\mathbf{x} = \mathbf{0}$$

   is $\mathbf{x} = \mathbf{0}$, that is, $\text{Nul}\{A\} = \{\mathbf{0}\}$ and by Exercise 6.5.19, $\text{Nul}\{A^T A\} = \{\mathbf{0}\}$. By the Invertible Matrix Theorem (see Section 2.9) this implies that $A^T A$ is invertible.

2. A has at least as many rows as columns if $m \geq n$. Note that $A^T A$ is of size $n \times n$ and we need its rank to be n (so that it can be inverted). The rank of a matrix meets:

   $$\text{Rank}\{A^T A\} = \text{Rank}\{A\} = \text{Rank}\{A^T\} = \text{Rank}\{AA^T\}$$

   Note also that the rank of A is at most the minimum between m and n, so if $A^T A$ is invertible, it must be $m \geq n$ because otherwise the rank of A would be $m < n$ and $A^T A$ would not be invertible.

3. See response to previous point, $\text{Rank}\{A\} = n$.

Lay, 6.5.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Find a formula for the least-squares solution of $A\mathbf{x} = \mathbf{b}$ when the columns of A are orthonormal.
Solution: The standard solution of the least-squares problem is

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$$

Consider the column decomposition of A and its implications in the computation of $A^T A$

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & ... & \mathbf{a}_n \end{pmatrix}$$

$$A^T = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ ... \\ \mathbf{a}_n^T \end{pmatrix}$$

$$A^T A = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ ... \\ \mathbf{a}_n^T \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & ... & \mathbf{a}_n \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & ... & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & ... & \mathbf{a}_2^T\mathbf{a}_n \\ ... & ... & ... & ... \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & ... & \mathbf{a}_n^T\mathbf{a}_n \end{pmatrix}$$

Since the columns of A are orthonormal all products $\mathbf{a}_i^T\mathbf{a}_j$ with $i \neq j$ are equal to 0 and the products $\mathbf{a}_i^T\mathbf{a}_i$ are equal to 1. Thus, we have

$$A^T A = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{pmatrix}$$

Then $(A^T A)^{-1} A^T = A^T$. Finally, the least-squares solution is

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b} = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ ... \\ \mathbf{a}_n^T \end{pmatrix} \mathbf{b} = \begin{pmatrix} \mathbf{a}_1^T\mathbf{b} \\ \mathbf{a}_2^T\mathbf{b} \\ ... \\ \mathbf{a}_n^T\mathbf{b} \end{pmatrix}$$

which is nothing more than the orthogonal projection of the vector $\mathbf{b}$ onto each one of the orthonormal columns of A.

Lay, 6.6.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line, that better fits the points (0,1), (1,1), (2,2), (3,2).
Solution: We need to solve the overdetermined equation system

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

which is of the form $X\beta = \mathbf{y}$. Its least-squares solution is

$$\hat{\beta} = (X^T X)^{-1} X^T \mathbf{y}$$

that in this case is

$$\hat{\beta} = \begin{pmatrix} 0.9 \\ 0.4 \end{pmatrix}$$

That is, the least-squares line is defined as

$$y = 0.9 + 0.4x$$

Lay, 6.6.5

Carlos Oscar Sorzano, Aug. 31st, 2013

Let X be the design matrix used to find the least-squares line to fit data (x_1, y_1), (x_2, y_2), ..., (x_n, y_n). Use a theorem in Section 6.5 to show that the normal equations have a unique solution if and only if the data include at least two points with different x-coordinates.

Solution: Theorem 6.5.15 states that if the columns of A are linearly independent, then A can be factorized as $A = QR$ and the least-squares solution of the problem $A\mathbf{x} = \mathbf{b}$ is unique and given by $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$.

If the data points do not have two different x-coordinates, then the design matrix of the least-squares will be of the form

$$A = \begin{pmatrix} 1 & x_1 \\ 1 & x_1 \\ ... & \\ 1 & x_1 \end{pmatrix}$$

It can be easily seen that its two columns are not linearly independent because $\mathbf{a}_2 = x_1 \mathbf{a}_1$.

Lay, 6.6.9

Carlos Oscar Sorzano, Aug. 31st, 2013

A certain experiment produces the data (1,7.9), (2,5.4) and (3,-0.9). Describe the model that produces a least-squares fit of these points by a function of the form

$$y = A\cos(x) + B\sin(x)$$

Solution: For each one of the data points we have a linear equation

$$\begin{aligned} 7.9 &= A\cos(1) + B\sin(1) \\ 5.4 &= A\cos(2) + B\sin(2) \\ -0.9 &= A\cos(3) + B\sin(3) \end{aligned}$$

This can be rewritten in matrix form as

$$\begin{pmatrix} \cos(1) & \sin(1) \\ \cos(2) & \sin(2) \\ \cos(3) & \sin(3) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 7.9 \\ 5.4 \\ -0.9 \end{pmatrix}$$

so we are back to the framework of least-squares fittings and we can solve for A and B by using the normal equations of the problem.

Lay, 6.6.19

Carlos Oscar Sorzano, Dec. 16th, 2014

Consider a design matrix X with two or more columns and a least-squares solution $\hat{\boldsymbol{\beta}}$ of $\mathbf{y} = X\boldsymbol{\beta}$. Consider the following numbers:

- $\|X\hat{\boldsymbol{\beta}}\|^2$: the sum of the squares of the "regression term". Denote this number by $SS(R)$.
- $\|\mathbf{y} - X\hat{\boldsymbol{\beta}}\|^2$: the sum of the squares for the "error term". Denote this number by $SS(E)$.
- $\|\mathbf{y}\|^2$: the total sum of the squares of the y-values. Denote this number by $SS(T)$.

To simplify matters, assume that the mean of the y-values is zero. In this case, $SS(T)$ is proportial to what is called the variance of the set of y-values.

Justify the equation

$$SS(T) = SS(R) + SS(E)$$

This equation is extremely important in Statistics, both in regression theory and in analysis of variance.

Solution: The equation

$$SS(T) = SS(R) + SS(E)$$

can be rewritten as

$$\|\mathbf{y}\|^2 = \|X\hat{\boldsymbol{\beta}}\|^2 + \|\mathbf{y} - X\hat{\boldsymbol{\beta}}\|^2$$

This is the orthogonal decomposition of the vector $\mathbf{y}$ on its projection on the column space of the matrix X (i.e., $X\hat{\boldsymbol{\beta}}$) and its residual $(\mathbf{y} - X\hat{\boldsymbol{\beta}})$, which is orthogonal to the column space of X. Thanks to the orthogonality of the residual to the column space of X, the Pythagorean theorem applies. This theorem is just

$$\|\mathbf{y}\|^2 = \|X\hat{\boldsymbol{\beta}}\|^2 + \|\mathbf{y} - X\hat{\boldsymbol{\beta}}\|^2$$

that is, the SS equation proposed by the problem.

Lay, 6.7.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be two vectors in $\mathbb{R}^2$. Let us define the inner product in $\mathbb{R}^2$ as

$$\mathbf{u} \cdot \mathbf{v} = 4u_1v_1 + 5u_2v_2$$

Let $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (5, -1)$.

a. Find $\|\mathbf{x}\|$, $\|\mathbf{y}\|$ and $|\mathbf{x} \cdot \mathbf{y}|^2$.

b. Describe all vectors that are orthogonal to $\mathbf{y}$.

Solution:

a. To find the required quantities we use their definition in terms of the dot product

$$\begin{gathered}\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{(1,1)\cdot(1,1)} = \sqrt{4\cdot 1\cdot 1+5\cdot 1\cdot 1} = \sqrt{9}\\ \|\mathbf{y}\| = \sqrt{\mathbf{y}\cdot\mathbf{y}} = \sqrt{(5,-1)\cdot(5,-1)} = \sqrt{4\cdot 5\cdot 5+5\cdot(-1)\cdot(-1)} = \sqrt{105}\\ \mathbf{x}\cdot\mathbf{y} = (1,1)\cdot(5,-1) = 4\cdot 1\cdot 5+5\cdot 1\cdot(-1) = 15\\ |\mathbf{x}\cdot\mathbf{y}|^2 = |15|^2 = 225\end{gathered}$$

b. Let $\mathbf{w} = (w_1, w_2)$ be an arbitrary vector in $\mathbb{R}^2$ orthogonal to $\mathbf{y}$. It must fulfill

$$\begin{gathered}\mathbf{w}\cdot\mathbf{y} = 0\\ \mathbf{w}\cdot(5,-1) = 0\\ 4w_1(5)+5w_2(-1) = 0\\ w_2 = 4w_1\end{gathered}$$

So, any vector $\mathbf{w}$ orthogonal to $\mathbf{y}$ according to the proposed inner product must be of the form $(w_1, 4w_1)$.

Lay, 6.7.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A by any invertible $n\times n$ matrix. Show that for $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the formula $\langle\mathbf{u},\mathbf{v}\rangle = (A\mathbf{u})^T(A\mathbf{v})$ defines an inner product in $\mathbb{R}^n$.

Solution: To show that the proposed operation is an inner product we need to show all the properties below:

a. $\langle\mathbf{u},\mathbf{v}\rangle = \langle\mathbf{v},\mathbf{u}\rangle$

$$\begin{array}{rcll}\langle\mathbf{u},\mathbf{v}\rangle & = & (A\mathbf{u})^T(A\mathbf{v}) & \text{[by definition]}\\ & = & \mathbf{u}^TA^TA\mathbf{v} & \\ & = & (\mathbf{u}^TA^TA\mathbf{v})^T & \text{the result of the inner product is a scalar}\\ & = & \mathbf{v}^TA^TA\mathbf{u} & \\ & = & (A\mathbf{v})^T(A\mathbf{u}) & \\ & = & \langle\mathbf{v},\mathbf{u}\rangle & \end{array}$$

b. $\langle\mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \langle\mathbf{u},\mathbf{w}\rangle + \langle\mathbf{v},\mathbf{w}\rangle$

$$\begin{array}{rcll}\langle\mathbf{u}+\mathbf{v},\mathbf{w}\rangle & = & (A(\mathbf{u}+\mathbf{v}))^T(A\mathbf{w}) & \text{[by definition]}\\ & = & (A\mathbf{u}+A\mathbf{v})^T(A\mathbf{w}) & \\ & = & (\mathbf{u}^TA^T+\mathbf{v}^TA^T)(A\mathbf{w}) & \\ & = & \mathbf{u}^TA^TA\mathbf{w}+\mathbf{v}^TA^TA\mathbf{w} & \\ & = & (A\mathbf{u})^T(A\mathbf{w})+(A\mathbf{v})^T(A\mathbf{w}) & \\ & = & \langle\mathbf{u},\mathbf{w}\rangle+\langle\mathbf{v},\mathbf{w}\rangle & \end{array}$$

c. $\langle c\mathbf{u},\mathbf{v}\rangle = c\langle\mathbf{u},\mathbf{v}\rangle$

$$\begin{array}{rcll}\langle c\mathbf{u},\mathbf{v}\rangle & = & (A(c\mathbf{u}))^T(A\mathbf{v}) & \text{[by definition]}\\ & = & (cA\mathbf{u})^T(A\mathbf{v}) & \\ & = & c\mathbf{u}^TA^TA\mathbf{v} & \\ & = & c(A\mathbf{u})^T(A\mathbf{v}) & \\ & = & c\langle\mathbf{u},\mathbf{v}\rangle & \end{array}$$

d. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$.

$$\begin{aligned} \langle \mathbf{u}, \mathbf{u} \rangle &= (A\mathbf{u})^T (A\mathbf{u}) \quad \text{[by definition]} \\ &= \|A\mathbf{u}\|^2 \geq 0 \end{aligned}$$

e. $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ iff $\mathbf{u} = \mathbf{0}$.

$$\langle \mathbf{u}, \mathbf{u} \rangle = 0 \Rightarrow \|A\mathbf{u}\|^2 = 0 \Rightarrow A\mathbf{u} = \mathbf{0}$$

According to the requirement of inner products, it must be that

$$A\mathbf{u} = \mathbf{0} \Leftrightarrow \mathbf{u} = \mathbf{0}$$

This means that it must be $\text{Nul}\{A\} = \{\mathbf{0}\}$. For an $n \times n$ matrix, this only happens if and only if A is invertible (as stated by the problem; see the Invertible Matrix Theorem).

Since the proposed inner product meets all the conditions, it is a true inner product.

Lay, 6.7.16

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if $S = \{\mathbf{u}, \mathbf{v}\}$ is an orthonormal set in V, then $\|\mathbf{u} - \mathbf{v}\| = \sqrt{2}$.
Solution: If S is orthonormal, then $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ and $\mathbf{u} \cdot \mathbf{v} = 0$. Then,

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\| &= \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v}} \\ &= \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}} \\ &= \sqrt{1 + 1 - 2 \cdot 0} \\ &= \sqrt{2} \end{aligned}$$

Lay, 6.7.18

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.
Solution:

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}\right) + \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}\right) \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2 \end{aligned}$$

Lay, 6.8.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the least-squares line $y = \beta_0 + \beta_1 x$ that best fits the data (-2,0), (-1,0), (0,2), (1,4), and (2,4), assuming that the first and last data point are less reliable. Weight them half as much as the three interior points.
Solution: The Weighted Least Squares solves the normal equations of the problem

$$WA\beta = W\mathbf{y}$$

being W the weight matrix, A the design matrix, β the unknown vector and $\mathbf{y}$ the observed vector. In this case,

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{pmatrix}$$

The normal equations of this problem are

$$(WA)^T WA\beta = (WA)^T W\mathbf{y}$$

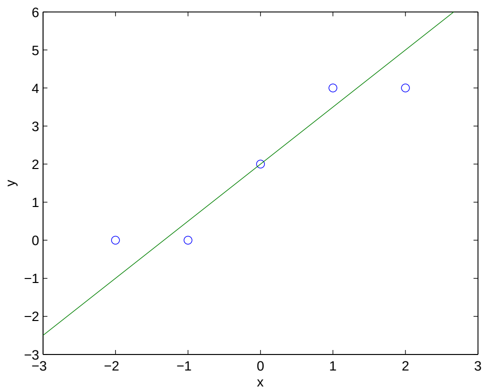
and its solution

$$\hat{\beta} = ((WA)^T WA)^{-1}(WA)^T W\mathbf{y}$$

In this particular case,

$$\hat{\beta} = \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix}$$

That is, the WLS line is $y = 2 + \frac{3}{2}x$ that is represented below along with the original data



Lay, 6.8.2

Carlos Oscar Sorzano, Dec. 16th, 2014

Suppose that 5 out of 25 measurements in a weighted least-squares problem have a y-measurement that is less reliable than the others, and they have to be weighted half as much as the other 20 points. One method is to weight the 20 points by a factor 1 and the other 5 by a factor $\frac{1}{2}$. A second method is to weight the 20 points by a factor 2 and the other 5 by a factor 1. Do the two methods produce different results? Explain.

Solution: The two methods produce the same solution. To show why let us call W_1 to the weight matrix of the first method and W_2 to the weight matrix of the second. We see that $W_2 = 2W_1$. The solution of the weighted least-squares problem is the least-square solution of the equation system

$$WA\mathbf{x} = W\mathbf{y}$$

In particular, for the second method

$$W_2 A\mathbf{x} = W_2\mathbf{y}$$
$$2W_1 A\mathbf{x} = 2W_1\mathbf{y}$$
$$W_1 A\mathbf{x} = W_1\mathbf{y}$$

That is, both methods are solving the same equation system.

Lay, 6.8.6

Carlos Oscar Sorzano, Aug. 31st, 2013

Let's define the inner product in the set of all continuous functions within the range $[0, 2\pi]$ as

$$\langle f(t), g(t)\rangle = \int\limits_0^{2\pi} f(t)g(t)dt$$

Show that the functions $\sin(mt)$ and $\cos(nt)$ are orthogonal for all positive integers m and n.

Solution: Let us solve first the indefinite integral

$$\int \sin(mt)\cos(nt)dt = -\frac{n\sin(mx)\sin(nx)+m\cos(mx)\cos(nx)}{m^2-n^2}$$

Let us compute now the inner product

$$\begin{array}{rcl} \langle \sin(mt), \cos(nt)\rangle & = & \int\limits_0^{2\pi} \sin(mt)\cos(nt)dt \\ & = & -\frac{n\sin(mx)\sin(nx)+m\cos(mx)\cos(nx)}{m^2-n^2}\Big|_0^{2\pi} \\ & = & -\frac{n\sin(2\pi m)\sin(2\pi n)+m\cos(2\pi m)\cos(2\pi n)}{m^2-n^2} - \left(-\frac{n\sin(0)\sin(0)+m\cos(0)\cos(0)}{m^2-n^2}\right) \\ & = & -\frac{m}{m^2-n^2} - \left(-\frac{m}{m^2-n^2}\right) \\ & = & 0 \end{array}$$

So the two functions $\sin(mt)$ and $\cos(nt)$ are orthogonal.

Lay, 6.8.8

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the third-order Fourier approximation to $f(t) = t - 1$ within the range $[0, 2\pi]$ with the inner product defined in Exercise 6.8.6.

Solution: The approximation we seek is of the form

$$f(t) \approx \frac{\langle f(t),1\rangle}{\|1\|^2} + \sum_{n=1}^{3}\left(\frac{\langle f(t),\cos(nt)\rangle}{\|\cos(nt)\|^2}\cos(nt) + \frac{\langle f(t),\sin(nt)\rangle}{\|\sin(nt)\|^2}\sin(nt)\right)$$

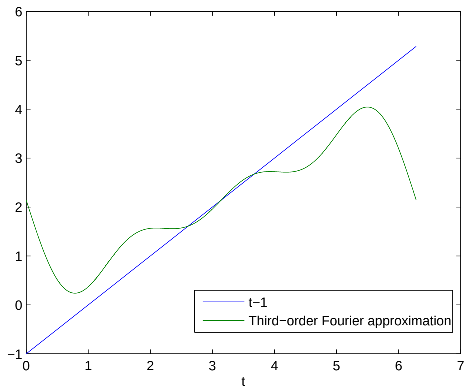
Let us calculate the different terms

$$\langle t-1,1\rangle = \int_0^{2\pi} (t-1)dt = 2\pi(\pi-1)$$
$$\|1\|^2 = \int_0^{2\pi} 1^2 dt = 2\pi$$
$$\langle t-1,\cos(nt)\rangle = \int_0^{2\pi} (t-1)\cos(nt)dt = 0$$
$$\langle t-1,\sin(nt)\rangle = \int_0^{2\pi} (t-1)\sin(nt)dt = -\frac{(\pi+1)}{n}$$
$$\|\sin(nt)\|^2 = \int_0^{2\pi} \sin^2(nt)dt = \pi$$

Gathering all together, we have

$$\begin{array}{rcl} t-1 & \approx & \frac{2\pi(\pi-1)}{2\pi} + \frac{-(\pi+1)}{\pi}\sin(t) + \frac{-\frac{(\pi+1)}{2}}{\pi}\sin(2t) + \frac{-\frac{(\pi+1)}{3}}{\pi}\sin(3t) \\ & = & (\pi-1) - \frac{\pi+1}{\pi}\left(\sin(t) + \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t)\right) \end{array}$$

We have both functions represented below



Lay, 6.8.11

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the third-order Fourier approximation to $\sin^2(t)$ without performing any integration calculations.

Solution: We know by trigonometric relationships that

$$\sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$$

This is in fact the Fourier approximation of order 2, in this case, the approximation is exact.

Lay, 6.Suppl.4

Carlos Oscar Sorzano, Jan. 19th 2015

Let U be a $n \times n$ orthogonal matrix. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is an orthogonal basis of $\mathbb{R}^n$, then so is $\{U\mathbf{v}_1, U\mathbf{v}_2, ..., U\mathbf{v}_n\}$

Solution: Let us calculate the dot product between any two vectors in the proposed set. Let us denote $\mathbf{w}_i = U\mathbf{v}_i$.

$$\langle \mathbf{w}_i, \mathbf{w}_j \rangle = (\mathbf{w}_i)^T (\mathbf{w}_j) = (U\mathbf{v}_i)^T (U\mathbf{v}_j) = \mathbf{v}_i^T U^T U \mathbf{v}_j$$

Since U is an orthogonal matrix, we have $U^T U = I$ and consequently

$$\langle \mathbf{w}_i, \mathbf{w}_j \rangle = \mathbf{v}_i^T \mathbf{v}_j = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$$

If the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is an orthogonal basis of $\mathbb{R}^n$, then so is $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n\}$ because as shown in the previous equation any pair of distinct $\mathbf{w}_i$ vectors are orthogonal to each other.

7 Chapter 7

Lay, 7.1.1

Marta Monsalve Buendía, Dic. 24th, 2014

Determine if the matrix $A = \begin{pmatrix} 3 & 5 \\ 5 & -7 \end{pmatrix}$ is symmetric.

Solution: A matrix A is symmetric if $A = A^T$. So A is symmetric.

Lay, 7.1.2

Marta Monsalve Buendía, Dic. 24th, 2014

Determine if the matrix $A = \begin{pmatrix} -3 & 5 \\ -5 & 3 \end{pmatrix}$ is symmetric.

Solution: A matrix A is symmetric if $A = A^T$. So A is not symmetric because $a_{12} \neq a_{21}$.

Lay, 7.1.3

Marta Monsalve Buendía, Dic. 24th, 2014

Determine if the matrix $A = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$ is symmetric.

Solution: A matrix A is symmetric if $A = A^T$. So A is not symmetric because $a_{12} \neq a_{21}$.

Lay, 7.1.4

Marta Monsalve Buendía, Dic. 24th, 2014

Determine if the matrix $A = \begin{pmatrix} 0 & 8 & 3 \\ 8 & 0 & -2 \\ 3 & -2 & 0 \end{pmatrix}$ is symmetric.

Solution: A matrix A is symmetric if $A = A^T$. So A is symmetric.

Lay, 7.1.5

Marta Monsalve Buendía, Dic. 24th, 2014

Determine if the matrix $A = \begin{pmatrix} -6 & 2 & 0 \\ 0 & -6 & 2 \\ 0 & 0 & -6 \end{pmatrix}$ is symmetric.

Solution: A matrix A is symmetric if $A = A^T$. So A is not symmetric because $a_{12} \neq a_{21}$ and $a_{23} \neq a_{32}$.

Lay, 7.1.6

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine if the matrix $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$ is symmetric.

Solution: A matrix A is symmetric if $A = A^T$. So one necessary condition to be symmetric is that A is a square matrix. Since the matrix in the problem is not square, it cannot be symmetric.

Lay, 7.1.7

Carlos Oscar Sorzano, Aug. 31st, 2013

Determine if the matrix $A = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$ is orthogonal. If it is, find its inverse.

Solution: A matrix A is orthogonal if all its columns are orthogonal to each other and they are of unit norm. In this case

$$\begin{gathered} \langle (0.6, 0.8), (0.8, -0.6) \rangle = 0.6 \cdot 0.8 + 0.8 \cdot (-0.6) = 0 \\ \|(0.6, 0.8)\|^2 = 0.6 \cdot 0.6 + 0.8 \cdot 0.8 = 1 \\ \|(0.8, -0.6)\|^2 = 0.8 \cdot 0.8 + (-0.6) \cdot (-0.6) = 1 \end{gathered}$$

Since the two columns are orthogonal to each other, A is an orthogonal matrix. The inverse of an orthogonal matrix is its transpose. In this case

$$A^{-1} = A^T = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$$

Lay, 7.1.9

Marta Monsalve Buendía, Dic. 24th, 2014

Determine if the matrix $A = \begin{pmatrix} -5 & 2 \\ 2 & 5 \end{pmatrix}$ is orthogonal. If it is, find its inverse.

Solution: A matrix A is orthogonal if all its columns are orthogonal to each other and they are of unit norm. In this case

$$\begin{gathered} \langle (-5, 2), (2, 5) \rangle = (-5) \cdot 2 + 2 \cdot 5 = 0 \\ \|(-5, 2)\|^2 = (-5) \cdot (-5) + 2 \cdot 2 = 29 \\ \|(2, 5)\|^2 = 2 \cdot 2 + 5 \cdot 5 = 29 \end{gathered}$$

The two columns of A are orthogonal to each other but they are not unit norm so A is not an orthogonal matrix.

Lay, 7.1.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Orthogonalize the matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ giving a matrix P and a diagonal matrix D.

Solution: Let's find first the eigenvalues of A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0$$

Let's find now the eigenvalues associated to each eigenspace
<u>Eigenspace $\lambda = 2$</u>
Let's solve the vector problem

$$(A - 2I)\mathbf{v} = \mathbf{0}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

whose solution are all vectors of the form $\mathbf{v} = (v_1, -v_1)$. In particular $\mathbf{v}_1 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ is unit vector of this subspace.
<u>Eigenspace $\lambda = 4$</u>

$$(A - 4I)\mathbf{v} = \mathbf{0}$$
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

whose solution are all vectors of the form $\mathbf{v} = (v_1, v_1)$. In particular $\mathbf{v}_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is unit vector of this subspace.
The eigendecomposition of matrix A is, therefore,

$$A = PDP^{-1}$$
$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Note that we can find an orthogonal matrix for P (and consequently $P^{-1} = P^T$) because A is a symmetric.

Lay, 7.1.17

Ana Sanmartin, Jan. 18th, 2015

Orthogonally diagonalize the matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

giving an orthogonal matrix P and a diagonal matrix D. The eigenvalues are: 5, 2, -2.
Solution: To orthogonally diagonalize a matrix, we follow the structure $A = PDP^T$ being P orthogonal ($P^{-1} = P^T$). Firstly, we have to look for the eigenvectors, following the formula

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

<u>$\lambda = 5$</u>:

$$(A - 5I)\mathbf{v}_1 = \mathbf{0} \Rightarrow \begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix} \mathbf{v}_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_1 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$\underline{\lambda = 2}$:

$$(A-2I)\mathbf{v}_2 = \mathbf{0} \Rightarrow \begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$\underline{\lambda = -2}$:

$$(A+2I)\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{pmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{pmatrix} \mathbf{v}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_3 = \mathbf{0} \Rightarrow \mathbf{v}_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Now, we can diagonalize A by setting the eigenvectors as the columns of the matrix P and the corresponding eigenvalues in the diagonal of D:

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = PDP^T$$

Lay, 7.1.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Verify that 5 is an eigenvalue of A and $\mathbf{u}$ is its eigenvector. Then, orthogonally diagonalize A.

Solution: Let's verify that $\mathbf{u}$ is an eigenvector of A

$$A\mathbf{u} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The other two vectors needed to orthogonally diagonalize A must be orthogonal to $\mathbf{u}$, let's call them $\mathbf{v}$ and $\mathbf{w}$. They must meet

$$\begin{array}{c} \mathbf{u} \cdot \mathbf{v} = (1,1,1) \cdot \mathbf{v} = v_1 + v_2 + v_3 = 0 \Rightarrow v_3 = -v_1 - v_2 \\ \mathbf{u} \cdot \mathbf{w} = (1,1,1) \cdot \mathbf{w} = w_1 + w_2 + w_3 = 0 \Rightarrow w_3 = -w_1 - w_2 \end{array}$$

Additionally, they must be orthogonal to each other so

$$\begin{array}{c} \mathbf{v} \cdot \mathbf{w} = (v_1, v_2, -v_1 - v_2) \cdot (w_1, w_2, -w_1 - w_2) = 2v_1w_1 + 2v_2w_2 + v_1w_2 + v_2w_1 = 0 \\ 2v_1w_1 + (2v_2 + v_1)w_2 + v_2w_1 = 0 \\ w_2 = -\frac{2v_1w_1 + v_2w_1}{2v_2 + v_1} \end{array}$$

So the two vectors must be of the form

$$\begin{array}{c} \mathbf{v} = (v_1, v_2, -v_1 - v_2) \\ \mathbf{w} = (w_1, -\frac{2v_1w_1 + v_2w_1}{2v_2 + v_1}, -w_1 + \frac{2v_1w_1 + v_2w_1}{2v_2 + v_1}) \end{array}$$

Giving the values $v_1 = 1$, $v_2 = 0$, and $w_1 = 1$, we get

$$\begin{array}{c} \mathbf{v} = (1, 0, -1) \\ \mathbf{w} = (1, -2, 1) \end{array}$$

The eigenvalue associated to these eigenvectors are 2 and 2 because

$$A\mathbf{v} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$
$$A\mathbf{w} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For an orthogonal diagonalization we need the vectors to be unitary so, we normalize them

$$\begin{aligned} \mathbf{u}' &= \tfrac{1}{\|\mathbf{u}\|}\mathbf{u} = \tfrac{1}{\sqrt{3}}(1,1,1) \\ \mathbf{v}' &= \tfrac{1}{\|\mathbf{v}\|}\mathbf{v} = \tfrac{1}{\sqrt{2}}(1,0,-1) \\ \mathbf{w}' &= \tfrac{1}{\|\mathbf{w}\|}\mathbf{w} = \tfrac{1}{\sqrt{6}}(1,-2,1) \end{aligned}$$

Finally, the orthogonal diagonalization of A is $A = PDP^T$ with

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \text{ and } D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Lay, 7.1.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that B^TAB, B^TB and BB^T are symmetric matrices.
Solution: Let's calculate the transpose of each one of the matrices and show that they are equal to the original matrices

$$\begin{aligned} (B^TAB)^T &= B^TA^T(B^T)^T = B^TAB \\ (B^TB)^T &= B^T(B^T)^T = B^TB \\ (BB^T)^T &= (B^T)^TB^T = BB^T \end{aligned}$$

Lay, 7.1.29

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.
Solution: If A is orthogonally diagonalizable, then $A = PDP^T$. Then,

$$A^{-1} = (PDP^T)^{-1} = (P^T)^{-1}D^{-1}P^{-1} = PD^{-1}P^T$$

So, A^{-1} is orthogonally diagonalizable.

Lay, 7.1.30

Carlos Oscar Sorzano, Feb. 15th, 2014

Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is orthogonally diagonalizable.
Solution: According to Theorem 7.1.2, a matrix is orthogonally diagonalizable if and only if it is symmetric. So, matrices A and B are symmetric. Let us compute $(AB)^T$:

$$(AB)^T = B^TA^T = BA$$

But according to the problem statement $BA = AB$, so $(AB)^T = AB$, that is, AB is symmetric, and consequently orthogonally diagonalizable.

Lay, 7.1.31

Carlos Oscar Sorzano, Dec. 16th, 2014

Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, and let λ be an eigenvalue of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

Solution: The diagonalization theorem (5.3.5) states that if A is diagonalizable, and in this case it is, then the columns of P are linearly independent eigenvectors of A, so P has exactly k eigenvectors corresponding to the eigenvalue λ. These k eigenvectors form a basis of the eigenspace associated to λ.

Lay, 7.1.35

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \mathbf{u}\mathbf{u}^T$.

a. Given any $\mathbf{x} \in \mathbb{R}^n$, compute $B\mathbf{x}$ and show that $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$, as described in Section 6.2.

b. Show that B is a symmetric matrix and $B^2 = B$.

c. Show that $\mathbf{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

Solution:

a. The orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$ is defined as

$$\mathrm{Proj}_{\mathbf{u}}\{\mathbf{x}\} = \frac{\mathbf{x}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}$$

Since $\mathbf{u}$ is unitary, $\|\mathbf{u}\|^2 = 1$, then

$$\begin{array}{rcll} \mathrm{Proj}_{\mathbf{u}}\{\mathbf{x}\} & = & (\mathbf{x}\cdot\mathbf{u})\mathbf{u} & \text{[inner product is commutative]} \\ & = & (\mathbf{u}\cdot\mathbf{x})\mathbf{u} & \text{[by definition of inner product]} \\ & = & (\mathbf{u}^T\mathbf{x})\mathbf{u} & [\mathbf{u}^T\mathbf{x} \text{ is a scalar}] \\ & = & \mathbf{u}(\mathbf{u}^T\mathbf{x}) & \text{[associativity of matrix multiplication]} \\ & = & (\mathbf{u}\mathbf{u}^T)\mathbf{x} & [B = \mathbf{u}\mathbf{u}^T] \\ & = & B\mathbf{x} & \end{array}$$

b. B is symmetric as shown in Exercise 7.1.27. Let's show now that $B^2 = B$

$$B^2 = (\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T) = \mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T$$

But $\mathbf{u}^T\mathbf{u} = 1$ because $\mathbf{u}$ is unitary. Then,

$$B^2 = \mathbf{u}\mathbf{u}^T = B$$

The fact that $B^2 = B$ implies that projecting $\mathrm{Proj}_{\mathbf{u}}\{\mathbf{x}\}$ onto $\mathbf{u}$ (applying the projection operation twice) has no effect.

c. Let's calculate the product $B\mathbf{u}$

$$B\mathbf{u} = (\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u}(\mathbf{u}^T\mathbf{u}) = \mathbf{u}$$

So, its eigenvalue is 1. The meaning of this latter property is that the orthogonal projection of **u** onto **u** is **u** itself.

Lay, 7.2.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, when $A = \begin{pmatrix} 5 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{pmatrix}$ and

a. $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

b. $\mathbf{x} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

c. $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Solution:

a. We simply need to perform all the multiplications

$$\begin{aligned} Q(\mathbf{x}) &= \mathbf{x}^T A\mathbf{x} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 5 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 5x_1 + \frac{1}{3}x_2 \\ \frac{1}{3}x_1 + x_2 \end{pmatrix} \\ &= 5x_1^2 + \frac{2}{3}x_1x_2 + x_2^2 \end{aligned}$$

b. We simply need to substitute $x_1 = 6$ and $x_2 = 1$ to obtain $Q(6,1) = 185$.

c. $Q(1,3) = 16$.

Lay, 7.2.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the matrix of the quadratic form. Assume **x** is in $\mathbb{R}^2$.

a. $Q(\mathbf{x}) = 10x_1^2 - 6x_1x_2 - 3x_2^2$

b. $Q(\mathbf{x}) = 5x_1^2 + 3x_1x_2$

Solution: We look for the matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. It can be easily verified that the solution of this problem is

a. $A = \begin{pmatrix} 10 & -3 \\ -3 & -3 \end{pmatrix}$

b. $A = \begin{pmatrix} 5 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{pmatrix}$

Lay, 7.2.5

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $Q(\mathbf{x}) = 8x_1^2 + 7x_2^2 - 3x_3^2 - 6x_1x_2 + 4x_1x_3 - 2x_2x_3$

b. $Q(\mathbf{x}) = 4x_1x_2 + 6x_1x_3 - 8x_2x_3$

Solution: We look for the matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. It can be easily verified that the solution of this problem is

a. $A = \begin{pmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{pmatrix}$

b. $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 0 \end{pmatrix}$

Lay, 7.2.6

Ignacio Sanchez Lopez, Dec. 29th, 2014

Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

a. $Q(\mathbf{x}) = 5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

b. $Q(\mathbf{x}) = x_3^2 - 4x_1x_2 + 4x_1x_3$

Solution: We look for the matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. It can be easily verified that the solution of this problem is

a. $A = \begin{pmatrix} 5 & 5/2 & -3/2 \\ 5/2 & -1 & 0 \\ -3/2 & 0 & 7 \end{pmatrix}$

b. $A = \begin{pmatrix} 0 & -2 & 2 \\ -2 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

Lay, 7.2.7

Carlos Oscar Sorzano, Aug. 31st, 2013

Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give P and the new quadratic form.
Solution: If we orthogonally diagonalize the quadratic form, we obtain $A = PDP^T$

$$A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^T$$

We need to do the change of variables

$$\mathbf{x} = P\mathbf{y} \Rightarrow \mathbf{y} = P^T\mathbf{x} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1 + x_2) \\ \frac{1}{\sqrt{2}}(-x_1 + x_2) \end{pmatrix}$$

In this new set of variables, we have that the quadratic form is

$$Q(\mathbf{y}) = \mathbf{y}^T D\mathbf{y} = 6y_1^2 - 4y_2^2$$

Lay, 7.2.8

Ana Sanmartin, Jan. 18th, 2015

Let A be the matrix of the quadratic form $9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$. It can be shown that the eigenvalues of A are $3, 9$, and 15. Find an orthogonal matrix P such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^T A\mathbf{x}$ into a quadratic form with no crossproduct term. Give P and the new quadratic form.
Solution: To find the matrix A, we have to know that the elements from the main diagonal are the coefficients going with x_i^2, and the ij-th entry is the half of the coefficient of x_ix_j. So the matrix A is:

$$A = \begin{pmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{pmatrix}$$

Now, we have to obtain the eigenvectors for each of the three eigenvalue of A.
$\mathbf{v}_1 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ corresponding to $\lambda = 3$
$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ corresponding to $\lambda = 9$
$\mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ corresponding to $\lambda = 15$
Because eigenvectors from different eigenvalues are orthogonal, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set. We obtain the orthonormal set by dividing each vector by its norm and putting them as columns of a matrix P

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & -1 & 2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

If we do the change of variable

$$\mathbf{x} = P\mathbf{y} \Rightarrow \mathbf{y} = P^T\mathbf{x} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & -2 & 1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \mathbf{x},$$

then the quadratic form can be expressed as

$$3y_1^2 + 9y_2^2 + 15y_3^2$$

Lay, 7.2.19

Carlos Oscar Sorzano, Aug. 31st, 2013

What is the largest possible value of the quadratic form $5x_1^2 + 8x_2^2$ if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^T\mathbf{x} = 1$, that is, if $x_1^2 + x_2^2 = 1$? Try some examples of $\mathbf{x}$.
Solution: The matrix associated to this quadratic form and its orthogonal diagonalization is

$$A = \begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T$$

The maximum value of the quadratic form in this constrained optimization problem is equal to the value of the maximum eigenvalue, in this case 8, that is achieved for $\mathbf{x} = (0, 1)$, the eigenvector associated to the maximum eigenvalue. We show below the value of the quadratic form for a few values of $\mathbf{x}$

$$\begin{array}{lcl} Q(1,0) & = & 5 \cdot 1^2 + 8 \cdot 0^2 = 5 \\ Q(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) & = & 5 \cdot (\frac{1}{\sqrt{2}})^2 + 8 \cdot (\frac{1}{\sqrt{2}})^2 = \frac{5}{2} + \frac{8}{2} = \frac{13}{2} \\ Q(0,1) & = & 5 \cdot 0^2 + 8 \cdot 1^2 = 8 \end{array}$$

Lay, 7.2.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Consider the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ when $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det\{A\} \neq 0$. If λ_1 and λ_2 are the eigenvalues of A, then the characteristic polynomial of A can be written in two ways: $\det\{A - \lambda I\}$ and $(\lambda - \lambda_1)(\lambda - \lambda_2)$. Use this fact to show that $\lambda_1 + \lambda_2 = a + d$ (the diagonal entries of A) and $\lambda_1\lambda_2 = \det\{A\}$.
Solution: We may express the characteristic polynomial as

$$\begin{array}{rcl} P(\lambda) & = & \det\{A - \lambda I\} = \det \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ & = & (a-\lambda)(d-\lambda) - bc \\ & = & \lambda^2 - (a+d)\lambda + (ad - bc) \\ & = & \lambda^2 - \mathrm{Trace}\{A\}\lambda + \det\{A\} \\ P(\lambda) & = & (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{array}$$

Identifying coefficients we see that

$$\begin{array}{c} \mathrm{Trace}\{A\} = a + d = \lambda_1 + \lambda_2 \\ \det\{A\} = ad - bc = \lambda_1\lambda_2 \end{array}$$

Lay, 7.2.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Consider the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ when $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det\{A\} \neq 0$. Verify the following statements:

a. Q is positive definite if $\det\{A\} > 0$ and $a + d > 0$.

b. Q is negative definite if $\det\{A\} > 0$ and $a + d < 0$.

c. Q is indefinite if $\det\{A\} < 0$.

Solution:

a. By definition, Q is positive definite if all its eigenvalues are positive. According to Exercise 7.2.23, $\det\{A\} = \lambda_1\lambda_2$. Then $\det\{A\} = \lambda_1\lambda_2 > 0$ implies that either both eigenvalues are positive or both are negative. If $a + d > 0$, then according to Exercise 7.2.23 $\lambda_1 + \lambda_2 > 0$, and both eigenvalues must be positive.

b. By definition, Q is negative definite if all its eigenvalues are negative. As in the previous point, $\det\{A\} = \lambda_1\lambda_2 > 0$ implies that either both eigenvalues are positive or both are negative. However, in this case, since $a+d < 0$, then $\lambda_1 + \lambda_2 < 0$ and both eigenvalues are negative.

c. By definition, Q is indefinite if it has positive and negative eigenvalues. If $\det\{A\} = \lambda_1\lambda_2 < 0$, then both eigenvalues have different sign, and consequently Q is indefinite.

Lay, 7.2.26

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if an $n \times n$ matrix A is positive definite, then there exists a positive definite matrix B such that $A = B^TB$. [*Hint*: Write $A = PDP^T$, with $P^T = P^{-1}$. Produce a diagonal matrix C such that $D = C^TC$, and let $B = PCP^T$. Show that B works.]

Solution: If A is positive definite, then it is symmetric and it can be orthogonally diagonalized as

$$A = PDP^T$$

Since it is positive definite, all its eigenvalues are larger than 0. So the diagonal matrix D has all its diagonal entries larger than 0. We now define

$$C = D^{\frac{1}{2}} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & ... & 0 \\ 0 & \lambda_2^{\frac{1}{2}} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n^{\frac{1}{2}} \end{pmatrix}$$

It can be easily verified that

$$C^TC = D$$

We now construct

$$B = PCP^T$$

Let's check that $B^TB = A$

$$B^TB = (PCP^T)^TPCP^T = (P^T)^TC^TP^TPCP^T = PC^TCP^T = PDP^T = A$$

Lay, 7.2.27

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A+B$ are all positive. [*Hint*: Consider quadratic forms.]
Solution: If A and B are symmetric matrices, then $C = A+B$ is also symmetric. Consider now the quadratic form

$$Q_C(\mathbf{x}) = \mathbf{x}^T C\mathbf{x} = \mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x}$$

We may define the quadratic forms

$$\begin{aligned} Q_A(\mathbf{x}) &= \mathbf{x}^T A\mathbf{x} \\ Q_B(\mathbf{x}) &= \mathbf{x}^T B\mathbf{x} \end{aligned}$$

So that $Q_C(\mathbf{x}) = Q_A(\mathbf{x}) + Q_B(\mathbf{x})$. Since A and B are symmetric matrices, these quadratic forms are well defined, and because all their eigenvalues are positive, then Q_A and Q_B are positive definite quadratic forms. This means that for any $\mathbf{x} \in \mathbb{R}^n$ it is verified that

$$\begin{aligned} Q_A(\mathbf{x}) &> 0 \\ Q_B(\mathbf{x}) &> 0 \end{aligned}$$

Consequently, $Q_C(\mathbf{x}) > 0$, that is Q_C is also positive definite and the eigenvalues of $C = A+B$ are all positive.

Lay, 7.3.1

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y}$ as shown

$$5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3 = 9y_1^2 + 6y_2^2 + 3y_3^2$$

Solution: Let A be

$$A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 7 \end{pmatrix}$$

We may orthogonally diagonalize it as

$$A = PDP^T = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}^T$$

The required change of variable is $\mathbf{x} = P\mathbf{y}$ with

$$P = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Lay, 7.3.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Let $Q(\mathbf{x}) = 5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3 = 9y_1^2 + 6y_2^2 + 3y_3^2$ (see Exercise 7.3.1).

a. Find the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$.

b. Find a unit vector $\mathbf{u}$ where this maximum is attained.

c. Find the maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$

Solution:

a. The maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is given by the maximum eigenvalue (see Exercise 7.3.1), which is 9.

b. The unit vector $\mathbf{u}$ where this maximum is attained is given by the eigenvector associated to this eigenvalue (see Exercise 7.3.1), that is, $\mathbf{u} = (-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$.

c. The maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$ is the second eigenvalue of A, that is, 6. This value is attained for its corresponding eigenvector, $\mathbf{x} = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$

Lay, 7.3.7

Let $Q(\mathbf{x}) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [Hint: The eigenvalues of the matrix of the quadratic form Q are $2, -1$ and -4].
Solution: Firstly, we need to construct the matrix corresponding to the quadratic form

$$A = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

We should know that $Q(\mathbf{x})$ will be maximized, with the constraint of being a unit vector, with the value of the highest eigenvalue. Given the eigenvalues $2, -1$ and -4, it is easy to say that the highest eigenvalue is 2. So we need to look for the eigenvector that is attached to the eigenvalue 2. We get the corresponding eigenvector solving the eigenvalue equation

$$(A - 2I)\mathbf{v} = \begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

The maximum value of Q subject to $\mathbf{x}^T\mathbf{x} = 1$ is obtained for $\mathbf{x} = \mathbf{v}$.

Lay, 7.3.9

Carlos Oscar Sorzano, Jan. 19th 2015

Find the maximum value of $Q(\mathbf{x}) = -3x_1^2 + 5x^2 - 2x_1x_2$ subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find the a vector where the maximum is attained.)
Solution: Let us express

$$Q(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} -3 & -1 \\ -1 & 5 \end{pmatrix} \mathbf{x}$$

This is a quadratic form defined by a symmetric matrix. We know that its maximum value subject to the constraint $\|\mathbf{x}\| = 1$ is given by the maximum eigenvalue of the matrix $\begin{pmatrix} -3 & -1 \\ -1 & 5 \end{pmatrix}$ which is given by the determinant

$$\det \begin{pmatrix} -3-\lambda & -1 \\ -1 & 5-\lambda \end{pmatrix} = (-3-\lambda)(5-\lambda) - 1 = \lambda^2 - 2\lambda - 16 = 0 \Rightarrow \lambda = 1 \pm \sqrt{17}$$

The largest eigenvalue is $1+\sqrt{17}$ and, consequently, the maximum sought is $1+\sqrt{17}$.

Lay, 7.3.12

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be a symmetric $n \times n$ matrix, let M and m denote the maximum and minimum values of the quadratic form $\mathbf{x}^T A\mathbf{x}$ subtject to $\|\mathbf{x}\| = 1$. Let λ be any eigenvalue of A. Justify that $m \leq \lambda \leq M$. (*Hint*: Find an $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = \lambda$.)

Solution: Thanks to the Principal Axes Theorem (Theorem 7.2.4) we know that by diagonalizing matrix $A = PDP^T$ we can express the quadratic form $\mathbf{x}^T A\mathbf{x}$ as

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + ... + \lambda_n y_n^2 \tag{10}$$

where we have made a change of variable $\mathbf{x} = P\mathbf{y}$. Let λ in the problem be the i-th eigenvalue of A and consider a unitary vector with a single 1 at the i-th positition ($\mathbf{y} = (0,0,...,0,1,0,...,0)$). It is obvious that

$$\lambda = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 0^2 + ... + \lambda_{i-1}0^2 + \lambda_i 1^2 + \lambda_{i+1}0^2 + ... + \lambda_n 0^2 = \lambda_i \tag{11}$$

This result is attained for $\mathbf{x} = P\mathbf{y} = \mathbf{u}_i$, that is, the eigenvector associated to the i-th eigenvalue.

Additionally, thanks to Theorem 7.3.6, we know that the quadratic form $\mathbf{x}^T A\mathbf{x}$ is bounded between the minimum and maximum eigenvalue of A when $\mathbf{x}$ is constrained to be unitary. Moreover, $m = \lambda_{min}$ and $M = \lambda_{max}$. Consequently, since $\lambda_{min} \leq \lambda \leq \lambda_{max}$, we have $m \leq \lambda \leq M$.

Lay, 7.3.13

Carlos Oscar Sorzano, Aug. 31st, 2013

Let A be a symmetric $n \times n$ matrix, let M and m denote the maximum and minimum values of the quadratic form $\mathbf{x}^T A\mathbf{x}$, and denote corresponding unit eigenvectors by $\mathbf{u}_1$ and $\mathbf{u}_n$. The following calculations show that given any number t between M and m, there is a unit vector $\mathbf{x}$ such that $t = \mathbf{x}^T A\mathbf{x}$. Verify that $t = (1-\alpha)m + \alpha M$ for some number α between 0 and 1. Then, let $\mathbf{x} = \sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1$, and show that $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T A\mathbf{x} = t$.

Solution: Let us first show that any number t between m and M can be written as

$$t = (1-\alpha)m + \alpha M$$

with $\alpha \in [0,1]$. If $\alpha = 0$, we get $t = m$. If $\alpha = 1$, we get $t = M$. We see that $t = (1-\alpha)m + \alpha M = m + \alpha(M-m)$ is a linear (and, therefore, continuous) function of α. So for any value t between 0 and 1, there exists a value of α such that $t = (1-\alpha)m + \alpha M$.

Now, we construct

$$\mathbf{x} = \sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1$$

Let's check that $\mathbf{x}^T\mathbf{x} = 1$, for which we will exploit the fact that $\mathbf{u}_1$ and $\mathbf{u}_n$ are unitary and orthogonal

$$\begin{array}{rcl}
\mathbf{x}^T\mathbf{x} & = & (\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1)^T(\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1) \\
& = & (\sqrt{1-\alpha}\mathbf{u}_n^T + \sqrt{\alpha}\mathbf{u}_1^T)(\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1) \\
& = & (1-\alpha)\mathbf{u}_n^T\mathbf{u}_n + \alpha\mathbf{u}_1^T\mathbf{u}_1 + \sqrt{1-\alpha}\sqrt{\alpha}\mathbf{u}_n^T\mathbf{u}_1 + \sqrt{1-\alpha}\sqrt{\alpha}\mathbf{u}_1^T\mathbf{u}_n \\
& = & (1-\alpha) + \alpha + 0 + 0 \\
& = & 1
\end{array}$$

Finally, we need to show that $\mathbf{x}^TA\mathbf{x} = t$, remind that $\mathbf{u}_1$ is the eigenvector associated to the eigenvalue M and that $\mathbf{u}_n$ is the eigenvector associated to eigenvalue m:

$$\begin{array}{rcl}
\mathbf{x}^TA\mathbf{x} & = & (\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1)^TA(\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1) \\
& = & (\sqrt{1-\alpha}\mathbf{u}_n^T + \sqrt{\alpha}\mathbf{u}_1^T)(\sqrt{1-\alpha}A\mathbf{u}_n + \sqrt{\alpha}A\mathbf{u}_1) \\
& = & (\sqrt{1-\alpha}\mathbf{u}_n^T + \sqrt{\alpha}\mathbf{u}_1^T)(m\sqrt{1-\alpha}\mathbf{u}_n + M\sqrt{\alpha}\mathbf{u}_1) \\
& = & (m(1-\alpha)\mathbf{u}_n^T\mathbf{u}_n + M\alpha\mathbf{u}_1^T\mathbf{u}_1 + M\sqrt{1-\alpha}\sqrt{\alpha}\mathbf{u}_n^T\mathbf{u}_1 + m\sqrt{1-\alpha}\sqrt{\alpha}\mathbf{u}_1^T\mathbf{u}_n \\
& = & m(1-\alpha) + M\alpha + 0 + 0 \\
& = & t
\end{array}$$

Lay, 7.4.3

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the singular values of the matrix $A = \begin{pmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{pmatrix}$

Solution: We compute A^TA

$$A^TA = \begin{pmatrix} \sqrt{6} & 0 \\ 1 & \sqrt{6} \end{pmatrix}\begin{pmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{pmatrix} = \begin{pmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 7 \end{pmatrix}$$

whose eigenvalues are

$$\begin{array}{c} \lambda_1 = 9 \\ \lambda_2 = 4 \end{array}$$

The singular values of A are

$$\begin{array}{c} \sigma_1 = \sqrt{\lambda_1} = 3 \\ \sigma_2 = \sqrt{\lambda_2} = 2 \end{array}$$

Lay, 7.4.10

Carlos Oscar Sorzano, Dec. 16th, 2014

Find a SVD decomposition of the matrix $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}$.

Solution: Let us calculate $A^T A$

$$A^t A = \begin{pmatrix} 4 & 2 & 0 \\ -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 20 & -10 \\ -10 & 5 \end{pmatrix}$$

Its eigenvalues and associated eigenvectors are

$$\lambda_1 = 25, \mathbf{v}_1 = \frac{1}{\sqrt{5}}(-2, 1)^T$$

$$\lambda_2 = 0, \mathbf{v}_2 = \frac{1}{\sqrt{5}}(1, 2)^T$$

We now construct the V and Σ matrices:

$$V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Let us construct now the U matrix. For doing so, we calculate

$$\mathbf{u}_1 = \frac{A\mathbf{v}_1}{\sigma_1} = \frac{\begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}}{5} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

We need now to extend the basis. All vectors perpendicular to $\mathbf{u}_1$ fulfill

$$\mathbf{u}_1 \cdot \mathbf{u} = 0 = -\frac{2}{\sqrt{5}} u_x - \frac{1}{\sqrt{5}} u_y + 0 u_z = 0 \Rightarrow u_y = -2u_x$$

That is, they are vectors of the form $\mathbf{u} = \begin{pmatrix} u_x \\ -2u_x \\ u_z \end{pmatrix} = u_x \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + u_z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. A possible basis of this orthogonal space is

$$\mathbf{u}_2 = \frac{1}{\sqrt{5}}(1, -2, 0)^T$$

$$\mathbf{u}_3 = (0, 0, 1)^T$$

Note that $\mathbf{u}_2$ and $\mathbf{u}_3$ are orthogonal to each other. Finally, the matrix U sought is

$$U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, the SVD decomposition is

$$A = U\Sigma V^T = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T$$

Lay, 7.4.11

Carlos Oscar Sorzano, Aug. 31st, 2013

Find the singular value decomposition of the matrix $A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix}$

Solution: We compute A^TA

$$A^TA = \begin{pmatrix} 81 & -27 \\ -27 & 9 \end{pmatrix}$$

Its eigenvalues and eigenvectors are

$$\begin{array}{c} \lambda_1 = 90,\ \mathbf{v}_1 = (0.9487, -0.3162) \\ \lambda_2 = 0,\ \mathbf{v}_2 = (0.3162, 0.9487) \end{array}$$

We now construct the matrices V and Σ as

$$\begin{array}{c} V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 0.9487 & 0.3162 \\ -0.3162 & 0.9487 \end{pmatrix} \\ \Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 9.4868 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

To construct the matrix U we calculate for the non-zero singular values

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = (-\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3})$$

We now need to extend the set $\{\mathbf{u}_1\}$ to become a basis of $\mathbb{R}^3$. To do so, we add the vectors

$$\begin{array}{c} \mathbf{u}_2 = (\tfrac{2}{3}, -\tfrac{1}{3}, \tfrac{2}{3}) \\ \mathbf{u}_3 = (\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}) \end{array}$$

The matrix U is

$$U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Finally, the SVD decomposition of A is

$$\begin{array}{c} A = U\Sigma V^T \\ \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 9.4868 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.9487 & -0.3162 \\ 0.3162 & 0.9487 \end{pmatrix} \end{array}$$

Lay, 7.4.15

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose the factorization below is an SVD of a matrix A, with the entries U and V rounded to two decimal places.

$$A = \begin{pmatrix} 0.40 & -0.78 & 0.47 \\ 0.37 & -0.33 & -0.87 \\ -0.84 & -0.52 & -0.16 \end{pmatrix} \begin{pmatrix} 7.10 & 0 & 0 \\ 0 & 3.10 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.30 & -0.51 & -0.81 \\ 0.76 & 0.64 & -0.12 \\ 0.58 & -0.58 & 0.58 \end{pmatrix}$$

a. What is the rank of A?

b. Use the decomposition of A, with no calculations, to write a basis for Col$\{A\}$ and a basis for Nul$\{A\}$. [*Hint*: First write the columns of V.]

Solution: The factorization above is of the form $A = U\Sigma V^T$. A is a 3×3 matrix ($n = 3$).

a. Since A has only two non-zero singular values, its rank is 2.

b. V is $\begin{pmatrix} 0.30 & 0.76 & 0.58 \\ -0.51 & 0.64 & -0.58 \\ -0.81 & -0.12 & 0.58 \end{pmatrix}$. Since A is of rank 2, the first two columns of U provide a basis for Col$\{A\}$

$$\text{Basis}\{\text{Col}\{A\}\} = \{(0.40, 0.37, -0.84), (-0.78, -0.33, -0.52)\}$$

Also, the last column ($n - r = 3 - 2 = 1$) of V provides a basis for Nul$\{A\}$

$$\text{Basis}\{\text{Nul}\{A\}\} = \{(0.58, -0.58, 0.58)\}$$

Lay, 7.4.17

Carlos Oscar Sorzano, Aug. 31st, 2013

Suppose A is square and invertible. Find a Singular Value Decomposition of A^{-1}

Solution: Let $A = U\Sigma V^T$ be a Singular Value Decomposition of the matrix A. Since A is invertible, Σ is full rank. Since A is square, U and V are square matrices, and they are always orthogonal matrices. So, we have

$$A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$$

that is a Singular Value Decomposition of A^{-1}.

Lay, 7.4.18

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if A is square, then $|\det\{A\}|$ is the product of the singular values of A.

Solution: Let $A = U\Sigma V^T$ be a Singular Value Decomposition of the matrix A. If A is square, then we may calculate its determinant and its absolute value as

$$|\det\{A\}| = |\det\{U\Sigma V^T\}| = |\det\{U\}\det\{\Sigma\}\det\{V^T\}| = \\ |\det\{U\}||\det\{\Sigma\}||\det\{V^T\}|$$

U and V are orthogonal matrices. This implies that their determinant is 1 or -1. Then,

$$|\det\{A\}| = |\det\{\Sigma\}|$$

But Σ is a diagonal matrix, so its determinant is the product of its diagonal entries, that are the singular values of A (which are all non-negative values)

$$|\det\{A\}| = \left|\prod_{i=1}^{n} \Sigma_{ii}\right| = \prod_{i=1}^{n} |\Sigma_{ii}| = \prod_{i=1}^{n} \sigma_i$$

Lay, 7.4.19

Carlos Oscar Sorzano, Aug. 31st, 2013

Given a SVD decomposition of a matrix A, $A = U\Sigma V^T$, show that the columns of V are eigenvectors of A^TA, the columns of U are eigenvectors of AA^T, and the diagonal entries of Σ are the singular values of A. [*Hint*: Use the SVD to compute A^TA and AA^T.]
Solution: Let us calculate A^TA

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^TU^TU\Sigma V^T = V(\Sigma^T\Sigma)V^T$$

But this is an eigendecomposition of A^TA because V is an orthogonal matrix $V^T = V^{-1}$ and $\Sigma^T\Sigma$ is a diagonal $n \times n$ matrix with r values σ_i^2 (being σ_i the singular values of A and r the number of non-zero singular values of A) and $n-r$ zeros. By the Diagonalization Theorem (Theorem 5.3.5), we have that the columns of V are the eigenvectors of A and the diagonal entries of $\Sigma^T\Sigma$ their corresponding eigenvalues.

Since the eigenvalues of A^TA are σ_i^2, σ_i are the singular values of the matrix A.

We can proceed analogously with AA^T

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^TV\Sigma^TU^T = U(\Sigma\Sigma^T)U^T$$

Similarly, the columns of U are the eigenvectors of the matrix AA^T and $\Sigma\Sigma^T$ is an $m \times m$ diagonal matrix with the eigenvalues of AA^T (r of them are non-zero and $m-r$ are zero).

Lay, 7.4.20

Carlos Oscar Sorzano, Aug. 31st, 2013

Show that if A is a positive definite matrix, then an orthogonal diagonalization $A = PDP^T$ is a singular value decomposition of A.
Solution: Let us calculate A^TA and consider the SVD $A = U\Sigma V^T$

$$A^TA = (PDP^T)^T(PDP^T) = PD^TP^TPDP^T = P(D^TD)P^T$$

Since A is positive definite D^TD is a diagonal entry whose ii-th entry is λ_i^2. So its singular value is $\sigma_i = \sqrt{\lambda_i^2} = \lambda_i$. That is, for an SVD, we have $\Sigma = D$.

By Exercise 7.4.19, we now that the columns of V are the eigenvectors of A^TA. Given the diagonalization $A^TA = P(D^TD)P^T$ and the Diagonalization Theorem (Theorem 5.3.5), we see that the columns of P are the eigenvectors of A^TA. So, we can make $V = P$

Similarly, if we calculate AA^T we have

$$AA^T = (PDP^T)(PDP^T)^T = PDP^TPD^TP^T = P(DD^T)P^T$$

Again, this decomposition along with the Diagonalization Theorem show that the columns of P are the eigenvectors of AA^T and by Exercise 7.4.19, we can make $U = P$.

Finally, the SVD decomposition of A becomes

$$A = U\Sigma V^T = PDP^T$$

Lay, 7.4.21

Carlos Oscar Sorzano, Jan. 20th, 2014

Show that if P is an orthogonal $m \times m$ matrix, then PA has the same singular values as A.

Solution: We know that the singular values of a matrix A are the square root of the eigenvalues of the matrix $A^T A$. The singular values of PA will be the square root of the eigenvalues of the matrix

$$(PA)^T(PA) = A^T P^T PA = A^T A \tag{12}$$

where we have made use of the fact that P is orthogonal and, consequently, $P^T P = I$.

Lay, 7.4.23

Carlos Oscar Sorzano, Aug. 31st, 2013

Given the Singular Value Decomposition theorem:

> Let $A \in \mathcal{M}_{m\times n}$ be a matrix with rank r. Then, there exists a matrix $\Sigma \in \mathcal{M}_{m\times n}$ whose diagonal entries are the first r singular values of A sorted in descending order ($\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$) and there exist orthogonal matrices $U \in \mathcal{M}_{m\times m}$ and $V \in \mathcal{M}_{n\times n}$ such that
>
> $$A = U\Sigma V^T$$

Let $U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & ... & \mathbf{u}_m \end{pmatrix}$ and $V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & ... & \mathbf{v}_n \end{pmatrix}$. Show that

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + ... + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

Solution: If we expand the SVD, we have

$$\begin{aligned}
A &= U\Sigma V^T \\
&= \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & ... & \mathbf{u}_m \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & ... & 0 & 0 & ... & 0 \\ 0 & \sigma_2 & ... & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & \sigma_r & 0 & ... & 0 \\ 0 & 0 & ... & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 0 & ... & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ ... \\ \mathbf{v}_n^T \end{pmatrix} \\
&= \begin{pmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & ... & \sigma_r\mathbf{u}_r & \mathbf{0} & ... & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ ... \\ \mathbf{v}_n^T \end{pmatrix} \\
&= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + ... + \sigma_r \mathbf{u}_r \mathbf{v}_r^T
\end{aligned}$$

Lay, 7.4.24

Carlos Oscar Sorzano, Aug. 31st, 2013

Using the notation of Exercise 7.4.23, show that $A^T \mathbf{u}_j = \sigma_j \mathbf{v}_j$.

Solution: Let's calculate first A^T

$A^T = (\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + ... + \sigma_r \mathbf{u}_r \mathbf{v}_r^T)^T = \sigma_1 \mathbf{v}_1 \mathbf{u}_1^T + \sigma_2 \mathbf{v}_2 \mathbf{u}_2^T + ... + \sigma_r \mathbf{v}_r \mathbf{u}_r^T$

Now, we can easily calculate $A^T \mathbf{u}_j$

$$A^T \mathbf{u}_j = \left(\sum_{i=1}^{r} \sigma_i \mathbf{v}_i \mathbf{u}_i^T \right) \mathbf{u}_j = \sum_{i=1}^{r} \sigma_i \mathbf{v}_i \mathbf{u}_i^T \mathbf{u}_j$$

Since the columns of U are orthogonal to each other all products $\mathbf{u}_i^T \mathbf{u}_j$ are 0 if $i \neq j$ and 1 if $i = j$. Then, the previous sum reduces to

$$A^T \mathbf{u}_j = \sigma_j \mathbf{v}_j$$