Chapter 0. Introduction to the Mathematical Method

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Outline

0. Mathematical language
   - Axioms, postulates, definitions and propositions (a)
   - Logical operators (a)
   - Qualifiers (b)
   - Mathematical proofs
     - Modus ponens (b)
     - Modus tollens (c)
     - Reductio ad absurdum (c)
     - Induction (c)
     - Case distinction (c)
     - Counterexample (c)
   - Common math mistakes (c)

References

M. de Guzmán Ozámiz. Cómo hablar, demostrar y resolver en Matemáticas. Anaya (2003)

A little bit of history

Modern logic is based on precise calculus rules and was born in the middle of the XIXth century with Gottfried Leibniz (1847), George Boole (1847), Augustus de Morgan (1847) and Bertrand Russell (1910).

To know more about the history of logic visit

- http://individual.utoronto.ca/pking/miscellaneous/history-of-logic.pdf
- http://en.wikipedia.org/wiki/History_of_logic

Outline

Axioms, postulates and propositions

Axioms, postulates and propositions

Mathematical language has to be **uniform** (everybody must use it in the same way) and **univocal** (i.e., without any kind of ambiguity). We start from some initial statements called **axioms, postulates and definitions**. These elements are not questioned, they are not true or false, they simply are, and they serve to build a logical reasoning.

Example

- Axiom: If A and B are equal to C, then A is equal to B.
- Postulate: For any two points, there is a unique straight line that joins them.
- Definition: A prime number is a natural number that can only be divided by 1 and itself.

Propositions

Propositions

Based on axioms, postulates and definitions, we can construct **propositions** that are statements that refer to already introduced objects. Propositions can be **true or false**. They are named with capital letters A, B, C, ...

Example

$2+3$ (is not a proposition)

A: $2+3=5$ (is a true proposition)

B: $2+3=7$ (is a false proposition)

Construction of new propositions

We can construct new propositions using already existing ones and logical operators

Example

A: 2+2=4 (true)

B: 2+3=5 (true)

C: 2+3=7 (false)

D: A y B (true)

E: A o C (true)

and quantifiers

Example

A: **Some** numbers are prime (true)

B: **All** even numbers can be divided by 2 (true)

C: **None** of odd numbers can be divided by 2 (true)

Outline

$\overline{A}$ (not A)

Definition

$\overline{A}$ is true if A is false, and $\overline{A}$ is false if A is true.

Example

A: 3+2=5 (true)

B: $\neg A \equiv 3 + 2 \neq 5$ (false)

C: 3+2=6 (false)

D: $\neg C \equiv 3 + 2 \neq 6$ (true)

Truth table

A	$\overline{A}$
F	T
T	F

Properties

$$\overline{\overline{A}} = A$$

$\overline{A}$ (not A)

A double negation is a positive statement.

Example

A: 3+2=5 (true)

B: $\overline{A} \equiv 3 + 2 \neq 5$ (false)

C: $\overline{B} \equiv 3 + 2 = 5$ (true)

Example

It is not true that John is not at home.

A: John is at home

B: $\overline{A} \equiv$ Not (John is at home) $\equiv$ John is not at home

C: $\overline{B} \equiv$ Not (John is not at home) $\equiv$ John is at home $\equiv$ A

If C is true, then A is true. Therefore, John is at home.

$A \cap B$ (A and B)

Truth table

A	B	$A \cap B$
F	F	F
F	T	F
T	F	F
T	T	T

Properties

$A \cap B = B \cap A$

Example

A: 3+2=5 (true)

B: 2+2=4 (true)

C: $A \cap B \equiv$ 3+2=5 and 2+2=4 (true)

D: 3+2=6 (false)

E: $D \cap B \equiv$ 3+2=6 and 2+2=4 (false)

$A \cap B$ (A and B)

The common language AND is sometimes equivalent to the mathematical AND

Example

Triangle ABC and triangle A'B'C' are equilateral $\Rightarrow$

A: ABC is equilateral

B: A'B'C' is equilateral

C: $A \cap B \equiv$ Triangle ABC is equilateral AND Triangle A'B'C' is equilateral

and sometimes not

Example

Triangle ABC and triangle A'B'C' are similar $\not\Rightarrow$

A: ABC is similar

B: A'B'C' is similar

C: $A \cap B \equiv$ Triangle ABC is similar AND Triangle A'B'C' is similar

$A \cup B$ (A or B; A and/or B)

15% discounts for customers having a student card or university card. Of course, people with both cards have a 15% discount. Inclusive OR.

Truth table

A	B	$A \cup B$
F	F	F
F	T	T
T	F	T
T	T	T

Properties

$A \cup B = B \cup A$

Example

A: 3+2=5 (true)

B: 2+2=4 (true)

C: $A \cup B \equiv$ 3+2=5 or 2+2=4 (true)

D: 3+2=6 (false)

E: $D \cup B \equiv$ 3+2=6 or 2+2=4 (true)

$A \oplus B$ (either A or B; A xor B (eXclusive or))

We'll go to Paris or Berlin. Either Paris or Berlin, we cannot go to both places at the same time. Exclusive OR.

Truth table

A	B	$A \oplus B$
F	F	F
F	T	T
T	F	T
T	T	F

Properties

$A \oplus B = B \oplus A$

Example

A: a<5

B: a=5

C: $A \oplus B \equiv a \leq 5$

If $a = 3$, then C is true. If $a = 6$, then C is false.

Negation of and

$\overline{A \cap B} = \overline{A} \cup \overline{B}$
This is one of Morgan's laws.

A	B	$A \cap B$	$\overline{A \cap B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cup \overline{B}$
F	F	F	**T**	T	T	**T**
F	T	F	**T**	T	F	**T**
T	F	F	**T**	F	T	**T**
T	T	T	**F**	F	F	**F**

Example

A: It rained on Monday

B: It rained on Tuesday

C: $\overline{A \cap B} \equiv$ It is not true that it rained on both days $\equiv$ Either it did not rain on Monday or it did not rain on Tuesday.

Negation of or

Inclusive OR: $\overline{A \cup B} = \overline{A} \cap \overline{B}$
This is another Morgan's law.

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
F	F	F	**T**	T	T	**T**
F	T	T	**F**	T	F	**F**
T	F	T	**F**	F	T	**F**
T	T	T	**F**	F	F	**F**

Exclusive OR: $\overline{A \oplus B} = (\overline{A} \cap B) \cup (A \cap \overline{B})$

A	B	$A \oplus B$	$\overline{A \oplus B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap B$	$A \cap \overline{B}$	$(\overline{A} \cap B) \cup (A \cap \overline{B})$
F	F	F	**T**	T	T	F	F	**T**
F	T	T	**F**	T	F	T	F	**F**
T	F	T	**F**	F	T	F	T	**F**
T	T	F	**T**	F	F	F	F	**T**

$A \Rightarrow B$ (A implies B)

Natural language

- A implies B
- A is sufficient for B
- A guarantees B
- B is necessary for A
- If A, then B
- If not B, then not A

Truth table

A	B	$A \Rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

$A \Rightarrow B$ (A implies B)

In natural language "If ..., then ..." is not used in the mathematical sense.

Example

If it rains, I'll stay at home.

If he is at home, is it raining?
We don't know, he didn't say what he would do if it was not raining.

Example

I'm going to the bank. If it is open, I'll bring 1000 euros.

If he is back with 1000 euros, is the bank open?
We don't know, maybe a very good friend of his gave him 1000 euros.

$A \Rightarrow B$ (If not B, then not A)

Example

I'm going to the bank. If it is open, I'll bring 1000 euros.

If I'm back without 1000 euros, is the bank open?
No, let's see why

A: Bank is open

B: I bring 1000 euros

A	B	$A \Rightarrow B$	Why
F	F	T	The bank was closed
F	T	T	A friend gave me
T	F	F	I lied
T	T	T	I withdrew 1000 euros from bank

There is only one situation in which my statement is true (I did not lie) and in which I do not bring 1000 euros (B is false) that is when the bank is closed (A is also false).

$A \Rightarrow B$ (If not B, then not A)

We can generally formulate this analysis as

Properties

$$A \Rightarrow B = \overline{B} \Rightarrow \overline{A}$$

A	B	$A \Rightarrow B$	$\overline{B}$	$\overline{A}$	$\overline{B} \Rightarrow \overline{A}$
F	F	**T**	T	T	**T**
F	T	**T**	F	T	**T**
T	F	**F**	T	F	**F**
T	T	**T**	F	F	**T**

$A \Rightarrow B$ (Not (A and not B))

Another interesting property

Properties

$$A \Rightarrow B = \overline{A \cap \overline{B}}$$
$$\overline{A \Rightarrow B} = A \cap \overline{B}$$

The proof of these properties is left to the reader.

Example

I'm going to the bank. If it is open, I'll bring 1000 euros.

It is equivalent to:

It will not be the case that (the bank is open (A) and I don't bring 1000 euros (not B)).

$A \Leftrightarrow B$ (A if and only if B)

Truth table

A	B	$A \Leftrightarrow B$
F	F	T
F	T	F
T	F	F
T	T	T

Properties

$$A \Leftrightarrow B = (A \Rightarrow B) \cap (B \Rightarrow A)$$

In plain language, we say:
A is necessary and sufficient for B
B is necessary and sufficient for A

Outline

Qualifiers

Example

There might be **a** person that reads **all** newspapers **every** day.
Every day, there might be **a** person that reads **all** newspapers.
Every one reads a newspaper **every** day.
Every day, there is **a** newspaper that **everybody** reads.

Example

We say that the limit of the function $f(x)$ when x goes to x_0 is y if and only if **for all** positive numbers (ϵ), **there exists** another positive number (δ) **such that** if the distance between x and x_0 is smaller than δ, then the distance between $f(x)$ and y is smaller than ϵ.

$$\lim_{x \to x_0} f(x) = y \Leftrightarrow \forall \epsilon > 0 \, \exists \delta > 0 \mid |x - x_0| < \delta \Rightarrow |f(x) - y| < \epsilon$$

$\forall$ (for all), $\exists$ (exists) and $\exists!$ (exists only one)

For all x in P	$\forall x, x \in P$; $\forall x \in P$
For any x in P	$\forall x, x \in P$; $\forall x \in P$
For each x with the property P	$\forall x, P(x)$
There **exists at least one** x in P	$\exists x, x \in P$; $\exists x \in P$
For at least one x in P	$\exists x, x \in P$; $\exists x \in P$
There **exists at least one** x with the property P	$\exists x, P(x)$
There **exists exactly one** x in P	$\exists! x, x \in P$; $\exists! x \in P$

Example

For all real numbers	$\forall x \in \mathbb{R}$
For all real numbers smaller than 4	$\forall x \in \mathbb{R}, x < 4$
There **exists at least one** real number	$\exists x \in \mathbb{R}$
There **exists at least one** real number greater than 2	$\exists x \in \mathbb{R}, x > 2$
There **exists a single** real number such that ...	$\exists! x \in \mathbb{R} \vert$ ...

| (such that, it is verified, verifying)

Example

There must be people that read all newspapers everyday. Let P be the set of all persons, let N be the set of all newspapers, and let D be the set of all days. Then, the previous sentence is formalized as

$\exists p \in P | \forall d \in D | \forall n \in N |$ p reads n on d.

Literal reading: There exist at least one person **such that** for all days and for all newspapers **it is verified that** p reads n on d.

Example

Every day, there must be someone that reads all newspapers.

$\forall d \in D | \exists p \in P | \forall n \in N |$ p reads n on d.

Literal reading: For all days **it is verified that** there exists at least one person **verifying that** for all newspapers **it is verified that** p reads n on d.

| (such that, it is verified, verifying)

Example

$$\lim_{x \to x_0} f(x) = y \Leftrightarrow \forall \epsilon > 0 \, \exists \delta > 0 \mid \; |x - x_0| < \delta \Rightarrow |f(x) - y| < \epsilon$$

Literal reading: the limit of $f(x)$ when x goes to x_0 is y if and only if for any ϵ greater than 0, there exists δ greater than 0 **such that** if $|x - x_0| < \delta$ is true, then $|f(x) - y| < \epsilon$ is also true.

Example

Fermat-Wiles Theorem:
$\forall n \in \mathbb{Z}, n > 2 | \forall (x, y, z) \in \mathbb{R}^3, x^n + y^n = z^n | xyz = 0$

Literal reading: For all integer numbers **it is verified that** for any real numbers x, y z with the property $x^n + y^n = z^n$ **it is verified that** at least one of the three numbers is 0.

Negation of qualifiers

Let's say we state that all elements in a given set S has a certain property ($\forall x \in S | P(x))$. The negation of this statement is that there exists at least one element of S that does not have that property $(\exists x \in S | \overline{P(x)})$.

Similarly, if we state that there exists at least one element in a given set S that has a certain property ($\exists x \in S | P(x))$. The negation of this statement is that none of the elements of S have that property $(\forall x \in S | \overline{P(x)})$.

Negation of qualifiers

Example

In a previous example we had: **There must be people that read all newspapers everyday**. Its negation is

$$\overline{\exists p \in P | \forall d \in D | \forall n \in N | \text{p reads n on d.}} =$$
$$\forall p \in P | \overline{\forall d \in D | \forall n \in N | \text{p reads n on d.}} =$$
$$\forall p \in P | \exists d \in D | \overline{\forall n \in N | \text{p reads n on d.}} =$$
$$\forall p \in P | \exists d \in D | \exists n \in N | \overline{\text{p reads n on d.}} =$$
$$\forall p \in P | \exists d \in D | \exists n \in N | \text{p does not read n on d.}$$

That is, **For everybody, there is at least one day and one paper, such that p did not read n on d.**

Outline

0 Mathematical language
- Axioms, postulates, definitions and propositions (a)
- Logical operators (a)
- Qualifiers (b)
- Mathematical proofs
  - Modus ponens (b)
  - Modus tollens (c)
  - Reductio ad absurdum (c)
  - Induction (c)
  - Case distinction (c)
  - Counterexample (c)
- Common math mistakes (c)

Modus ponens

The following proofs follow a reasoning model called *Modus ponens* which is formally written as

$$(A \cap (A \Rightarrow B)) \Rightarrow B.$$

The intuitive meaning is that if A is true and $A \Rightarrow B$, then B is also true. Most proofs follow this way of reasoning. They can be performed in a forward way

$$A \Rightarrow B_1 \Rightarrow B_2 \Rightarrow ... \Rightarrow B$$

or in a backward way

$$B \Leftarrow B_n \Leftarrow B_{n-1} \Leftarrow ... \Leftarrow A.$$

Forward proofs ($A \Rightarrow B$; B is necessary for A)

Example

Prove that the third power of an odd number is odd.

Proof

Let there be the following propositions:

A: x is odd.

B: x^3 is odd.

We need to prove that $A \Rightarrow B$ (B is necessary for A).

Proof $A \Rightarrow B$

Since x is an odd number we can write $x = 2k + 1$ for some integer number k. Then,
$x^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 = 2k' + 1$.
For $k' = 4k^3 + 6k^2 + 3k$, which is another integer number. Therefore, x^3 is odd.

Forward proofs ($A \Rightarrow B$; B is necessary for A)

Example

A necessary condition for a natural number to be a multiple of 360 is that it is a multiple of 3 and 120.

Proof

Let there be the following propositions:

A: To be multiple of 360

B: To be multiple of 3 and 120

We need to prove that $A \Rightarrow B$ (B is necessary for A).

Proof $A \Rightarrow B$

Let x be a multiple of 360 (A) $\Rightarrow$ There exists a natural number k such that $x = 360 \cdot k \Rightarrow x = 120 \cdot 3 \cdot k$. From this factorization, it is obvious that x is a multiple of 120 and a multiple of 3 (B).

Forward proofs ($A \Leftarrow B$; B is sufficient for A)

Example

A sufficient condition for a natural number to be a multiple of 360 is that it is a multiple of 3 and 120. *Proof*

Let there be the following propositions:

A: To be multiple of 360

B1: To be multiple of 3

B2: To be multiple of 120

B: B1 $\cap$ B2

We need to prove that $B \Rightarrow A$ (B is sufficient for A).

Proof $B \Rightarrow A$

We can easily prove that $B \not\Rightarrow A$ with a counterexample. Let us consider $x = 240$. It is a multiple of 3 (B1). It is a multiple of 120 (B2). Therefore, B is true. However, 240 is not a multiple of 360 (A is false). Therefore, we have proved that $B \not\Rightarrow A$.

Forward proofs ($A \Rightarrow B$)

Example

Show that $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ is solution of the equation $ax^2 + bx + c = 0$

Proof

Let there be the following propositions:

A: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$

B: $ax^2 + bx + c = 0$

We need to prove that $A \Rightarrow B$.

If $A \Rightarrow B$ is true, then it must also be true that $A \Rightarrow B_1$

$$B_1\colon\ a\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)^2 + b\frac{-b+\sqrt{b^2-4ac}}{2a} + c = 0$$

that we can rewrite as

$$B_1\colon\ a\left(\frac{b^2}{4a^2} + \frac{b^2-4ac}{4a^2} - \frac{2b\sqrt{b^2-4ac}}{4a^2}\right) + b\frac{-b+\sqrt{b^2-4ac}}{2a} + c = 0$$

Forward proofs ($A \Rightarrow B$)

Example (continued)

that we can simplify to

$$B_1\colon\ \frac{b^2}{4a} + \frac{b^2}{4a} - c - \frac{b\sqrt{b^2-4ac}}{2a} + \frac{-b^2}{2a} + \frac{b\sqrt{b^2-4ac}}{2a} + c = 0$$

$$B_1\colon\ \cancel{\frac{b^2}{4a}} + \cancel{\frac{b^2}{4a}} - \cancel{c} - \cancel{\frac{b\sqrt{b^2-4ac}}{2a}} + \cancel{\frac{-b^2}{2a}} + \cancel{\frac{b\sqrt{b^2-4ac}}{2a}} + \cancel{c} = 0$$

$$B_1\colon\ 0 = 0$$

Since B_1 is always true (a statement that is always true is called a tautology), then $A \Rightarrow B_1$ is true, as we wanted.

Forward proofs ($A \Leftrightarrow B$)

In this case we have to prove both directions: $A \Rightarrow B$ and $B \Rightarrow A$.

Example

A necessary and sufficient condition for a natural number to be a multiple of 360 is that it is a multiple of 5 and 72.

Proof

Let there be the following propositions:

A: To be multiple of 360

B1: To be multiple of 5

B2: To be multiple of 72

B: B1 $\cap$ B2

We need to prove that $A \Leftrightarrow B$, that is, $A \Rightarrow B$ and $B \Rightarrow A$

Proof $A \Rightarrow B$

Let x be a multiple of 360 (A) $\Rightarrow$ There exists a natural number k such that $x = 360 \cdot k \Rightarrow x = 72 \cdot 5 \cdot k$. From this factorization, it is obvious that x is a multiple of 72 and a multiple of 5 (B).

Forward proofs ($A \Leftrightarrow B$)

Example (continued)

Proof $B \Rightarrow A$

$B1 \Rightarrow$ There is a natural number k_1 such that $x = 5 \cdot k_1$

$B2 \Rightarrow$ There is a natural number k_2 such that $x = 72 \cdot k_2$

Therefore, $5k_1 = 72k_2 \Rightarrow k_1 = \frac{72}{5}k_2$. But k_1 is a natural number not a rational number, therefore, k_2 needs to be a multiple of 5, i.e., there exists a natural number k_3 such that $k_2 = 5 \cdot k_3$. Consequently, considering B2, we have $x = 72 \cdot 5 \cdot k_3 = 360 \cdot k_3$. That is x is a multiple of 360. Therefore, we have proved that $A \Rightarrow B$.

More forward proofs

If I want to prove that A does not imply B ($A \Rightarrow B$ is false), I have to prove that B is false, but A is true.

Example

In our example, I have to prove that you did not bring 1000 euros (B is false), but the bank is open (A is true). I don't have to prove that

- B is false (you did not bring 1000 euros)
- A is false (the bank is closed)
- B is true but A is false (you brought 1000 euros, but the bank is closed)
- A and B are false (you did not bring 1000 euros, and the bank is closed)

More forward proofs

If I know that B is false, and I want to proof that A implies B ($A \Rightarrow B$ is true), then I have to prove that A is also false.

A	B	$A \Rightarrow B$
F	**F**	**T**
F	T	T
T	F	F
T	T	T

Example

If I know that you did not bring 1000 euros (B is false), all I have to prove to show that $A \Rightarrow B$ is true, is that the bank is closed (A is false).

More forward proofs

If I want to proof that A implies B or C ($A \Rightarrow B \cup C$ is true), and I prove that it is false that $A \Rightarrow B \cap C$, have I finished? No,let's see why

A	B	C	$B \cup C$	$A \Rightarrow B \cup C$	$B \cap C$	$A \Rightarrow B \cap C$
F	F	F	F	T	F	T
F	F	T	T	T	F	T
F	T	F	T	T	F	T
F	T	T	T	T	T	T
T	F	F	F	F	F	F
T	F	T	T	T	F	F
T	T	F	T	T	F	F
T	T	T	T	T	T	T

More forward proofs

If I prove that $A \Rightarrow B \cap C$ is false, that amounts to selecting the following rows from the table

A	B	C	$B \cup C$	$A \Rightarrow B \cup C$	$B \cap C$	$A \Rightarrow B \cap C$
T	F	F	F	F	F	F
T	F	T	T	T	F	F
T	T	F	T	T	F	F

In those lines, $A \Rightarrow B \cup C$ is true for two of the A, B, C combinations (that's good), but false for the other (that's bad). Therefore, we have not finished yet and we have to prove that either B or C is true, so that we can finally reduce the table to

A	B	C	$B \cup C$	$A \Rightarrow B \cup C$	$B \cap C$	$A \Rightarrow B \cap C$
T	F	T	T	T	F	F
T	T	F	T	T	F	F

in which $A \Rightarrow B \cup C$ is true, and consequently, we have proved that $A \Rightarrow B \cup C$.

Backward proofs ($A \Rightarrow B$)

Example

Show that if $x > 0$, then $x + \frac{1}{x} \geq 2$.

Proof

Let there be the following propositions:

A: $x > 0$

B: $x + \frac{1}{x} \geq 2$

It is obvious that $C_1 \Rightarrow B$, $C_2 \Rightarrow C_1$, $C_3 \Rightarrow C_2$ being

C1: $x + \frac{1}{x} - 2 \geq 0$

C2: $\frac{x^2+1-2x}{x} \geq 0$

C3: $\frac{(x-1)^2}{x} \geq 0$

It is also obvious that $A \Rightarrow C_3$ and, in this way, we have proved that $A \Rightarrow B$. We can simplify the writing of this proof as:

$x + \frac{1}{x} \geq 2 \Leftarrow x + \frac{1}{x} - 2 \geq 0 \Leftarrow \frac{x^2+1-2x}{x} \geq 0 \Leftarrow \frac{(x-1)^2}{x} \geq 0 \Leftarrow x > 0$

Backward proofs ($A \Rightarrow B$)

Example

If $x, y \in \mathbb{R}$, $x, y > 0$, then $\sqrt{xy} \leq \frac{x+y}{2}$

Proof

$\sqrt{xy} \leq \frac{x+y}{2} \Leftarrow \sqrt{xy} - \frac{x+y}{2} \leq 0 \Leftarrow \frac{x+y}{2} - \sqrt{xy} \geq 0$

Since x and y are positive numbers, we can write them as $x = a^2$ and $y = b^2$.

Then, $\frac{x+y}{2} - \sqrt{xy} \geq 0 \Leftarrow \frac{a^2+b^2}{2} - ab \geq 0 \Leftarrow a^2 + b^2 - 2ab \geq 0 \Leftarrow (a-b)^2 \geq 0$

This last proposition is always true, therefore $\sqrt{xy} \leq \frac{x+y}{2}$ is also true.

Outline

Modus tollens

The following proofs follow a reasoning model called *Modus tollens* which is formally written as

$$\left(\overline{B} \cap (A \Rightarrow B)\right) \Rightarrow \overline{A}.$$

The intuitive meaning is that if $A \Rightarrow B$ is true and B is false, then A must also be false. Another way of writing this reasoning is

$$(A \Rightarrow B) \Leftrightarrow (\overline{B} \Rightarrow \overline{A}).$$

That is if we want to prove $A \Rightarrow B$, it is enough to prove $\overline{B} \Rightarrow \overline{A}$.

Modus tollens

Example

Show that if x^3 is even, then x is even.

Proof

Let there be the following propositions:

A: x^3 is even

B: x is even

We want to prove that $A \Rightarrow B$. Instead, we'll prove that $\overline{B} \Rightarrow \overline{A}$, with

$\overline{B}$: x is odd

$\overline{A}$: x^3 is odd

But we already proved this in a previous example. Therefore, $A \Rightarrow B$ is true.

Modus tollens

Example

Show that if c is odd, then the equation $n^2 + n - c = 0$ has no integer solution.

Proof

Let there be the following propositions:

A: c is odd

B: $n^2 + n - c = 0$ has no integer solution

We want to prove that $A \Rightarrow B$. Instead, we'll prove that $\overline{B} \Rightarrow \overline{A}$, with

$\overline{B}$: $n^2 + n - c = 0$ has an integer solution

$\overline{A}$: c is even

Proof $\overline{B} \Rightarrow \overline{A}$

Let's assume that $n \in \mathbb{Z}$ is solution of $n^2 + n - c = 0$.
If n is even, then c is even because $c = n^2 + n = (2k)^2 + 2k = 2(2k^2 + k)$.
If n is odd, then c is also even because $c = n^2 + n = (2k+1)^2 + (2k+1) = 2(2k^2 + 3k + 1)$.

Outline

Reductio ad absurdum

The following proofs follow a reasoning model called *Reductio ad absurdum* which is formally written as

$$A \Rightarrow B \Leftrightarrow (A \cap \overline{B} \Rightarrow \text{absurdum}).$$

Absurdum is a statement that is always false, like $P \cap \overline{P}$. Let's analyze the truth table for this proposition

Truth table

A	B	$A \Rightarrow B$	$A \cap \overline{B}$	$P \cap \overline{P}$	$A \cap \overline{B} \Rightarrow (P \cap \overline{P})$
F	F	T	F	F	T
F	T	T	F	F	T
T	F	F	T	F	F
T	T	T	F	F	T

We see that the third and sixth columns are identical.

Reductio ad absurdum

Example

Show that $\sqrt{2}$ is irrational.

Proof

It does not appear in the form $A \Rightarrow B$ but it can be put with

A: All facts we know about numbers

B: $\sqrt{2}$ is irrational

Let's assume that $\sqrt{2}$ is rational ($\overline{B}$), that is $\exists p, q \in \mathbb{Z} | \sqrt{2} = \frac{p}{q}$ and p, q are irreducible (they don't have any common factor). If this is true, then $2q^2 = p^2$, i.e., 2 must be a factor of p and consequently p must be $p = 2r$. Substituting this knowledge into $2q^2 = p^2$ we obtain $2q^2 = (2r)^2 \Rightarrow q^2 = 2r^2$. Consequently, 2 is another factor of q. But we presumed that

P: p and q were irreducible

So, if $\sqrt{2}$ is rational, then we have P and $\overline{P}$ at the same time, which is a contradiction, and therefore $\sqrt{2}$ cannot be rational.

Reductio ad absurdum

Example

Show that there are infinite prime numbers.

Proof

Let's presume they is a finite list of prime numbers (in ascending order):

$$2, 3, 5, 7, ..., P$$

Now we construct the number $M = 2 \cdot 3 \cdot 5 \cdot 7 \cdot ... \cdot P + 1$.
If M is prime, then we have a contradiction is M is prime and is larger than P.
If M is not prime, then it has as a factor at least one of the prime numbers in the list. Let's assume it is 3, that is

$$M = 3H = 2 \cdot 3 \cdot 5 \cdot 7 \cdot ... \cdot P + 1 \Rightarrow 1 = 3(H - 2 \cdot 5 \cdot 7 \cdot ... \cdot P)$$

that means that 3 is a factor of 1, which is an absurdum.

Outline

Weak induction

This is a strategy to prove a property of a natural number, $P(n)$. We follow the strategy below:

1. Prove that $P(k)$ is true.
2. Prove that if $P(n-1)$ is true, then $P(n)$ is also true

Example

Show that $S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof

1. $S_1 = \sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1$, which is obviusly true.
2. Let's assume that $S_{n-1} = \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$. Then, we need to prove that $S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. But $S_n = S_{n-1} + n = \frac{(n-1)n}{2} + n = n\left(\frac{n-1}{2} + 1\right) = \frac{n(n+1)}{2}$. q.e.d.

Strong induction

The goal is similar to the previous method, but now in the second step we assume that the property is true for all previous integers

1. Prove that $P(k)$ is true.
2. Prove that if $P(k)$ is true and $P(k+1)$ is true and ... $P(n-1)$ is true, then $P(n)$ is also true

Example: Fundamental theorem of arithmetics

Show that for all natural numbers larger than 1 either it is prime or it is the product of prime numbers

Proof

1. The property is true for 2.
2. Let's assume that it is true for $2, 3, 4, ..., n-1$.
   If n is prime, then the property is also true for n.
   If n is not prime, then it can be written as the product of several numbers between 2 and $n-1$. But the property is true for all these numbers, and therefore, the property is also true for n.

Outline

Case distinction

For each case we follow a different strategy.

Example: Triangular inequality

Show that $\forall a, b \in \mathbb{R} | |a+b| \leq |a| + |b|$

Proof

We remind that the absolute value is a function defined by parts:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Case $a+b \geq 0$: $a+b \leq |a|+|b|$

For all real numbers it is obvious that $x \leq |x|$. Therefore, we have $a \leq |a|$ and $b \leq |b|$. Consequently, $a+b \leq |a|+|b|$.

Case $a+b < 0$: $-(a+b) \leq |a|+|b|$

For all real numbers it is also true that $-x \leq |x|$. Therefore, we have $-a \leq |a|$ and $-b \leq |b|$. Consequently, $-(a+b) = -a-b \leq |a|+|b|$.

Outline

Counterexample

To prove that something is not true, it is enough to show that it is not true for one example. This example is called a counterexample.

Example

Show that $\forall x, y, z \in \mathbb{R}^+$ and $\forall n \in \mathbb{Z}, n \geq 2$ it is verified that $x^n + y^n \neq z^n$

Proof

The proposition is false because, for instance, for $x = 3$, $y = 4$, $z = 5$ and $n = 2$ we have

$$3^2 + 4^2 = 5^2$$

Outline

Common math mistakes

Avoid some common mathematical mistakes (many of them, algebraic):

- Common math mistakes Video 1:
  http://www.youtube.com/watch?v=VHo_sfVdieM
- Common math mistakes PDF:
  http://tutorial.math.lamar.edu/pdf/Common_Math_Errors.pdf
- Common math mistakes Video 2:
  http://www.youtube.com/watch?v=qHSUU_q_2wA
- Common math mistakes Video 3:
  http://www.youtube.com/watch?v=cTiuocJfyCs
- Common math mistakes Video 4:
  http://www.youtube.com/watch?v=r5Yro2GdJ6w

Outline

0. Mathematical language
   - Axioms, postulates, definitions and propositions (a)
   - Logical operators (a)
   - Qualifiers (b)
   - Mathematical proofs
     - Modus ponens (b)
     - Modus tollens (c)
     - Reductio ad absurdum (c)
     - Induction (c)
     - Case distinction (c)
     - Counterexample (c)
   - Common math mistakes (c)