Chapter 1. Vectors

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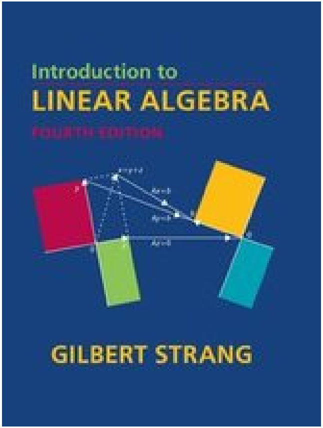
Biomedical Engineering

December 3, 2013

Outline

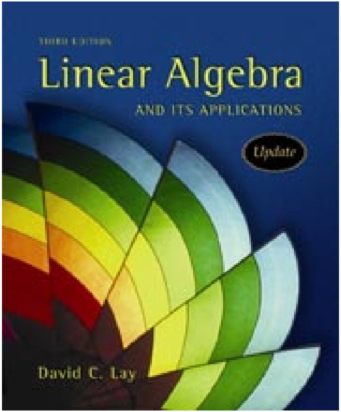
1. Vectors
   - Vectors and basic operations (a)
   - Linear combination (a)
   - Inner product or dot product (b)
   - Norm, vector length and unit vectors (b)
   - Distances and angles (b)
   - Multiplication by matrices (b)

References



G. Strang. Introduction to linear algebra (4th ed). Wellesley Cambridge Press (2009). Chapter 1.

References



D. Lay. Linear algebra and its applications (3rd ed). Pearson (2006). Chapter 1.

A little bit of history

Vectors were developed during the XIX$^{\text{th}}$ century by mathematicians and physicists like Carl Friedrich Gauss (1799), William Rowan Hamilton (1837), and James Clerk Maxwell (1873), mostly as a tool to represent complex numbers, and later as a tool to perform geometrical reasoning. Their modern algebra was formalized by Josiah Willard Gibbs (1901), a university professor at Yale.

To know more about the history of vectors visit

- http://www.math.mcgill.ca/labute/courses/133f03/VectorHistory.html
- https://www.math.ucdavis.edu/~temple/MAT21D/SUPPLEMENTARY-ARTICLES/Crowe_History-of-Vectors.pdf

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What is a vector?

Definition 1.1

*Informally, a **vector** is a collection of n numbers of the same type. We say it has n components (1,2,...,n)*

We'll see that this definition is terribly simplistic since many other things (like functions, infinite sequences, etc.) can be vectors. But, for the time being, let's stick to this simple definition.

Example

$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^3$ is a collection of 3 integer numbers

$\begin{pmatrix} -1.1 \\ 1.1 \end{pmatrix} \in \mathbb{Q}^2$ is a collection of 2 rational numbers

$\begin{pmatrix} -1.1 \\ \sqrt{2} \end{pmatrix} \in \mathbb{R}^2$ is a collection of 2 real numbers

Matlab:

```
[-1.1; sqrt(2)]
```

Transpose

We distinguish between column vectors (for instance **v** below) and row vectors (**w**). In the first case, we say **v** is a $n \times 1$ vector, while in the second, we say **w** is a $1 \times n$ vector.

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix} \text{ and } \mathbf{w} = (w_1 w_2 \dots w_n).$$

Definition 1.2

*The **transpose** is the operation that transforms a column vector into a row vector and viceversa.*

Example

$$(-1\ 1)^T = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Matlab:

```
[-1 1]'
```

Addition of vectors

Definition 1.3

Given two vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}$ *and* $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$ *the* ***sum*** *of these two vectors is another vector defined as* $\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \dots \\ v_n + w_n \end{pmatrix}$. *Note that you can only add two column vectors or two row vectors, but not a column and a row vector.*

Example

$$\begin{pmatrix} -1.1 \\ 1.1 \end{pmatrix} + \begin{pmatrix} -1.1 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} -2.2 \\ 1.1 + \sqrt{2} \end{pmatrix}$$

Matlab:

```
[-1.1; 1.1]+[-1.1; sqrt(2)]
```

Properties 1.1

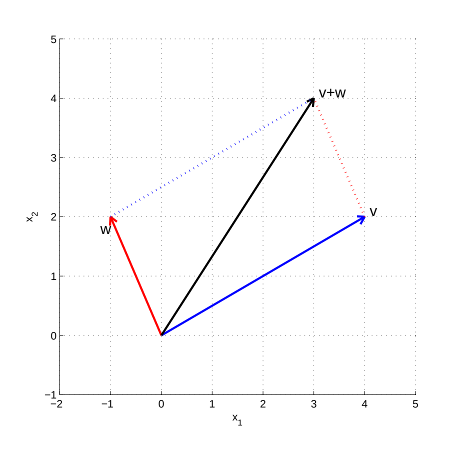
Commutativity:

$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

Addition of vectors

Example

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$



Product by scalar

Definition 1.4

Given a vector $\mathbf{v}$ *and a scalar* c*, the* ***multiplication*** *of* c *and* $\mathbf{v}$ *is defined as*

$$c\mathbf{v} = \begin{pmatrix} cv_1 \\ cv_2 \\ \ldots \\ cv_n \end{pmatrix}$$

Example

$$2\begin{pmatrix} -1.1 \\ 1.1 \end{pmatrix} = \begin{pmatrix} -2.2 \\ 2.2 \end{pmatrix}$$

$$-\begin{pmatrix} -1.1 \\ 1.1 \end{pmatrix} = \begin{pmatrix} 1.1 \\ -1.1 \end{pmatrix}$$

Matlab:

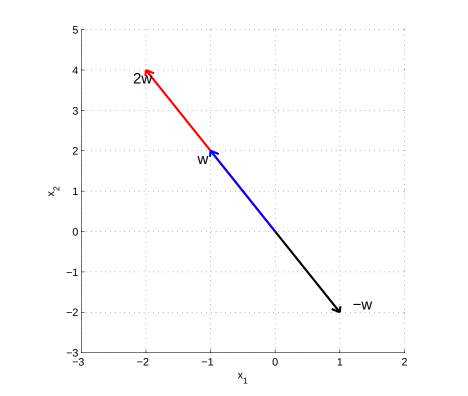
```
2*[-1.1; 1.1] -[1.1; 1.1]
```

Product by scalar

Example

$$\mathbf{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

What is the shape of all scaled vectors of the form $c\mathbf{w}$?
If $\mathbf{w} = \mathbf{0}$, then it is a single point ($\mathbf{0}$). If $\mathbf{w} \neq \mathbf{0}$, then it is the straight line that passes through $\mathbf{0}$ and $\mathbf{w}$.



Properties

For simplification we will present them as properties for $\mathbb{R}^n$, but they apply to all vector spaces. Given any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and any two scalars $c, d \in \mathbb{R}$, we have

Vector operation properties

Regarding the sum of vectors:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutativity
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associativity
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ Existence of neutral element
4. $\mathbf{u} + -\mathbf{u} = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ Existence of symmetric element

Regarding the sum of vectors and scalar product:

5. $\mathrm{c}(\mathbf{u} + \mathbf{v}) = c\mathbf{v} + c\mathbf{u}$ Distributivity with respect to the sum of vectors
6. $(\mathrm{c+d})\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributivity with respect to the sum of scalars

Regarding the scalar product:

7. $\mathrm{c}(\mathrm{d}\mathbf{u}) = (cd)\mathbf{u}$ Associativity
8. $1\mathbf{u} = \mathbf{u}$ Existence of neutral element

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Linear combination

Definition 2.1

*Given a collection of p scalars (x_i, $i = 1, 2, ..., p$) and p vectors ($\mathbf{v}_i$), the **linear combination** of the p vectors using the **weights** given by the p scalars is defined as*

$$\sum_{i=1}^{p} x_i \mathbf{v}_i = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + ... + x_p \mathbf{v}_p$$

Example

$$\frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} \\ -\frac{11}{6} \end{pmatrix}$$

Matlab:

```
    format rational
-1/2*[-1; 1]-2/3*[2; 2]
```

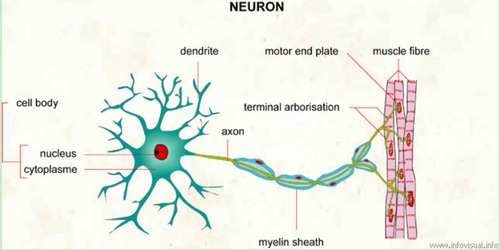
Linear combination

Example

A very basic model of the activity of neurons is

$$output = f(\sum_i weight_i input_i)$$

where $f(x)$ is a non-linear function. In fact, this is the model used in artificial neuron networks.

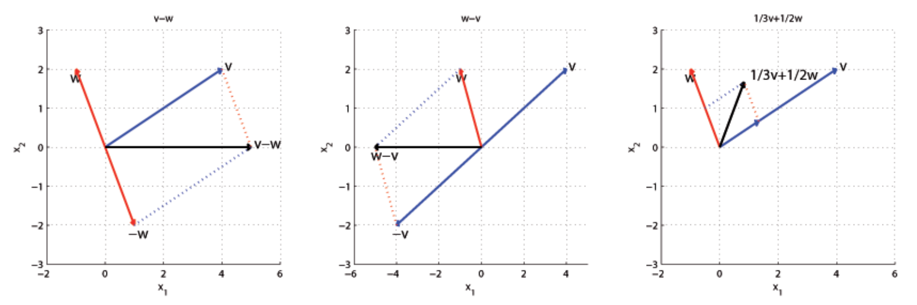


The human brain has in the order of 10^{11} neurons and about 10^{18} connections. See https://www.youtube.com/watch?v=zLp-edwiGUU.

Linear combination

Example

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \ \mathbf{w} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$



We may think of the weight coefficients as the "travelling" instructions. For instance, for the figure in the right, the instructions say: *"Travel $\frac{1}{3}$ of* **v** *along* **v***, then travel $\frac{1}{2}$ of* **w** *along* **w***"*.

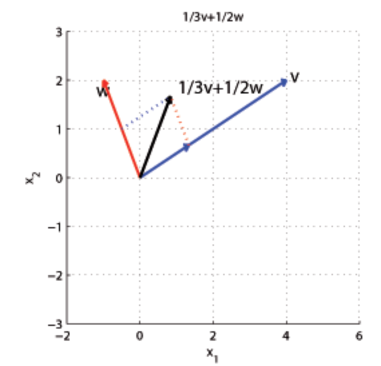
Linear combination

What is the shape of all linear combinations of the form $c\mathbf{v} + d\mathbf{w}$

If the two vectors are not collinear (i.e., $\mathbf{w} \neq k\mathbf{v}$), then it is the whole plane passing by $\mathbf{0}$, $\mathbf{v}$ and $\mathbf{w}$. We can think of it as the sum of all vectors belonging to the line $\overline{\mathbf{0v}}$ and $\overline{\mathbf{0w}}$.

The plane generated by **v** and **w** is the set of all vectors that can be generated as a linear combination of both vectors.

$\Pi = \{\mathbf{r} | \mathbf{r} = c\mathbf{v} + d\mathbf{w} \, \forall c, d \in \mathbf{R}\}$



Linear combination

The previous example prompts the following definition:

Definition 2.2 (Spanned subspace)

The subspace spanned by the vectors $\mathbf{v}_i$, $i = 1, 2, ..., p$, is the set of all vectors that can be expressed as the linear combination of them. Formally,

$$\langle \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p \rangle = \mathrm{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\right\} \triangleq \left\{\mathbf{v} \in \mathbb{R}^n | \mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_p\mathbf{v}_p\right\}$$

Example

Assuming all vectors below are linearly independent:
$\mathrm{Span}\left\{\mathbf{v}_1\right\}$ is a straight line.
$\mathrm{Span}\left\{\mathbf{v}_1, \mathbf{v}_2\right\}$ is a plane.
$\mathrm{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}\right\}$ is a hyperplane.

Properties

$$\mathbf{0} \in \mathrm{Span}\left\{\cdot\right\}$$

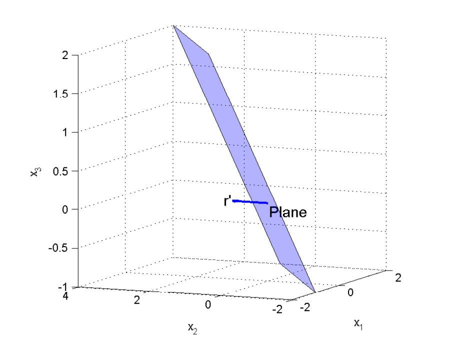
Linear combination

Outside the plane

Let $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (0, 1, 1)$. The linear combinations of $\mathbf{v}$ and $\mathbf{w}$ fill a plane in 3D. All points belonging to this plane are of the form

$$\Pi = \{\mathbf{r} | \mathbf{r} = c(1, 1, 0) + d(0, 1, 1)\, \forall c, d \in \mathbf{R}\} = \{\mathbf{r} = (c, c + d, d)\, \forall c, d \in \mathbf{R}\}$$

It is clear that the vector $\mathbf{r}' = (0, 1, 0) \notin \Pi$, therefore, it is outside the plane.

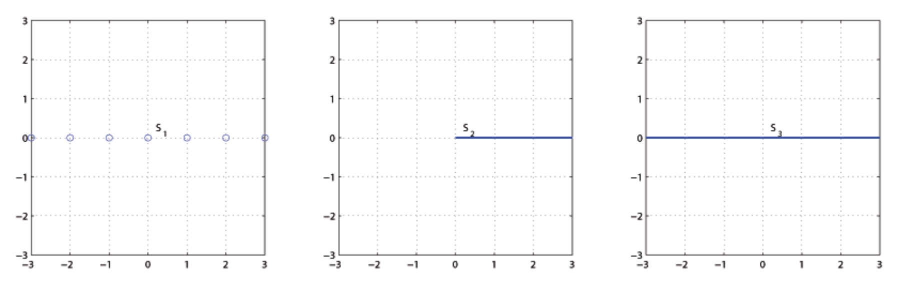


Linear combination

Sets of points

Let $\mathbf{v} = (1, 0)$.

1. $S_1 = \{\mathbf{r} = c\mathbf{v}\ \forall c \in \mathbb{Z}\}$ is a set of points
2. $S_2 = \{\mathbf{r} = c\mathbf{v}\ \forall c \in \mathbb{R}^+\}$ is a semiline
3. $S_3 = \{\mathbf{r} = c\mathbf{v}\ \forall c \in \mathbb{R}\}$ is a line

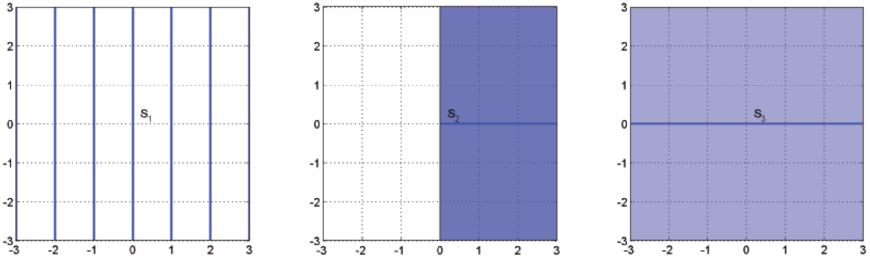


Linear combination

Sets of points

Let $\mathbf{v} = (1,0)$ and $\mathbf{w} = (0,1)$.

1. $S_1 = \{\mathbf{r} = c\mathbf{v} + d\mathbf{w}\ \forall c \in \mathbb{Z}, \forall d \in \mathbb{R}\}$ is a set of lines
2. $S_2 = \{\mathbf{r} = c\mathbf{v} + d\mathbf{w}\ \forall c \in \mathbb{R}^+, \forall d \in \mathbb{R}\}$ is a semiplane
3. $S_3 = \{\mathbf{r} = c\mathbf{v} + d\mathbf{w}\ \forall c, d \in \mathbb{R}\}$ is a plane



Linear combination

Combination coefficients

Let $\mathbf{v} = (2, -1)$, $\mathbf{w} = (-1, 2)$ and $\mathbf{b} = (1, 0)$. Find c and d such that $\mathbf{b} = c\mathbf{v} + d\mathbf{w}$.

Solution

We need to find c and d such that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c \begin{pmatrix} 2 \\ -1 \end{pmatrix} + d \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2c - d \\ 2d - c \end{pmatrix}$$

This gives a simple equation system

$$\begin{aligned} 2c - d &= 1 \\ 2d - c &= 0 \end{aligned}$$

whose solution is $c = \frac{2}{3}$ and $d = \frac{1}{3}$. We can easily check it with Matlab:

```
2/3*[2 -1]'+1/3*[-1 2]'
```

Exercises

Exercises

From Lay (4th ed.), Chapter 1, Section 3:

- 1.3.1
- 1.3.3
- 1.3.6
- 1.3.7
- 1.3.25
- 1.3.27
- 1.3.29
- 1.3.31

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Inner product

Definition 3.1

Given two vectors **v** *and* **w** *the* ***inner or dot product*** *between* **v** *and* **w** *is defined as*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} \triangleq \mathbf{v}^T \mathbf{w} = \sum_{i=1}^{n} v_i w_i = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

Mathematically, the concept of inner product is much more general, and this operational definition is just a particularization for vectors in $\mathbb{R}^n$. Although, the introduced inner product is the most common, it is not the only one that can be defined in $\mathbb{R}^n$. But, let's leave these generalization for the moment.

Example

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 4 \cdot (-1) + 2 \cdot 2 = 0$$

Matlab:

```
dot([4; 2],[-1; 2])
```

Properties 3.1

Commutativity:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$

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Vector norm and vector length

Definition 4.1

Given a vector $\mathbf{v}$, *its* ***length or norm*** *is defined as*

$$\|\mathbf{v}\| \triangleq \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

In the particular case of working with the previously introduced inner product, this definition boils down to

$$\|\mathbf{v}\| \triangleq \sqrt{\mathbf{v}^T \mathbf{v}} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

that is known as the ***Euclidean norm*** *of vector* $\mathbf{v}$.

Properties 4.1

$$\begin{aligned} \|-\mathbf{v}\| &= \|\mathbf{v}\| \\ \|c\mathbf{v}\| &= |c|\|\mathbf{v}\| \end{aligned}$$

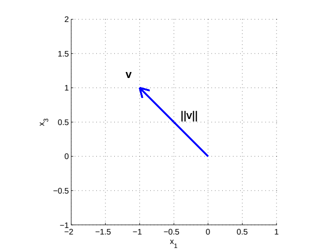
Vector norm and vector length

Example

$$\|(-1,0,1)\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

Matlab:

```
norm([-1;0;1])
```



Unit vectors

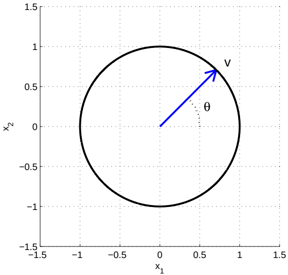
Definition 4.2

$\mathbf{v}$ *is* ***unitary*** *iff* $\|\mathbf{v}\| = 1$.

Example

$\mathbf{e}_1 = (1, 0)$
$\mathbf{e}_2 = (0, 1)$
$\mathbf{e}_\theta = (\cos(\theta), \sin(\theta))$
Matlab:

```
   theta=pi/4;
e_theta=[cos(theta);sin(theta)];
norm(e_theta)
```



Unit vectors

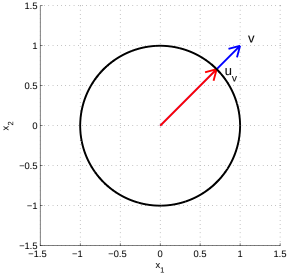
Definition 4.3 (Construction of a unit vector)

Given any vector $\mathbf{v}$ *(whose norm is not null), we can always construct a unitary vector with the same direction of* $\mathbf{v}$ *as* $\mathbf{u_v} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$.

Example

$\mathbf{v} = (1, 1)$

$\mathbf{u_v} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(1,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



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Distance and angle between two vectors

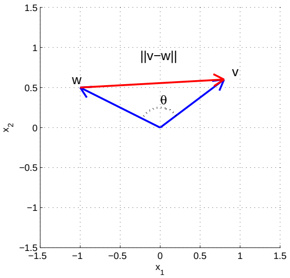
Definition 5.1

Given two vectors **v** *and* **w**, *the* ***distance*** *between both is defined as*

$$d(\mathbf{v},\mathbf{w}) \triangleq \|\mathbf{v}-\mathbf{w}\|$$

and their ***angle*** *is*

$$\angle(\mathbf{v},\mathbf{w}) \triangleq \mathrm{acos}\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \theta$$



Definition 5.2

Two vectors are ***orthogonal*** *(perpendicular) iff their inner product is 0. We then write* $\mathbf{v} \perp \mathbf{w}$. *In this case,* $\angle(\mathbf{v},\mathbf{w}) = \frac{\pi}{2}$.

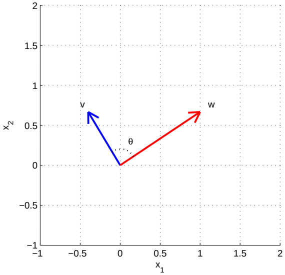
Distance and angle between two vectors

Example

Let $\mathbf{v} = (-\frac{2}{5}, \frac{2}{3})$ and $\mathbf{w} = (1, \frac{2}{3})$. The angle between these two vectors can be calculated as

$$\mathbf{v} \cdot \mathbf{w} = (-\tfrac{2}{5})1 + \tfrac{2}{3}\tfrac{2}{3} = \tfrac{2}{45}$$
$$\|\mathbf{v}\| = \sqrt{(-\tfrac{2}{5})^2 + (\tfrac{2}{3})^2} = \frac{\sqrt{136}}{15}$$
$$\|\mathbf{w}\| = \sqrt{(1)^2 + (\tfrac{2}{3})^2} = \frac{\sqrt{13}}{3}$$
$$\angle(\mathbf{v}, \mathbf{w}) = \operatorname{acos}\frac{\frac{2}{45}}{\frac{\sqrt{136}}{15}\frac{\sqrt{13}}{3}} = 87.27^\circ$$

$\mathbf{v}$ and $\mathbf{w}$ are almost orthogonal.



Example

Let $\mathbf{v} = (1, 0, 0, 1, 0, 0, 1, 0, 0, 1)$ and $\mathbf{w} = (0, 1, 1, 0, 1, 1, 0, 1, 1, 0)$. These two vectors in a 10-dimensional space are orthogonal because

$$\mathbf{v} \cdot \mathbf{w} = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0$$

Distance and angle between two vectors

Example

Search for a vector that is orthogonal to $\mathbf{v} = (-\frac{2}{5}, \frac{2}{3})$

<u>*Solution*</u>

Let the vector $\mathbf{w} = (w_1, w_2)$ be such a vector. Since it is orthogonal to $\mathbf{v}$ it must meet

$$\langle \mathbf{v}, \mathbf{w} \rangle = 0 = (-\tfrac{2}{5})w_1 + \tfrac{2}{3}w_2 \Rightarrow w_2 = \tfrac{3}{5}w_1$$

That is, any vector of the form $\mathbf{w} = (w_1, \frac{3}{5}w_1) = w_1(1, \frac{3}{5})$ is perpendicular to $\mathbf{v}$. This is the line passing by the origin and with direction $(1, \frac{3}{5})$. In particular, for $w_1 = \frac{2}{3}$ we have that $\mathbf{w} = (\frac{2}{3}, \frac{2}{5})$ and for $w_1 = -\frac{2}{3}$ we have $\mathbf{w} = (-\frac{2}{3}, -\frac{2}{5})$.

This is a general rule in 2D. Given a vector $\mathbf{v} = (a, b)$, the vectors $\mathbf{w} = (b, -a)$ and $\mathbf{w} = (-b, a)$ are orthogonal to $\mathbf{v}$.

$$(a, b) \perp (b, -a) \text{ and } (a, b) \perp (-b, a)$$

Distance and angle between two vectors

Theorem 5.1 (Pythagorean theorem)

If $\mathbf{v} \perp \mathbf{w}$*, then* $\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

Proof

$$\|\mathbf{v}-\mathbf{w}\|^2 = (\mathbf{v}-\mathbf{w})^T(\mathbf{v}-\mathbf{w}) = \mathbf{v}^T\mathbf{v} - \mathbf{v}^T\mathbf{w} - \mathbf{w}^T\mathbf{v} + \mathbf{w}^T\mathbf{w} = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\langle \mathbf{v}, \mathbf{w}\rangle$$

But, because $\mathbf{v} \perp \mathbf{w}$*, we have* $\langle \mathbf{v}, \mathbf{w}\rangle = 0$*, and consequently*

$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$ *(q.e.d.)*

Corollary 5.1

- *If* $\langle \mathbf{v}, \mathbf{w}\rangle < 0$*, then* $\frac{\pi}{2} < \theta \leq \pi$.
- *If* $\langle \mathbf{v}, \mathbf{w}\rangle > 0$*, then* $0 \leq \theta < \frac{\pi}{2}$.
- *For two unit vectors,* $\mathbf{u}_1$ *and* $\mathbf{u}_2$*, we have* $\cos\theta = \langle \mathbf{u}_1, \mathbf{u}_2\rangle$*, and as a consequence* $-1 \leq \langle \mathbf{u}_1, \mathbf{u}_2\rangle \leq 1$.

Distance and angle between two vectors

Theorem 5.2 (Cosine formula)

For any two vectors, $\mathbf{v}$ *and* $\mathbf{w}$*, such that* $\|\mathbf{v}\| \neq 0$ *and* $\|\mathbf{w}\| \neq 0$*, we have*

$$\langle \mathbf{v}, \mathbf{w} \rangle = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$$

<u>Proof</u>

By use of Definition 4.3, we can construct the unit vectors associated to $\mathbf{v}$ *and* $\mathbf{w}$*, that is* $\mathbf{u_v}$ *and* $\mathbf{u_w}$*. Then by Corollary 5.1 we know that*

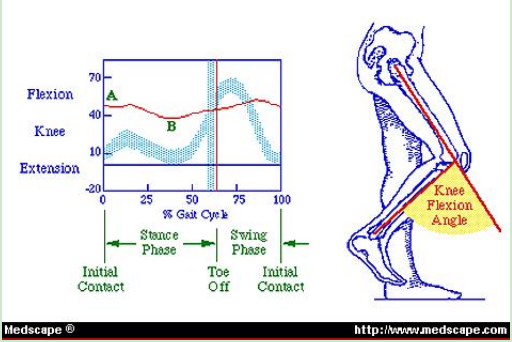
$$\cos\theta = \langle \mathbf{u_v}, \mathbf{u_w} \rangle = \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)^T \left(\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) = \frac{1}{\|\mathbf{u}\|\|\mathbf{w}\|}\mathbf{u}^T\mathbf{w} = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{u}\|\|\mathbf{w}\|}$$

From this point it is trivial to deduce that $\langle \mathbf{v}, \mathbf{w} \rangle = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$ *(q.e.d.)*

Distance and angle between two vectors

Example

To compute the knee flexion angle, we need to calculate the dot product between the vectors aligned with the leg before and after the knee.



Distance and angle between two vectors

Theorem 5.3 (Cauchy-Schwarz inequality)

For any two vectors, $\mathbf{v}$ *and* $\mathbf{w}$*, it is verified that*

$$|\langle \mathbf{v}, \mathbf{w} \rangle| < \|\mathbf{v}\| \|\mathbf{w}\|$$

Proof

From the cosine formula (Theorem 5.2), we know that

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle &= \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta \Rightarrow \\ |\langle \mathbf{v}, \mathbf{w} \rangle| &= |\|\mathbf{v}\| \|\mathbf{w}\| \cos\theta| = \|\mathbf{v}\| \|\mathbf{w}\| \, |\cos\theta| \leq \|\mathbf{v}\| \|\mathbf{w}\| \end{aligned}$$

Example

Let $\mathbf{v} = (-\frac{2}{5}, \frac{2}{3})$ and $\mathbf{w} = (1, \frac{2}{3})$. We already know that $\mathbf{v} \cdot \mathbf{w} = \frac{2}{45}$, $\|\mathbf{v}\| = \frac{\sqrt{136}}{15}$, and $\|\mathbf{w}\| = \frac{\sqrt{13}}{3}$. Let us check Cauchy-Schwarz inequality

$$|\tfrac{2}{45}| < \frac{\sqrt{136}}{15} \frac{\sqrt{13}}{3} \Leftrightarrow 0.0444 < 0.9344$$

Distance and angle between two vectors

Example

Show that for any two positive numbers, x and y, the geometric mean ($\sqrt{xy}$) is always smaller or equal than the arithmetic mean ($\frac{x+y}{2}$). For instance, the statement is verified for $x = 2$ and $y = 3$: $\sqrt{6} \leq \frac{5}{2} \Leftrightarrow 2.4495 \leq 2.5$.

Proof

Let there be vectors $\mathbf{v} = (a, b)$ and $\mathbf{w} = (b, a)$. Then, by Cauchy-Schwarz inequality we know that

$$|\langle \mathbf{v}, \mathbf{w} \rangle| < \|\mathbf{v}\| \|\mathbf{w}\| \Rightarrow |2ab| \leq a^2 + b^2$$

Since x and y are positive numbers, we may consider them to be $x = a^2$ and $y = b^2$. Consequently, we can rewrite the previous expression as

$$2\sqrt{x}\sqrt{y} \leq x + y \Rightarrow \sqrt{xy} \leq \frac{x+y}{2} \text{ (q.e.d.)}$$

In fact, the geometric mean is nothing more than the arithmetic mean in logarithmic units

$$\log(\sqrt{xy}) = \log(xy)^{\frac{1}{2}} = \frac{1}{2}(\log x + \log y) = \frac{\log x + \log y}{2}$$

Distance and angle between two vectors

Theorem 5.4 (Triangular inequality)

For any two vectors, $\mathbf{v}$ *and* $\mathbf{w}$*, it is verified that*

$$\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

Proof

By definition we know that

$$\|\mathbf{v}+\mathbf{w}\|^2 = (\mathbf{v}+\mathbf{w})^T(\mathbf{v}+\mathbf{w}) = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle$$

Applying the Cauchy-Schwarz inequality (Theorem 5.3), we have

$$\|\mathbf{v}+\mathbf{w}\|^2 \leq \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| = (\|\mathbf{v}\| + \|\mathbf{w}\|)^2$$

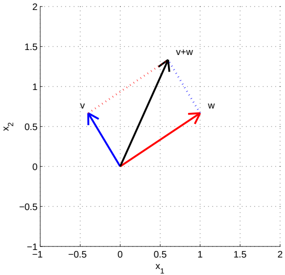
Taking the square root we have

$$\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

Example

Let $\mathbf{v} = (-\frac{2}{5}, \frac{2}{3})$ and $\mathbf{w} = (1, \frac{2}{3})$. We already know that $\|\mathbf{v}\| = \frac{\sqrt{136}}{15}$ and $\|\mathbf{w}\| = \frac{\sqrt{13}}{3}$. Let us check the triangular inequality

$$\mathbf{v} + \mathbf{w} = (\tfrac{3}{5}, \tfrac{4}{3}) \Rightarrow \|\mathbf{v} + \mathbf{w}\| = \frac{\sqrt{481}}{15}$$
$$\frac{\sqrt{481}}{15} \leq \frac{\sqrt{136}}{15} + \frac{\sqrt{13}}{3} \Leftrightarrow 1.4621 \leq 1.9793$$



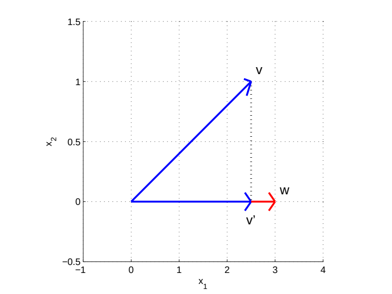
Distance and angle between two vectors

Orthogonal projections

Let us consider the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$.

$$\mathbf{v}' = \langle \mathbf{v}, \mathbf{w} \rangle \frac{\mathbf{w}}{\|\mathbf{w}\|^2} = \frac{\langle \mathbf{v},\mathbf{w} \rangle}{\|\mathbf{w}\|} \frac{\mathbf{w}}{\|\mathbf{w}\|}$$

The length of this vector is $\frac{\langle \mathbf{v},\mathbf{w} \rangle}{\|\mathbf{w}\|}$



Example

Let $\mathbf{v} = (\frac{5}{2}, 1)$ and $\mathbf{w} = (3, 0)$. Then, $\mathbf{v}' = \frac{\frac{5}{2}3+1\cdot 0}{3}(1, 0) = (\frac{5}{2}, 0)$. See the figure above.

Outline

1. Vectors
   - Vectors and basic operations (a)
   - Linear combination (a)
   - Inner product or dot product (b)
   - Norm, vector length and unit vectors (b)
   - Distances and angles (b)
   - Multiplication by matrices (b)

Multiplication by matrices

Example

Let's consider three vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Let's consider the linear combination

$$\mathbf{y} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = x_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{pmatrix}$$

I can obtain the same result by constructing a matrix

$$A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

And making the multiplication

$$\mathbf{y} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{pmatrix}$$

Multiplication by matrices

Example

We can also achieve the same result by calculating **y** as the inner product of the rows of the matrix A and the weight vector.

$$\mathbf{y} = \begin{pmatrix} \left\langle (1,0,0), (x_1, x_2, x_3) \right\rangle \\ \left\langle (-1,1,0), (x_1, x_2, x_3) \right\rangle \\ \left\langle (0,-1,1), (x_1, x_2, x_3) \right\rangle \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{pmatrix}$$

Matlab:

```
 syms x1 x2 x3
x=[x1; x2; x3]
A=[1 0 0; -1 1 0; 0 -1 1];
y=A*x
```

Multiplication by matrices

Matrix multiplication as a linear combination

This is a general rule: a matrix multiplication can be seen as the linear combination of the columns of the matrix.

$$A = (\mathbf{c}_1\ \mathbf{c}_2\ ...\mathbf{c}_p) \Rightarrow \mathbf{y} = A\mathbf{x} = \sum_{i=1}^{p} x_i \mathbf{c}_i$$

Matrix multiplication as inner products

Also, a matrix multiplication can be seen as the dot product of the weight vector with the rows of the matrix.

$$A = \begin{pmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ ... \\ \mathbf{r}_n^T \end{pmatrix} \Rightarrow \mathbf{y} = A\mathbf{x} = \begin{pmatrix} \langle \mathbf{r}_1, \mathbf{x} \rangle \\ \langle \mathbf{r}_2, \mathbf{x} \rangle \\ ... \\ \langle \mathbf{r}_n, \mathbf{x} \rangle \end{pmatrix}$$

Multiplication by matrices

Properties of multiplication by matrices

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$$
$$A(c\mathbf{u}) = c(A\mathbf{u})$$

Outline

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