

Chapter 2. Linear equation systems

C.O.S. Sorzano

Biomedical Engineering

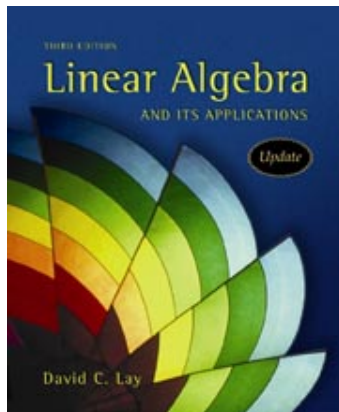
September 24, 2013



CEU

*Universidad
San Pablo*

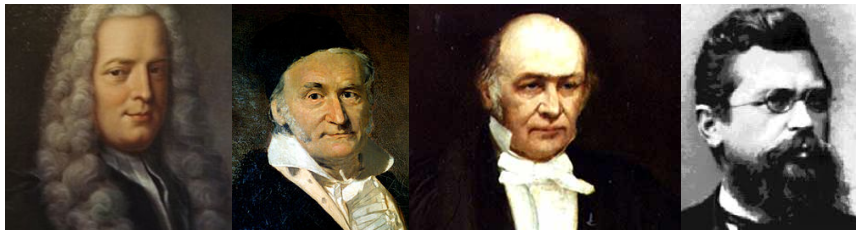
- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - Linear independence (c)
 - Linear transformations (d)
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)



D. Lay. Linear algebra and its applications (3rd ed). Pearson (2006). Chapter 1.

A little bit of history

Linear equations in their modern form are known since the middle of the XVIIIth century and they were strongly developed during the XIXth century with important contributions of people like **Gabriel Cramer** (1750), **Carl Friedrich Gauss** (1801), Sir **William Rowan Hamilton** (1843) and **Wilhelm Jordan** (1873). They were mostly developed to explain the mechanics of celestial objects.

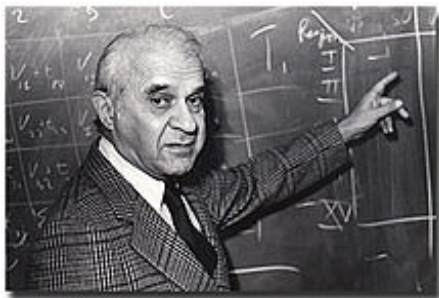


To know more about the history of linear equations visit

- <http://hom.wikidot.com/cramer-s-method-and-cramer-s-paradox>

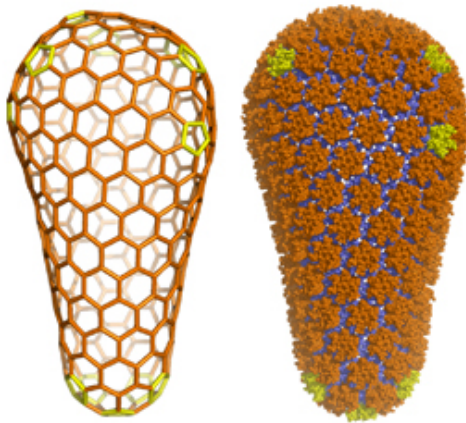
A little bit of history

Wassily Leontief was a Russian-American economist that worked in Harvard. In 1949 he performed an analysis with the early computers at Harvard using data from the U.S. Bureau of Labor Statistics to classify the U.S. economy into 500 sectors, that were later simplified to 42. He used linear equation systems to do so. It took 56 hours in Mark II (one of the first computers) to solve it. He was awarded the Nobel prize in 1970 for his work on input-output tables that analyze how outputs from some industries are inputs to some other industries.



A little bit of history

Currently, we need about two weeks in a supercomputer (128 cores) to solve the structure of a macromolecular assembly (in the figure, the HIV virus capsid). We have 1,000 million equations with about 3 million unknowns.



- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - Linear independence (c)
 - Linear transformations (d)
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)

What is a linear equation system?

Definition 1.1 (Linear equation system)

A **linear equation** is one that can be expressed in the form

$$\sum_{i=1}^n a_i x_i = b$$
$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$
$$\langle \mathbf{a}, \mathbf{x} \rangle = b$$

The unknowns are x_i ($i = 1, 2, \dots, n$) while a_i 's and b are coefficients. When we have several of these equations, we have a **linear equation system**.

Example

Examples of linear equations

$$7x_1 - 2x_2 = 4$$
$$7(x_1 - \sqrt{3}x_2) = \frac{1}{\sqrt{2}}x_1 \Rightarrow$$
$$(7 - \frac{1}{\sqrt{2}})x_1 - 7\sqrt{3}x_2 = 0$$

Examples of non-linear equations

$$x_1 + x_2 + x_1 x_2 = 1$$
$$\sqrt{x_1} + x_2 = 1$$

Set of solutions of a linear equation

Definition 1.2 (Set of solutions of a linear equation system)

The set of solutions of a linear equation system $S \subseteq \mathbb{R}^n$ is the set of all those values that we can assign to x_1, x_2, \dots, x_n such that the equation system is fulfilled.

Example

Consider the following equation system

$$\begin{aligned}2x_1 - x_2 &= 7 \\ x_1 + 2x_2 &= 11\end{aligned}$$

$\mathbf{x} = (5, 3)$ is a solution to this equation system because

$$\begin{aligned}2 \cdot 5 - 3 &= 7 \\ 5 + 2 \cdot 3 &= 11\end{aligned}$$

In fact it is its unique solution and, therefore, $S = \{(5, 3)\} \subset \mathbb{R}^2$.

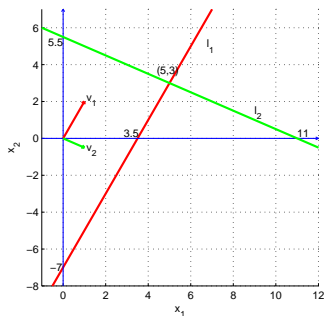
Geometric interpretation

Example

$$l_1: 2x_1 - x_2 = 7 \Rightarrow x_2 = 2x_1 - 7 \Rightarrow \mathbf{v}_1 = (1, 2)$$

$$l_2: x_1 + 2x_2 = 11 \Rightarrow x_2 = 11 - \frac{1}{2}x_1 \Rightarrow \mathbf{v}_2 = (1, -\frac{1}{2})$$

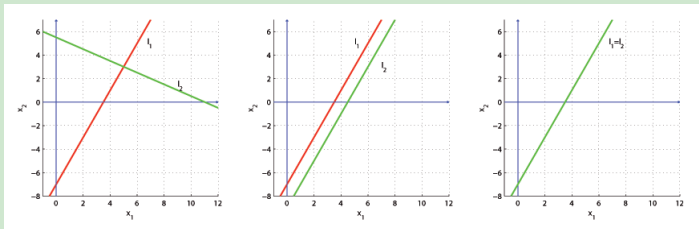
Each one of the equations is actually representing a line, and both lines, in this case intersect at the point $(5, 3)$, the unique solution of this equation system.



Geometric interpretation

Example

There can be a single solution (left), no solution (middle), or infinite ($l_1 = l_2$; right)



In general

With linear equations we can represent:

a **line** in 2D: $a_1x_1 + a_2x_2 = b$

a **plane** in 3D: $a_1x_1 + a_2x_2 + a_3x_3 = b$

a **hyperplane** in nD: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

Matrix notation

Example

The equation system

$$\begin{array}{rclcl} x_1 & -2x_2 & +x_3 & = & 0 \\ & 2x_2 & -8x_3 & = & 8 \\ -4x_1 & +5x_2 & +9x_3 & = & -9 \end{array}$$

can be represented as

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right) [\tilde{A}]$$

or

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix} \quad [A\mathbf{x} = \mathbf{b}]$$

Matrix notation

In general

$A \in \mathcal{M}_{m \times n}$ is called the **system matrix** of an equation system with m equations and n unknowns.

$\tilde{A} \in \mathcal{M}_{m \times (n+1)}$ is called the **augmented system matrix** of an equation system with m equations and n unknowns.

Basic row iterations

To solve the equation system with the augmented system matrix, we used the so-called basic row operations:

Substitution: $\mathbf{r}_i \leftarrow k_i \mathbf{r}_i + k_j \mathbf{r}_j$: Row i is substituted by a linear combination of rows i and j

Swapping: $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$: Row i swapped with row j

Scaling: $\mathbf{r}_i \leftarrow k_i \mathbf{r}_i$: Row i is multiplied by a scale factor

All these operations transform the equation system into an **equivalent** system (with the same set of solutions). The two matrices (original and transformed) are said to be **row equivalent**.

Solving the equation system

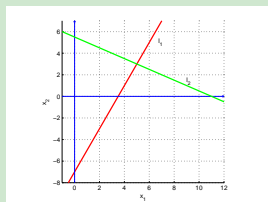
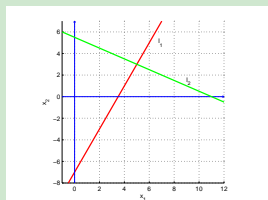
Example

In the following example we will see how linear combinations are actually changing the equation system to a different one, while scaling is not.

$$\begin{array}{rcl} 2x_1 & -x_2 & = 7 \\ x_1 & +2x_2 & = 11 \end{array} \quad \left(\begin{array}{cc|c} 2 & -1 & 7 \\ 1 & 2 & 11 \end{array} \right)$$

$$\mathbf{r}_1 \leftarrow \frac{1}{2}\mathbf{r}_1$$

$$\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{7}{2} \\ 1 & 2 & 11 \end{array} \right)$$



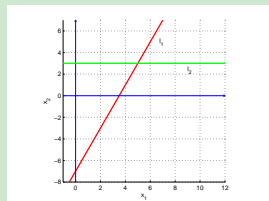
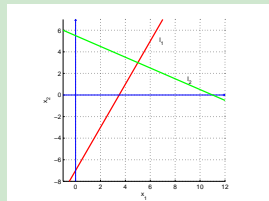
Solving the equation system

Example

$$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \mathbf{r}_1$$

$$\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{7}{2} \\ 1 & \frac{1}{2} & 11 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & \frac{5}{2} & \frac{15}{2} \end{array} \right)$$



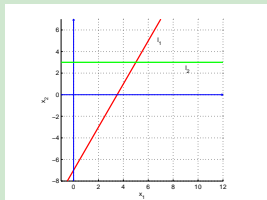
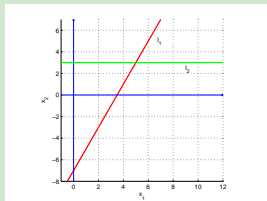
Solving the equation system

Example

$$\mathbf{r}_2 \leftarrow \frac{2}{5}\mathbf{r}_2$$

$$\left(\begin{array}{cc|c} 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & \frac{15}{2} & \frac{15}{2} \end{array} \right)$$

$$\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & 3 \end{array} \right)$$



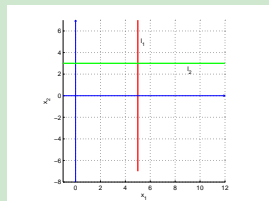
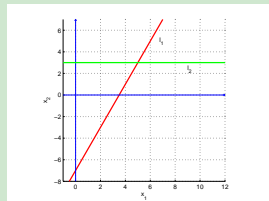
Solving the equation system

Example

$$\mathbf{r}_1 \leftarrow \mathbf{r}_1 + \frac{1}{2}\mathbf{r}_2$$

$$\left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & 3 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \end{array} \right)$$



Existence and uniqueness of solutions

Example

$$\begin{array}{rclcl} x_1 & -2x_2 & +x_3 & = & 0 \\ & 2x_2 & -8x_3 & = & 8 \\ -4x_1 & +5x_2 & +9x_3 & = & -9 \end{array} \sim \dots \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

I can solve for x_3 ($x_3 = 3$), then use this value in the second equation to solve for x_2 , and finally use these two values in the first equation to solve for x_1 . Thus, the equation system **has a solution and it is unique**. We say the equation system is **compatible**. The set of solutions is $S = \{(29, 16, 3)\}$.

Matlab:

```
A=[1 -2 1; 0 2 -8; -4 5 9];  
b=[0; 8; -9];  
x=A\b
```

Existence and uniqueness of solutions

Example

$$\begin{array}{rclcl} & x_2 & -4x_3 & = & 8 \\ 2x_1 & -3x_2 & +2x_3 & = & 1 \\ 5x_1 & -8x_2 & +7x_3 & = & 1 \end{array} \sim \dots \sim \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array} \right)$$

Last equation implies $0 = \frac{5}{2}$ which is impossible. Consequently, there is **no solution** and we say that the equation system is **incompatible**. The set of solutions is $S = \emptyset$.

Example

$$\begin{array}{rclcl} x_1 & +x_2 & = & 1 \\ 2x_1 & +2x_2 & = & 2 \end{array} \sim \dots \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

There are **infinite** solutions. The system is **compatible indeterminate**. The set of solutions is $S = \{(x_1, 1 - x_1)\}$.

Exercises

From Lay (4th ed.), Chapter 1, Section 1:

- 1.1.11
- 1.1.4
- 1.1.15
- 1.1.18
- 1.1.25
- 1.1.26
- 1.1.33

- 2 Linear equation system
 - Introduction (a)
 - **Gauss-Jordan algorithm (b)**
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - Linear independence (c)
 - Linear transformations (d)
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)

Echelon matrices

Example

The following matrices are echelon matrices:

$$A_1 = \begin{pmatrix} \diamond & \heartsuit & \heartsuit & \heartsuit \\ 0 & \diamond & \heartsuit & \heartsuit \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0 & \diamond & \heartsuit & \heartsuit \\ 0 & 0 & 0 & \diamond \end{pmatrix}$$

In the previous matrices we have marked with \diamond the **leading elements** (the first ones different from 0 in their row), and with \heartsuit the rest of the elements different from 0.

Echelon matrices

Definition 2.1 (Echelon matrix)

A rectangular matrix has an **echelon** form iif:

- 1 Within each row, the first element different from zero (called the **leading entry**) is in a column to the right of the leading entry of the previous row.
- 2 Within each column, all values below a leading entry are zero.
- 3 All rows without a leading entry (i.e., they only have zeros) are below all the rows in which at least one element is not zero.

Definition 2.2 (Reduced echelon matrix)

A rectangular matrix has a **reduced echelon** form iif:

- 1 It is echelon.
- 2 The leading entry of each row is 1.
- 3 The leading entry is the only 1 in its column.

Echelon matrices

Theorem 2.1

Each matrix is row equivalent to one and only one reduced echelon matrix.

Example

$$\begin{array}{l} \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 4\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_1 \\ \\ \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 \\ \\ \mathbf{r}_3 \leftrightarrow \mathbf{r}_3 + 3\mathbf{r}_2 \end{array} \left| \begin{array}{c} \left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & -1 & 0 \end{array} \right) \\ \left(\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 1 & 3 \end{array} \right) \\ \left(\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & -3 & -6 \end{array} \right) \\ \left(\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{array} \right) \end{array} \right. \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 - 2\mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \frac{1}{3}\mathbf{r}_3 \\ \\ \mathbf{r}_1 \leftarrow \mathbf{r}_1 + 3\mathbf{r}_3 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_3 \end{array} \left| \begin{array}{c} \left(\begin{array}{ccc} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array} \right) \\ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \end{array} \right.$$

Echelon matrices

Example (continued)

Matlab:

```
A=[1 2 3; 4 5 6; -1 -1 0]
A(2,:)=A(2,:)-4*A(1,:)
A(3,:)=A(3,:)+A(1,:)
aux=A(2,:); A(2,:)=A(3,:); A(3,:)=aux
A(3,:)=A(3,:)+3*A(2,:)
A(1,:)=A(1,:)-2*A(2,:)
A(3,:)=1/3*A(3,:)
A(1,:)=A(1,:)+3*A(3,:)
A(2,:)=A(2,:)-3*A(3,:)
```

Echelon matrices

Example

Now, we'll repeat the same example using different row operations:

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_3 \\ \\ \mathbf{r}_1 \leftarrow -\mathbf{r}_1 \\ \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 4\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1 \end{array} \left| \begin{array}{c} \left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & -1 & 0 \end{array} \right) \\ \\ \left(\begin{array}{ccc} 1 & 1 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \right) \\ \\ \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 1 & 3 \end{array} \right) \end{array} \right. \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_2 \\ \\ \mathbf{r}_3 \leftarrow -\frac{1}{3}\mathbf{r}_3 \\ \\ \mathbf{r}_1 \leftarrow \mathbf{r}_1 + 6\mathbf{r}_3 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 6\mathbf{r}_3 \end{array} \left| \begin{array}{c} \left(\begin{array}{ccc} 1 & 0 & -6 \\ 0 & 1 & 6 \\ 0 & 0 & -3 \end{array} \right) \\ \\ \left(\begin{array}{ccc} 1 & 0 & -6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{array} \right) \\ \\ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \end{array} \right.$$

Gauss-Jordan algorithm

Definition 2.3 (Pivot and pivot column)

A pivot element is the element of a matrix that is used to perform certain calculations. For the Gauss-Jordan algorithm it corresponds to the first element different from zero in a given row. A pivot column is a column that contains a pivot.

Step 1

Choose the left-most pivot column. The pivot element (marked in red) is any value within this column different from 0. Note: Normally, we should take the one with maximum absolute value to avoid numerical errors.

Example

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

Gauss-Jordan algorithm

Step 2

Sort rows if necessary so that the pivot is as high as possible.

Example

$$\mathbf{r}_3 \leftrightarrow \mathbf{r}_1 \quad \left| \quad \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix} \right.$$

Step 3

Use row operations to force the elements below the pivot to be 0.

Example

$$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \mathbf{r}_1 \quad \left| \quad \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix} \right.$$

Gauss-Jordan algorithm

Step 4

Repeat Steps 1 to 3 with the rows below the pivot.

Example

$$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \frac{2}{3}\mathbf{r}_2 \left(\begin{array}{c} \left(\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{array} \right) \\ \left(\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & -\frac{8}{3} \end{array} \right) \\ \left(\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & -\frac{8}{3} \end{array} \right) \end{array} \right)$$

Gauss-Jordan algorithm

Step 5

Starting from the lowest and right-most pivot, force the elements above that pivot to be zero. If the pivot is not 1, then rescale the row. Repeat with the next pivot on the left.

Example

$$\begin{array}{l} r_3 \leftarrow -\frac{3}{2}r_3 \\ r_2 \leftarrow r_2 - 4r_3 \\ r_1 \leftarrow r_1 - 6r_3 \\ r_2 \leftarrow \frac{1}{3}r_2 \\ r_1 \leftarrow r_1 + 9r_2 \\ r_1 \leftarrow \frac{1}{3}r_1 \end{array} \quad \left| \begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 3 & -6 & 6 & 0 & -21 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right.$$

Computing the inverse of a $n \times n$ matrix costs in the order of n^3 operations ($O(n^3)$). However, calculating the reduced echelon form is only in the order of n^2 ($O(n^2)$). This difference is more and more important as n grows.

Existence and uniqueness of solutions (revisited)

We can now review the issue of existence and uniqueness under the light of the reduced echelon matrix.

Example

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

The system is compatible and the set of solutions is formed by a single point $S = \{(1, 4, 0)\}$.

Example

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

There are **infinite** solutions. The system is **compatible indeterminate**. The set of solutions is $S = \{(1, 4 - x_3, x_3) \mid \forall x_3 \in \mathbb{R}^3\}$. Because the set of solutions depends on a single variable, the set of solutions is a line.

Existence and uniqueness of solutions (revisited)

Example

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

There are **infinite** solutions. The system is **compatible indeterminate**. The set of solutions is $S = \{(1, 4 - x_3 - x_4, x_3, x_4) \mid \forall x_3, x_4 \in \mathbb{R}^3\}$. Now, the set of solutions depends on 2 variables and, consequently, it is a plane.

Example

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

The system is incompatible since the last equation is $0 = 1$. The set of solutions is the empty set, $S = \emptyset$.

Exercises

From Lay (4th ed.), Chapter 1, Section 2:

- 1.2.2
- 1.2.8
- 1.2.19
- 1.2.33
- 1.2.34

- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - **Interpretation as a subspace (b)**
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - Linear independence (c)
 - Linear transformations (d)
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)

Interpretation as a subspace

Subspace spanned by columns

Consider the equation system given by the matrix equation $A\mathbf{x} = \mathbf{b}$, where $A \in \mathcal{M}_{n \times p}$. Let us call the p columns of A as $\mathbf{c}_i \in \mathbb{R}^n$. The previous equation can be rewritten as

$$(\mathbf{c}_1 \ \mathbf{c}_2 \ \dots \ \mathbf{c}_p) \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_p \end{pmatrix} = \mathbf{b} \Rightarrow \sum_{i=1}^p x_i \mathbf{c}_i = \mathbf{b}$$

That is, $A\mathbf{x}$ is the subspace spanned by the columns of matrix A .

$$\text{Span} \{ \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_p \} = \{ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} = A\mathbf{x} \ \forall \mathbf{x} \in \mathbb{R}^p \}$$

The equation system $A\mathbf{x} = \mathbf{b}$ poses the question: **Find the weight coefficients x_i such that vector \mathbf{b} belongs to $\text{Span} \{ \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_p \}$.**

Interpretation as a subspace

Example

The equation system

$$\begin{array}{rclcl} x_1 & +2x_2 & -x_3 & = & 4 \\ & -5x_2 & +3x_3 & = & 1 \end{array}$$

can be represented as

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

That is, which are the weight coefficients x_1 , x_2 and x_3 such that the vector $(4, 1)$ belongs to the subspace generated by the vectors $(1, 0)$, $(2, -5)$, and $(-1, 3)$.

Interpretation as a subspace

Theorem 3.1

The matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution as the vector equation $\sum_{i=1}^p x_i \mathbf{c}_i = \mathbf{b}$ and as the equation system whose augmented matrix is $\tilde{A} = (A|\mathbf{b})$.

Theorem 3.2

For any $A \in \mathcal{M}_{n \times p}$ and vector $\mathbf{b} \in \mathbb{R}^n$, the following four statements are equivalent, that is, $P_1 \Leftrightarrow P_2 \Leftrightarrow P_3 \Leftrightarrow P_4$

P_1 : The equation $A\mathbf{x} = \mathbf{b}$ has a solution.

P_2 : \mathbf{b} is a linear combination of the columns of A .

P_3 : The columns of A span all \mathbb{R}^n , i.e., $\text{Span}\{\mathbf{c}_i\} = \mathbb{R}^n$.

P_4 : A has a pivot in each row.

Exercises

From Lay (4th ed.), Chapter 1, Section 4:

- 1.4.13
- 1.4.18
- 1.4.26
- 1.4.27
- 1.4.32
- 1.4.39
- 1.4.41 (bring computer)

- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - **Existence and uniqueness of solutions (c)**
 - Applications (c)
 - Linear independence (c)
 - Linear transformations (d)
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)

Existence and uniqueness of solutions (revisited once again)

Let us consider the **homogeneous system** $Ax = \mathbf{0}$. It obviously has the trivial solution $\mathbf{x} = \mathbf{0}$. Non-trivial solutions can be found through the echelon matrix

Example

$$\begin{array}{rclcl} 3x_1 & +5x_2 & -4x_3 & = & 0 \\ -3x_1 & -2x_2 & +4x_3 & = & 0 \\ 6x_1 & +x_2 & -8x_3 & = & 0 \end{array} \Rightarrow \dots \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This is a compatible indeterminate system whose set of solutions is $S = \left\{ \left(-\frac{4}{3}x_3, 0, x_3 \right) \forall x_3 \in \mathbb{R} \right\}$, or what is the same

$$S = \text{Span} \left\{ \left(-\frac{4}{3}, 0, 1 \right) \right\}.$$

That is, any of the infinite points in the straight line whose director vector is $\left(-\frac{4}{3}, 0, 1 \right)$ is a solution of the equation system.

Existence and uniqueness of solutions (revisited once again)

Let us consider the **non-homogeneous system** $Ax = b$.

Example

$$\begin{array}{rclcl} 3x_1 & +5x_2 & -4x_3 & = & 7 \\ -3x_1 & -2x_2 & +4x_3 & = & -1 \\ 6x_1 & +x_2 & -8x_3 & = & -4 \end{array} \Rightarrow \dots \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This is a compatible indeterminate system whose set of solutions is

$$S = \left\{ \left(-\frac{4}{3}x_3, 2, x_3 \right) \forall x_3 \in \mathbb{R} \right\}, \text{ or what is the same}$$

$$S = \left\{ (0, 2, 0) + \left(-\frac{4}{3}x_3, 0, x_3 \right) \forall x_3 \in \mathbb{R} \right\} = (0, 2, 0) + \text{Span} \left\{ \left(-\frac{4}{3}, 0, 1 \right) \right\}.$$

That is, any of the infinite points in the straight line whose director vector is $\left(-\frac{4}{3}, 0, 1 \right)$ and passes through the point $(0, 2, 0)$ is a solution of the equation system.

Existence and uniqueness of solutions (revisited once again)

Consider the following homogeneous equation system

Example

$$10x_1 - 3x_2 - 2x_3 = 0 \Rightarrow \dots \sim (10 \quad -3 \quad -2 \mid 0)$$

This is a compatible indeterminate system whose set of solutions is

$$S = \left\{ \left(\frac{3}{10}x_2 + \frac{1}{5}x_3, x_2, x_3 \right) \mid \forall x_2, x_3 \in \mathbb{R} \right\}, \text{ or what is the same}$$

$$S = \text{Span} \left\{ \left(\frac{3}{10}, 1, 0 \right), \left(\frac{1}{5}, 0, 1 \right) \right\}.$$

That is, any of the infinite points in the plane containing the vectors $\left(\frac{3}{10}, 1, 0 \right)$ and $\left(\frac{1}{5}, 0, 1 \right)$ is a solution of the equation system.

Existence and uniqueness of solutions (revisited once again)

Consider now the following non-homogeneous equation system

Example

$$10x_1 - 3x_2 - 2x_3 = 10 \Rightarrow \dots \sim (10 \quad -3 \quad -2 \mid 10)$$

This is a compatible indeterminate system whose set of solutions is

$$S = \left\{ \left(1 + \frac{3}{10}x_2 + \frac{1}{5}x_3, x_2, x_3 \right) \mid \forall x_2, x_3 \in \mathbb{R} \right\}, \text{ or what is the same}$$

$$S = \left\{ (1, 0, 0) + \left(\frac{3}{10}x_2 + \frac{1}{5}x_3, x_2, x_3 \right) \mid \forall x_2, x_3 \in \mathbb{R} \right\} = \\ (1, 0, 0) + \text{Span} \left\{ \left(\frac{3}{10}, 1, 0 \right), \left(\frac{1}{5}, 0, 1 \right) \right\}.$$

Existence and uniqueness of solutions (revisited once again)

Corollary 4.1

Consider the compatible, non-homogeneous equation system given by $A\mathbf{x} = \mathbf{b}$ and its homogeneous counterpart $A\mathbf{x} = \mathbf{0}$. Let S_h be the set of solutions of the homogeneous equation system. Then, the set of solutions of the non-homogeneous equation system is of the form

$$S_{nh} = \mathbf{x}_0 + S_h$$

For some $\mathbf{x}_0 \in \mathbb{R}^n$.

Definition 4.1 (Null space of A)

S_h is called the null space of the matrix A . It has the property that given an equation system $A\mathbf{x} = \mathbf{b}$, if \mathbf{x}_0 is a solution of the equation system, then $\mathbf{x}_0 + \mathbf{x}_h$ is also a solution, for any $\mathbf{x}_h \in S_h$.

Existence and uniqueness of solutions (revisited once again)

nature.com ▶ Publications A-Z index ▶ Browse by subject

Search [Advanced search](#)

Nature-standard editing and advice on your scientific manuscripts
Submit today!

MACMILLAN
SCIENCE COMMUNICATION

My account
Submit manuscript
Register
Subscribe

nature COMMUNICATIONS

Home About the journal Authors and referees Browse archive Search

◉ Login Cart

nature.com ▶ journal home ▶ archive by date ▶ october ▶ abstract

ARTICLE PREVIEW

[view full access options](#) ▶

NATURE COMMUNICATIONS | ARTICLE

MACMILLAN
SCIENCE COMMUNICATION

Nature-standard editing and advice on your scientific manuscripts



Science jobs

naturejobs.com

Post doctoral fellow - Molecular evolution of membrane bound proteins
Uppsala University, Department of Neuroscience

PhD Studentships / Postdoctoral Research
University of Lethbridge

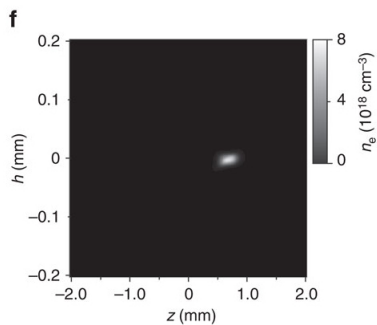
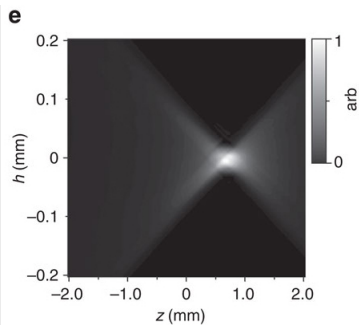


Single-shot ultrafast tomographic imaging by spectral multiplexing

N.H. Matlis¹ A. Axley¹ W.P. Leemans^{1,2}

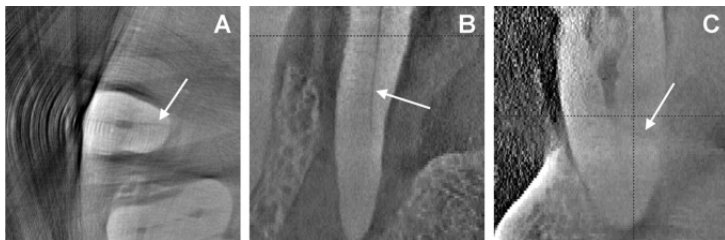
Existence and uniqueness of solutions (revisited once again)

In this example, the authors describe how to solve a problem appearing in the tomographic use of a certain microscope due to the absence of some measurements (resulting in an important null space of the tomographic problem).



Existence and uniqueness of solutions (revisited once again)

In this example, the authors describe how the exact location of a tooth fracture is uncertain (Fig. C) due to the artifacts introduced by the null space of the tomographic problem.



Mora, M. A.; Mol, A.; Tyndall, D. A., Rivera, E. M. *In vitro* assessment of local computed tomography for the detection of longitudinal tooth fractures.

Oral Surg Oral Med Oral Pathol Oral Radiol Endod, **2007**, 103, 825-829.

Exercises

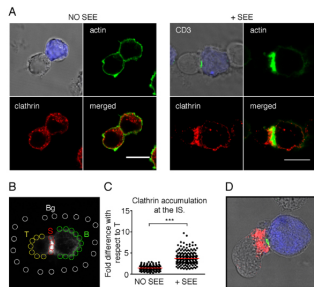
From Lay (4th ed.), Chapter 1, Section 5:

- 1.5.11
- 1.5.13
- 1.5.19
- 1.5.21
- 1.5.25
- 1.5.26
- 1.5.36
- 1.5.39

- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - **Applications (c)**
 - Linear independence (c)
 - Linear transformations (d)
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)

Applications

In fluorescence microscopy, we can quantitatively measure the amount of fluorescence coming from each source with a linear equation system.



$$\begin{pmatrix} Bg \\ B \\ T \\ S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{B}_g \\ \hat{B} \\ \hat{T} \\ \hat{T}_s \end{pmatrix}$$

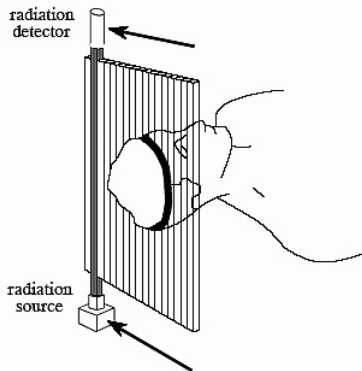
C. Calabia-Linares, M. Pérez-Martínez, N. Martín-Cofreces, M. Alfonso-Pérez, C. Gutiérrez-Vázquez, M. Mittelbrunn, S. Ibiza, F.R. Urbano-Olmos, C. Aguado-Ballano, C.O.S. Sorzano, F. Sánchez-Madrid, E. Veiga. *Clathrin drives actin accumulation at the immunological synapse*. *J. Cell Science*, 124: 820-830 (2011)

Applications

In computed tomography, a simple model (but widely used) for data collection states that the data observed is the sum of the values of the density found along the X-ray path.

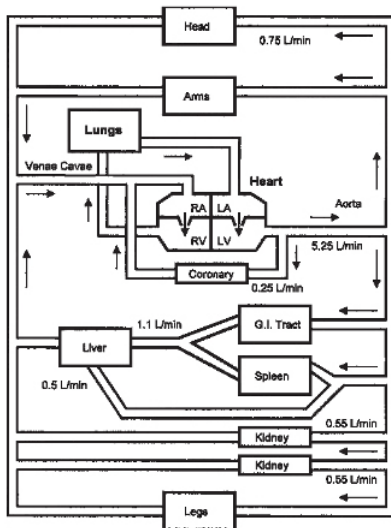
FIGURE 25-14

CT data acquisition. A simple CT system passes a narrow beam of x-rays through the body from source to detector. The source and detector are then translated to obtain a complete view. The remaining views are obtained by rotating the source and detector in about 1° increments, and repeating the translation process.



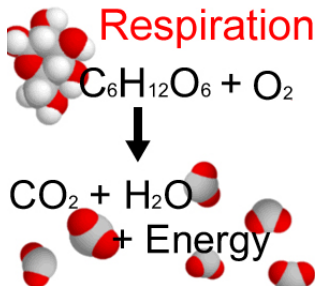
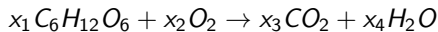
Applications

In the blood system, at each node, the sum of output flows must be equal to the sum of input flows.



Applications

In a very simplified model, respiration is the burning of glucose that can be written as



$$C: 6x_1 = x_3$$

$$H: 6x_1 = 2x_4$$

$$O: 6x_1 + 2x_2 = 2x_3 + x_4$$

Exercises

From Lay (4th ed.), Chapter 1, Section 6:

- 1.6.5
- 1.6.7
- 1.6.12

- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - **Linear independence (c)**
 - Linear transformations (d)
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)

Definition 6.1 (Linear independence)

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is **linearly independent** if

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0} \Rightarrow x_1 = x_2 = \dots = x_p = 0$$

That is the only solution of the previous equation is the trivial solution $\mathbf{x} = \mathbf{0}$.
The set is **linearly dependent** if at least two x_i 's are different from 0.

Linear independence

Example

Determine if the vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (4, 5, 6)$, and $\mathbf{v}_3 = (2, 1, 0)$ are linearly independent.

Solution

The augmented matrix associated to the equation system in Definition 6.1 is

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array} \right) \sim \dots \sim \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Since the system is compatible indeterminate, there exists a solution apart from the trivial solution and, therefore, the vectors are linearly dependent.

Linear independence

Example

If possible, find a linear relationship among the three vectors. Solution

We continue transforming the augmented matrix to its reduced echelon form

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \sim \dots \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

From which $x_1 = 2x_3$ and $x_2 = -x_3$. Simply by choosing $x_3 = 1$, we obtain have that a possible solution to the equation system in Definition 6.1 is $x_1 = 2$, $x_2 = -1$ and $x_3 = 1$, consequently we have that

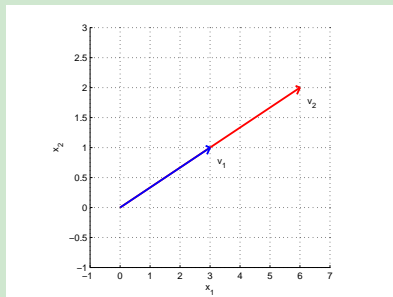
$$2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

Linear independence

Example

$\mathbf{v}_1 = (3, 1)$ and $\mathbf{v}_2 = (6, 2)$ are linearly dependent because

$$\mathbf{v}_2 = 2\mathbf{v}_1 \Rightarrow -2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \frac{1}{2}\mathbf{v}_2$$

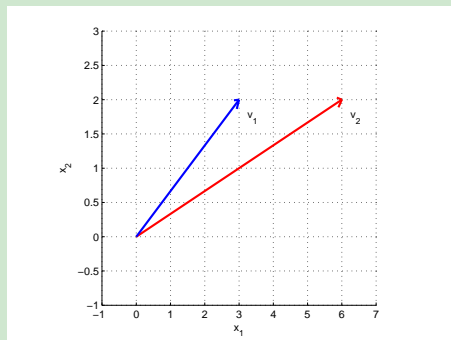


If two vectors are linearly dependent of each other, then any one of them is a multiple of the other.

Linear independence

Example

$\mathbf{v}_1 = (3, 2)$ and $\mathbf{v}_2 = (6, 2)$ are linearly independent

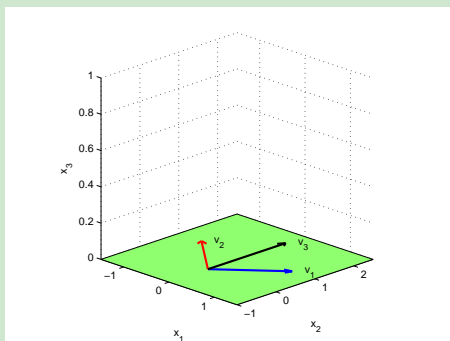


Linear independence

Example

$\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (-1, 1, 0)$ and $\mathbf{v}_3 = (0, 2, 0)$ are linearly dependent because

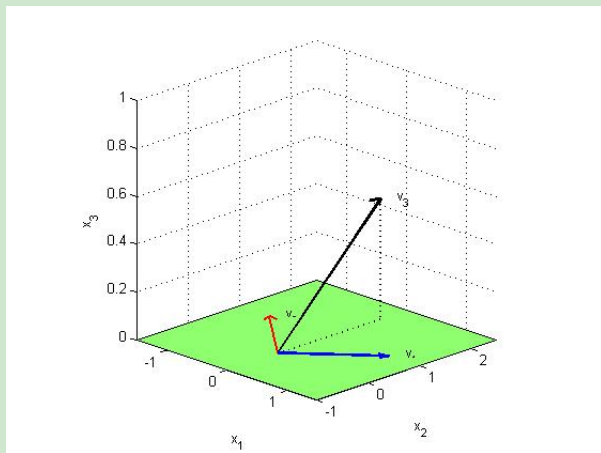
$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$$



Linear independence

Example

$\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (-1, 1, 0)$ and $\mathbf{v}_3 = (0, 2, 1)$ are linearly independent



Linear independence

Theorem 6.1 (Linear independence of matrix columns)

The columns of the matrix A are linearly independent iff the only solution of $A\mathbf{x} = \mathbf{0}$ is the trivial one.

Proof

Let $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_p]$ so that the columns of the matrix A are the vectors \mathbf{a}_i . According to Definition 6.1 these vectors are linearly independent iff

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_p\mathbf{a}_p = \mathbf{0} \Rightarrow x_1 = x_2 = \dots = x_p = 0$$

or what is the same

$$A\mathbf{x} = \mathbf{0} \Rightarrow x_1 = x_2 = \dots = x_p = 0$$

as stated by the theorem (q.e.d.)

Linear independence

Theorem 6.2

Any set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ with $\mathbf{v}_i \in \mathbb{R}^n$ is linearly dependent if $p > n$.

Proof

Let $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_p]$ and let us consider the equation system $A\mathbf{x} = \mathbf{0}$. If $p > n$ there are more unknowns than equations, and consequently, there are free variables and the system is compatible indeterminate. Thus, there are more solutions apart from the trivial one and the set of vectors is linearly dependent.

Theorem 6.3

If any set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ with $\mathbf{v}_i \in \mathbb{R}^n$ contains the vector $\mathbf{0}$, then the set of vectors is linearly dependent.

Proof

We can assume, without loss of generality, that $\mathbf{v}_1 = \mathbf{0}$. Then, we can set $x_1 = 1$, $x_2 = x_3 = \dots = x_p = 0$ so that the following equation is met:

$$1\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_p = \mathbf{0} \text{ (q.e.d.)}$$

Linear independence

Theorem 6.4

A set of vectors is linearly dependent iff at least 1 of the vectors is linearly dependent on the rest

Proof

Proof \Leftarrow

Let us assume that \mathbf{v}_j is a linear combination of the rest of the vectors, that is,

$$\mathbf{v}_j = \sum_{k \neq j} x_k \mathbf{v}_k$$

Then, we can write $\mathbf{v}_j - \sum_{k \neq j} x_k \mathbf{v}_k = \mathbf{0} \Rightarrow$

$$-x_1 \mathbf{v}_1 - x_2 \mathbf{v}_2 - \dots - x_{j-1} \mathbf{v}_{j-1} + \mathbf{v}_j - x_{j+1} \mathbf{v}_{j+1} - \dots - x_p \mathbf{v}_p = \mathbf{0}$$

And consequently there exists a non-trivial solution of the equation of Definition 6.1.

Linear independence

Proof \Rightarrow

If $\mathbf{v}_1 = \mathbf{0}$, then we have already a vector that is a trivial combination of the rest ($\mathbf{v}_1 = 0\mathbf{v}_2 + 0\mathbf{v}_3 + \dots + 0\mathbf{v}_p$).

If $\mathbf{v}_1 \neq \mathbf{0}$, then there exist some coefficients such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

Let j be the largest index for which $x_j \neq 0$ (that is, $x_{j+1} = x_{j+2} = \dots = x_p = 0$).

If $j = 1$, then $x_1\mathbf{v}_1 = \mathbf{0}$, but this is not possible because $\mathbf{v}_1 \neq \mathbf{0}$. Then, $j > 1$ and consequently

$$\begin{aligned}x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_j\mathbf{v}_j + 0\mathbf{v}_{j+1} + \dots + 0\mathbf{v}_p &= \mathbf{0} \Rightarrow \\x_j\mathbf{v}_j &= -x_1\mathbf{v}_1 - x_2\mathbf{v}_2 - \dots - x_{j-1}\mathbf{v}_{j-1} \Rightarrow \\ \mathbf{v}_j &= -\frac{x_1}{x_j}\mathbf{v}_1 - \frac{x_2}{x_j}\mathbf{v}_2 - \dots - \frac{x_{j-1}}{x_j}\mathbf{v}_{j-1} \text{ (q.e.d.)}\end{aligned}$$

Exercises

From Lay (4th ed.), Chapter 1, Section 7:

- 1.7.9
- 1.7.39
- 1.7.40
- 1.7.41 (bring computer)

- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - Linear independence (c)
 - **Linear transformations (d)**
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)

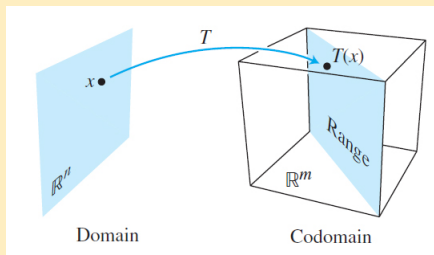
Linear transformations

Definition 7.1 (Transformation)

A **transformation** (or **function** or **mapping**), T , from \mathbb{R}^n to \mathbb{R}^m is a rule that assigns to each vector of \mathbb{R}^n a vector of \mathbb{R}^m .

$$\begin{aligned} T : \mathbb{R}^n &\rightarrow \mathbb{R}^m \\ \mathbf{x} &\rightarrow T(\mathbf{x}) \end{aligned}$$

\mathbb{R}^n is called the **domain** of the transformation, and \mathbb{R}^m its **codomain**. $T(\mathbf{x})$ is the image of vector \mathbf{x} under the action of T . The set of all images is the **range** of T .



Linear transformations

Definition 7.2 (Matrix transformation)

T is a **matrix transformation** iff $T(\mathbf{x}) = A\mathbf{x}$ for some matrix $A \in \mathcal{M}_{m \times n}$.

Example

Let us consider $A = \begin{pmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix}$ and the matrix transformation $\mathbf{y} = A\mathbf{x}$. For instance, the image of $\mathbf{x} = (1, 1, 1, 1)$ is

$$\mathbf{y} = \begin{pmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

The equation system $A\mathbf{x} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$ looks for all those \mathbf{x} , if any, such that

$T(\mathbf{x}) = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$. The domain of this transformation is \mathbb{R}^4 and its codomain \mathbb{R}^2 .

Linear transformations

Example

Let us consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the matrix transformation $\mathbf{y} = A\mathbf{x}$. The domain of this transformation is \mathbb{R}^2 and its codomain \mathbb{R}^3 . However, not all points in \mathbb{R}^3 need to be an image of some point $\mathbf{x} \in \mathbb{R}^2$, only a subset of them may be. In this case,

$$\mathbb{R}^3 \supset \text{Range}(T) = \langle (1, 0, 0), (0, 1, 0) \rangle$$

In general, the range of the transformation T is the subspace spanned by the columns of the matrix A .

Linear transformations

Example

Let us consider $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$ and the matrix transformation $\mathbf{y} = A\mathbf{x}$.

- ① What is the image of $\mathbf{u} = (2, -1)$ under T ?

$$T(\mathbf{u}) = A\mathbf{u} = (5, 1, 9)$$

- ② Let $\mathbf{b} = (3, 2, -5)$. Which is \mathbf{x} such that $T(\mathbf{x}) = \mathbf{b}$?

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{array} \right)$$

From which we deduce $\mathbf{x} = (\frac{3}{2}, -\frac{1}{2})$.

- ③ Is there any other \mathbf{x} such that $T(\mathbf{x}) = \mathbf{b}$?

No, the previous solution is unique because the equation system is definite compatible.

Linear transformations

Example

- 4 Does $\mathbf{c} = (3, 2, 5)$ belong to $\text{Range}(T)$?

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -35 \end{array} \right)$$

Since the system is incompatible, we deduce that there is no vector \mathbf{x} whose image is \mathbf{c} and, consequently, $\mathbf{c} \notin \text{Range}(T)$.

- 5 Which is the function $\mathbf{y} = T(\mathbf{x})$?

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{pmatrix}$$

- 6 Which is $\text{Range}(T)$?

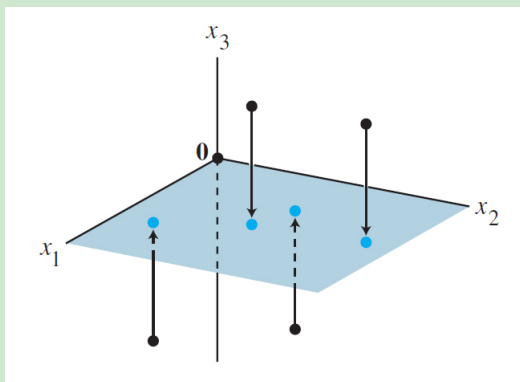
$$\begin{aligned} \text{Range}(T) &= \langle (1, 3, -1), (-3, 5, 7) \rangle = \\ &= \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = x_1 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} \quad \forall x_1, x_2 \in \mathbb{R} \right\} \end{aligned}$$

Because $(1, 3, -1)$ and $(-3, 5, 7)$ are linearly independent, $\text{Range}(T)$ is a plane.

Linear transformations

Example

Consider the transformation $T(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$. This is a projection transformation that projects any 3D point onto the XY plane.



Linear transformations

Definition 7.3 (Linear transformation)

T is a **linear transformation** iff $\forall \mathbf{x}_1, \mathbf{x}_2 \in \text{Dom}(T), \forall c \in \mathbb{R}$

- 1 $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$
- 2 $T(c\mathbf{x}_1) = cT(\mathbf{x}_1)$

Theorem 7.1

If $T(\mathbf{x})$ is a linear transformation, then

- 1 $T(\mathbf{0}) = \mathbf{0}$
- 2 $T(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1T(\mathbf{x}_1) + c_2T(\mathbf{x}_2) \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \text{Dom}(T), \forall c_1, c_2 \in \mathbb{R}$

Proof

- 1 $T(\mathbf{0}) = T(0\mathbf{x}_1) = [(2), \text{Def. 7.3}] = 0T(\mathbf{x}_1) = \mathbf{0} \quad (q.e.d.)$
- 2 $T(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = [(1), \text{Def. 7.3}] = T(c_1\mathbf{x}_1) + T(c_2\mathbf{x}_2) = [(2), \text{Def. 7.3}]$
 $c_1T(\mathbf{x}_1) + c_2T(\mathbf{x}_2) \quad (q.e.d.)$

Theorem 7.2

If $\forall \mathbf{x}_1, \mathbf{x}_2 \in \text{Dom}(T), \forall c_1, c_2 \in \mathbb{R}$ it is verified that $T(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1T(\mathbf{x}_1) + c_2T(\mathbf{x}_2)$, then $T(\mathbf{x})$ is a linear transformation.

Proof

- 1 Let us consider the case $c_1 = c_2 = 1$, then according to the assumption of the theorem we have $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$, which implies (1) in Def. 7.3.
- 2 Let us consider the case $c_2 = 0$, then according to the assumption of the theorem we have $T(c_1\mathbf{x}_1) = c_1T(\mathbf{x}_1)$, which implies (2) in Def. 7.3.

(q.e.d.)

Corollary: Principle of superposition

If $\forall \mathbf{x}_i \in \text{Dom}(T), \forall c_i \in \mathbb{R}$ it is verified that $T\left(\sum_i c_i \mathbf{x}_i\right) = \sum_i c_i T(\mathbf{x}_i)$.

Proof

Apply the previous theorem multiple times. (q.e.d.)

Linear transformations

Example

Show that $T(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a linear transformation.

Proof

- ① Show that $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$

$$\text{On one side we have } T(\mathbf{x}_1 + \mathbf{x}_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} + x_{21} \\ x_{12} + x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} + x_{21} \\ -x_{12} - x_{22} \end{pmatrix}$$

On the other side we have $T(\mathbf{x}_1) + T(\mathbf{x}_2) =$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} \\ -x_{12} \end{pmatrix} + \begin{pmatrix} x_{21} \\ -x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} + x_{21} \\ -x_{12} - x_{22} \end{pmatrix}$$

Obviously, these two calculations give the same result.

- ② Show that $T(c_1\mathbf{x}_1) = c_1 T(\mathbf{x}_1)$

$$\begin{aligned} T(c_1\mathbf{x}_1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1x_{11} \\ c_1x_{12} \end{pmatrix} = \begin{pmatrix} c_1x_{11} \\ -c_1x_{12} \end{pmatrix} = c_1 \begin{pmatrix} x_{11} \\ -x_{12} \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = c_1 T(\mathbf{x}_1) \end{aligned}$$

Theorem 7.3

Any matrix transformation is a linear transformation.

Proof

① Show that $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$

$$T(\mathbf{x}_1 + \mathbf{x}_2) = A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = T(\mathbf{x}_1) + T(\mathbf{x}_2) \text{ (q.e.d.)}$$

② Show that $T(c_1\mathbf{x}_1) = c_1T(\mathbf{x}_1)$

$$T(c_1\mathbf{x}_1) = A(c_1\mathbf{x}_1) = c_1(A\mathbf{x}_1) = c_1T(\mathbf{x}_1) \text{ (q.e.d.)}$$

Reinterpreting the columns of a matrix

Example

Consider $T(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{pmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix}$. Consider the standard canonical basis of \mathbb{R}^4 formed by the vectors $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, and $\mathbf{e}_4 = (0, 0, 0, 1)$. Let us consider the transformation of each one of these vectors

$$T(\mathbf{e}_1) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad T(\mathbf{e}_2) = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad T(\mathbf{e}_3) = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad T(\mathbf{e}_4) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In general, we note that the transformation of \mathbf{e}_i is the i -th column of matrix A .

Corollary

The columns of the matrix $A \in \mathcal{M}_{m \times n}$ can be understood as the transformations of the canonical basis of \mathbb{R}^n :

$$A = (\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n) = (T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n))$$

Reinterpreting the columns of a matrix

Example (continued)

In the previous example consider transforming the vector $\mathbf{x} = (1, -2, 3, 5)$. This vector is equal to

$$\mathbf{x} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 + 5\mathbf{e}_4$$

Then, we have

$$\begin{aligned} T(\mathbf{x}) &= T(\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 + 5\mathbf{e}_4) = T(\mathbf{e}_1) - 2T(\mathbf{e}_2) + 3T(\mathbf{e}_3) + 5T(\mathbf{e}_4) \\ &= \begin{pmatrix} 4 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} -3 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + 5 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 28 \\ 22 \end{pmatrix} \end{aligned}$$

Exercises

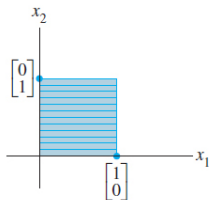
From Lay (4th ed.), Chapter 1, Section 8:

- 1.8.23
- 1.8.25
- 1.8.26
- 1.8.30
- 1.8.34

- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - Linear independence (c)
 - Linear transformations (d)
 - **Geometrical transformations (e)**
 - Classification of functions (e)
 - More applications (e)

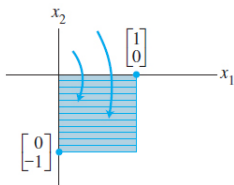
Geometrical transformations

Certain matrix transformations are used to transform the unit square into different shapes. The following table shows some of such transformations.



Transformation	Image of the Unit Square	Standard Matrix
----------------	--------------------------	-----------------

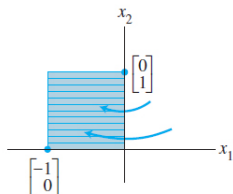
Reflection through the x_1 -axis		
------------------------------------	--	--



		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
--	--	---

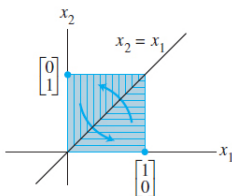
Geometrical transformations

Reflection through
the x_2 -axis



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

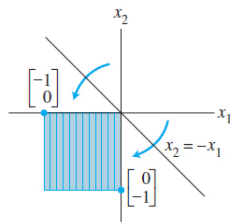
Reflection through
the line $x_2 = x_1$



$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

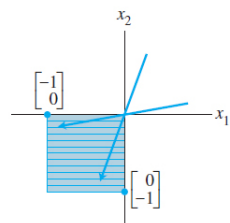
Geometrical transformations

Reflection through
the line $x_2 = -x_1$



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

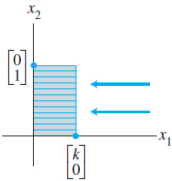
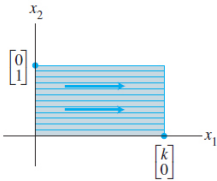
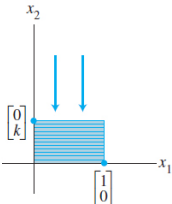
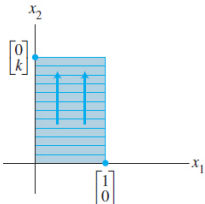
Reflection through
the origin



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

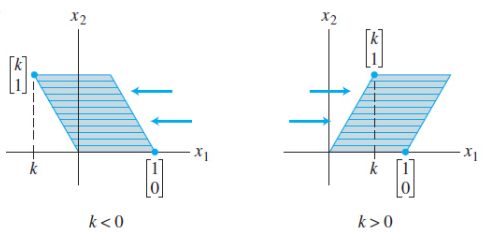
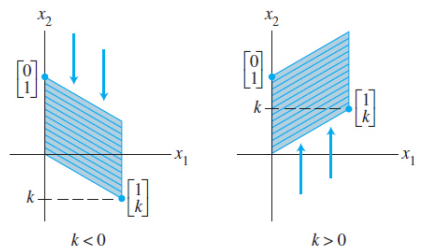
Geometrical transformations

TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
	 $k > 1$	
Vertical contraction and expansion	 $0 < k < 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
	 $k > 1$	

Geometrical transformations

TABLE 3 Shears

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 <p>The left diagram shows a unit square in the x_1-x_2 plane with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. The image of this square is a parallelogram with vertices at $(0,0)$, $(1,0)$, $(1+k,1)$, and $(k,1)$. Blue arrows point from the original square towards the new parallelogram, indicating a leftward shear. The label $k < 0$ is below the diagram.</p> <p>The right diagram shows a unit square in the x_1-x_2 plane with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. The image of this square is a parallelogram with vertices at $(0,0)$, $(1,0)$, $(1+k,1)$, and $(k,1)$. Blue arrows point from the original square towards the new parallelogram, indicating a rightward shear. The label $k > 0$ is below the diagram.</p>	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 <p>The left diagram shows a unit square in the x_1-x_2 plane with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. The image of this square is a parallelogram with vertices at $(0,0)$, $(1,0)$, $(1,1+k)$, and $(0,1+k)$. Blue arrows point from the original square towards the new parallelogram, indicating a downward shear. The label $k < 0$ is below the diagram.</p> <p>The right diagram shows a unit square in the x_1-x_2 plane with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. The image of this square is a parallelogram with vertices at $(0,0)$, $(1,0)$, $(1,1+k)$, and $(0,1+k)$. Blue arrows point from the original square towards the new parallelogram, indicating an upward shear. The label $k > 0$ is below the diagram.</p>	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Geometrical transformations

TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1 -axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2 -axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - Linear independence (c)
 - Linear transformations (d)
 - Geometrical transformations (e)
 - **Classification of functions (e)**
 - More applications (e)

Classification of functions

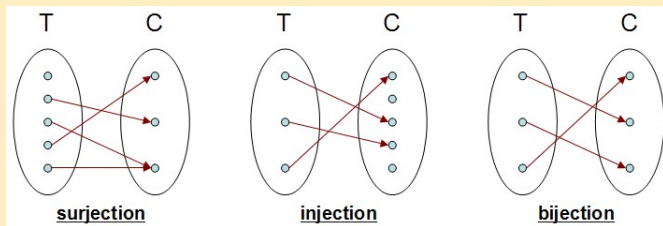
Definition 9.1

Functions can be classified as **surjective**, **injective** or **bijective**:

Surjective: A function is surjective if every point of the codomain has **at least one** point of the domain that maps onto it. They are also called **onto** functions.

Injective: A function is injective if every point of the codomain has **at most one** point in the domain that maps onto it. They are also called **one-to-one** functions.

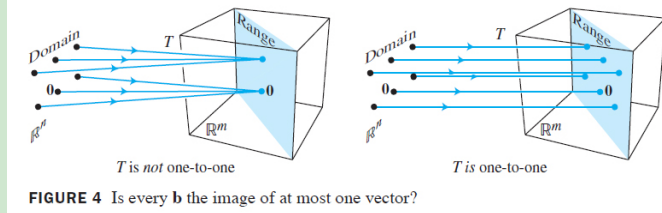
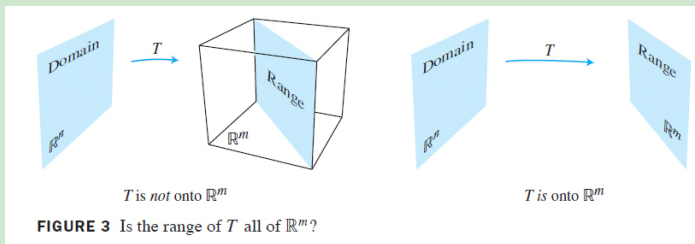
Bijective: A function is bijective if it is injective and surjective.



Classification of functions

Example

Here we have some examples of the classification of functions applied to linear transformations



Classification of functions

Example

Consider $T(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$. This is a transformation from \mathbb{R}^4 onto \mathbb{R}^3 . The columns of A \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_4 are linearly independent and span \mathbb{R}^3 (that is, the function is surjective). Therefore, there must be points in \mathbb{R}^3 that come from several points in \mathbb{R}^4 (the function is not injective). Let us find some of these points.

$$\left(\begin{array}{cccc|c} 1 & -4 & 8 & 1 & y_1 \\ 0 & 2 & -1 & 3 & y_2 \\ 0 & 0 & 0 & 5 & y_3 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 6 & 0 & y_1 - 2y_2 - \frac{4}{5}y_3 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2}y_2 - \frac{3}{10}y_3 \\ 0 & 0 & 0 & 1 & \frac{1}{5}y_3 \end{array} \right) \Rightarrow$$
$$\begin{aligned} x_1 &= y_1 - 2y_2 - \frac{4}{5}y_3 - 6x_3 \\ x_2 &= \frac{1}{2}y_2 - \frac{3}{10}y_3 + \frac{1}{2}x_3 \\ x_4 &= \frac{1}{5}y_3 \end{aligned}$$

Since x_3 is a free variable, we have that for each point in the codomain, there is a straight line that maps onto it (the equation of the line is the one given above).

Classification of functions

Theorem 9.1

Let $T(\mathbf{x})$ be a linear transformation. $T(\mathbf{x})$ is an injective function iff $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Proof

Proof \Rightarrow

If T is injective, then, by definition, every point of the codomain, in particular $\mathbf{0}$ is the mapping of at most one point in the domain. We already know that for any linear transformation $T(\mathbf{0}) = \mathbf{0}$, therefore, $\mathbf{x} = \mathbf{0}$ must be the unique solution of the equation $T(\mathbf{x}) = \mathbf{0}$.

Classification of functions

Proof \Leftarrow

For any linear transformation we know that $T(\mathbf{0}) = \mathbf{0}$. Let us assume that the statement is false, that is $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution, but T is not injective. IF T is not injective there exist a point \mathbf{y} in the codomain that is the image of two points in the domain

$$T(\mathbf{x}_1) = \mathbf{y}$$

$$T(\mathbf{x}_2) = \mathbf{y}$$

If we know subtract the two equations we have

$$T(\mathbf{x}_1) - T(\mathbf{x}_2) = \mathbf{0}$$

$$T(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0} \quad T \text{ is linear}$$

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0} \quad \text{There is only one solution of } T(\mathbf{x}) = \mathbf{0}$$

$$\mathbf{x}_1 = \mathbf{x}_2 \quad \text{contradiction (q.e.d.)}$$

Classification of functions

Theorem 9.2

Let $T(\mathbf{x}) = A\mathbf{x}$ be a linear transformation. Then:

- ① $\text{Range}(T) = \mathbb{R}^m$ iff $\text{Span}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) = \mathbb{R}^m$.
- ② T is injective iff all columns of A are linearly independent.

Proof

- ① According to Theorem 3.2, the columns of A span \mathbb{R}^m if for each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent, that is, if there exists at least one solution of $T(\mathbf{x}) = \mathbf{b}$. If this is true, then $\text{Range}(T) = \mathbb{R}^m$.
- ② According to Theorem 9.1, T is injective iff $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution, or what is the same iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens only if the columns of A are linearly independent as stated by Theorem 6.1.

Classification of functions

Example

$$\text{Let } T(\mathbf{x}) = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{pmatrix}:$$

- 1 Show that it is a linear transformation
- 2 Does it map \mathbb{R}^2 onto \mathbb{R}^3 ?

Solution

- 1 The transformation is of the form $T(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix}$ and,

therefore, the transformation is linear.

- 2 The columns of A are linearly independent (because they are not multiples of each other), then, by the previous theorem, the transformation is injective. However, they do not span all \mathbb{R}^3 (since they are only two vectors and for spanning all \mathbb{R}^3 we need at least 3 vectors). Consequently, the transformation is not surjective, and it does not map \mathbb{R}^2 onto \mathbb{R}^3 .

Exercises

From Lay (4th ed.), Chapter 1, Section 9:

- 1.9.1
- 1.9.3
- 1.9.17
- 1.9.33
- 1.9.36
- 1.9.37
- 1.9.39

2 Linear equation system

- Introduction (a)
- Gauss-Jordan algorithm (b)
- Interpretation as a subspace (b)
- Existence and uniqueness of solutions (c)
- Applications (c)
- Linear independence (c)
- Linear transformations (d)
- Geometrical transformations (e)
- Classification of functions (e)
- More applications (e)

More applications

Construction of a diet

Given the following nutritional information:

Amounts (g) Supplied per 100 g of Ingredient				Amounts (g) Supplied by Cambridge Diet in One Day
Nutrient	Nonfat milk	Soy flour	Whey	
Protein	36	51	13	33
Carbohydrate	52	34	74	45
Fat	0	7	1.1	3

What is the amount of nonfat milk, soy flour and whey needed to provide the protein carbohydrate and fat planned for one day?

Solution

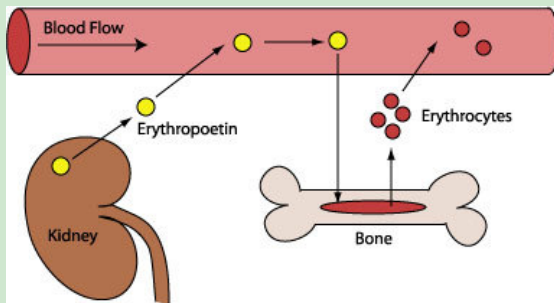
$$\left(\begin{array}{ccc|c} 36 & 51 & 13 & 33 \\ 52 & 34 & 74 & 45 \\ 0 & 7 & 11 & 3 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0.277 \\ 0 & 1 & 0 & 0.392 \\ 0 & 0 & 1 & 0.233 \end{array} \right)$$

That is, we need $x_1 = 0.277 \cdot 100\text{g} = 277\text{g}$ of non-fat milk, $x_2 = 392\text{g}$ of soy flour and $x_3 = 233\text{g}$ of whey.

More applications

Dynamic systems: difference equations

In a simplistic model red blood cells (erythrocytes) are created in the bone marrow, then some of them pass to the blood. After some time, old red blood cells are destroyed in the spleen (bazo).



Dynamic systems: difference equations (continued)

Let's say that at every time interval:

- 5% of the erythrocytes in the marrow leave to the blood stream.
- 2% of the erythrocytes in the blood stream are destroyed by the spleen.
- 1M new red blood cells are created at the marrow.

The following equation can be used to determine the amount of erythrocytes at any moment

$$\begin{pmatrix} x_{\text{marrow}}^{(k+1)} \\ x_{\text{blood}}^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0.95 & 0 \\ 0.05 & 0.98 \end{pmatrix} \begin{pmatrix} x_{\text{marrow}}^{(k)} \\ x_{\text{blood}}^{(k)} \end{pmatrix} + \begin{pmatrix} 10^6 \\ 0 \end{pmatrix}$$

This kind of models is called difference equations.

- 2 Linear equation system
 - Introduction (a)
 - Gauss-Jordan algorithm (b)
 - Interpretation as a subspace (b)
 - Existence and uniqueness of solutions (c)
 - Applications (c)
 - Linear independence (c)
 - Linear transformations (d)
 - Geometrical transformations (e)
 - Classification of functions (e)
 - More applications (e)