Chapter 5. Vector spaces

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Outline

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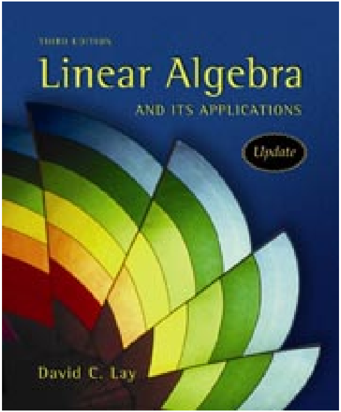
   - Rank of a matrix (d)
   - Change of basis (d)

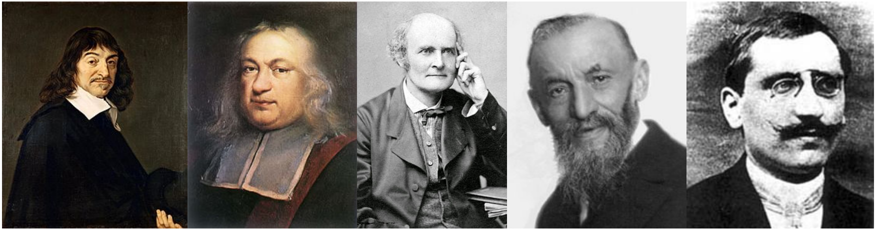
References



D. Lay. Linear algebra and its applications (3rd ed). Pearson (2006). Chapter 4.

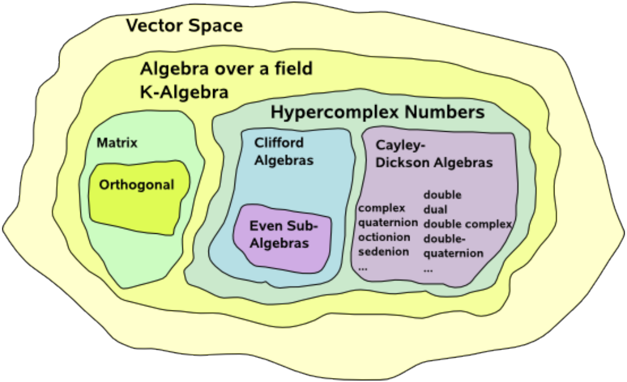
A little bit of history

Vectors were first used about 1636 in 2D and 3D to describe geometrical operations by René Descartes and Pierre de Fermat. In 1857 the notation of vectors and matrices was unified by Arthur Cayley. Giuseppe Peano was the firsst to give the modern definition of vector space in 1888, and Henri Lebesgue (about 1900) applied this theory to describe functional spaces as vector spaces.



Applications

It is difficult to think a mathematical tool with more applications than vector spaces. Thanks to them we may sum forces, control devices, model complex systems, denoise images, ... They underlie all these processes and it is thank to them that we can "nicely" operate with vectors. They are a mathemtical structure that generalizes many other useful structures.



Outline

Vector space

Definition 1.1 (Vector space)

A ***vector space*** *is a non-empty set, V, of objects (called* ***vectors****) in which we define two operations: the sum among vectors and the multiplication by a scalar (an element of any field, $\mathbb{K}$), and that $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall c, d \in \mathbb{K}$ it is verified that*

1. $\mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. $\exists \mathbf{0} \in V | \mathbf{u} + \mathbf{0} = \mathbf{u}$
5. $\forall \mathbf{u} \in V \quad \exists ! \mathbf{w} \in V | \mathbf{u} + \mathbf{w} = \mathbf{0}$ *(we normally write $\mathbf{w} = -\mathbf{u}$)*
6. $c\mathbf{v} \in V$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

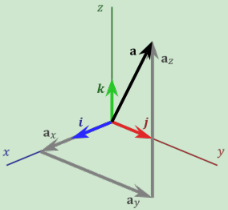
Vector space

Theorem 1.1 (Other properties)

11. $0\mathbf{u} = \mathbf{0}$
12. $c\mathbf{0} = \mathbf{0}$
13. $-\mathbf{u} = (-1)\mathbf{u}$

Watch out that 0 and 1 refer respectively to the neutral elements of the sum and multiplication in the field $\mathbb{K}$. -1 is the opposite number in $\mathbb{K}$ of 1 with respect to the sum of scalars.

Example: $\mathbb{R}^n$



$\mathbb{R}^n$ is a vector space of finite dimension for any n. As well as $\mathbb{C}^n$.

Vector space

Example: Force fields in Physics

Consider V to be the set of all arrows (directed line segments) in 3D. Two arrows are regarded as equal if they have the same length and direction. Define the sum of arrows and the multiplication by a scalar as shown below:

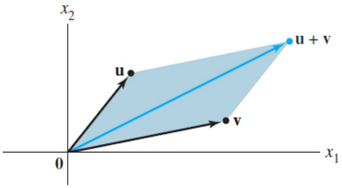
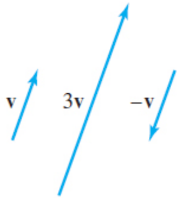


FIGURE 3 The parallelogram rule.



Example: Force fields in Physics (continued)

Here is an example of the application of some of the properties of vector spaces

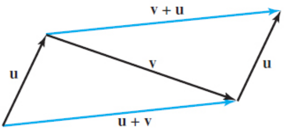


FIGURE 2 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

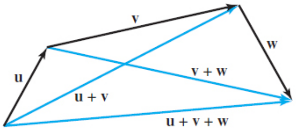
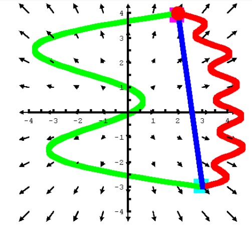


FIGURE 3 $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

With a force field we may define at every point in 3D space, which is the force that is applied.

Conservative force field



Vector space

Example: Infinite sequences

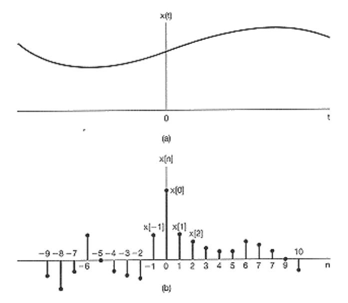
Let S be the set of all infinite sequences of numbers

$$\mathbf{u} = (..., u_{-2}, u_{-1}, u_0, u_1, u_2, ...)$$

Define the sum among two vectors and the multiplication by a scalar as

$$\mathbf{u}+\mathbf{v} = (..., u_{-2}+v_{-2}, u_{-1}+v_{-1}, u_0+v_0, u_1+v_1, u_2+v_2, ...)$$
$$c\mathbf{u} = (..., cu_{-2}, cu_{-1}, cu_0, cu_1, cu_2, ...)$$

Digital Signal Processing



Vector space

Example: Polynomials of degree n ($\mathbb{P}_n$)

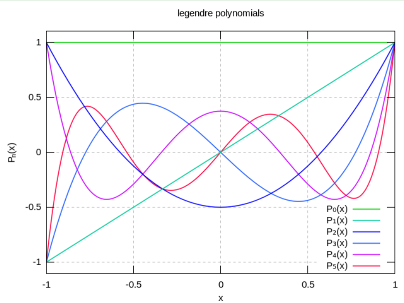
Let $\mathbb{P}_n$ be the set of all polynomials of degree n

$$u(x) = u_0 + u_1 x + u_2 x^2 + \ldots + u_n x^n$$

Define the sum among two vectors and the multiplication by a scalar as

$$\begin{aligned}(u+v)(x) &= (u_0+v_0) + (u_1+v_1)x + (u_2+v_2)x^2 + \ldots + (u_n+v_n)x^n \\ (cu)(x) &= cu_0 + cu_1 x + cu_2 x^2 + \ldots + cu_n x^n\end{aligned}$$

Legendre polynomials

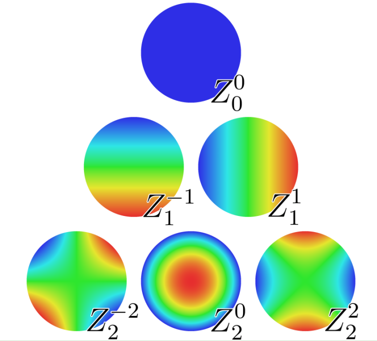


Example: Set of real functions defined in some domain

Let $\mathbb{F}$ be the set of all real valued functions defined in some domain ($f : D \rightarrow \mathbb{R}$)
Define the sum among two vectors and the multiplication by a scalar as

$$(u+v)(x) = u(x) + v(x)$$
$$(cu)(x) = cu(x)$$

Ex: $u(x) = 3 + x$
Ex: $v(x) = \sin x$
Ex: Zernike polynomials



Outline

5 Vector spaces

Vector subspace

Sometimes we don't need to deal with the whole vector space, but only a part of it. It would be nice if it also has the space properties.

Definition 2.1 (Vector subspace)

Let V be a vector space, and $H \subseteq V$ a part of it. H is ***vector subspace*** *iff*

a) $\mathbf{0} \in H$

b) $\forall \mathbf{u}, \mathbf{v} \in H \quad \mathbf{u} + \mathbf{v} \in H$ *(H is closed with respect to sum)*

c) $\forall \mathbf{u} \in H, \ \forall c \in \mathbb{K} \quad c\mathbf{u} \in H$ *(H is closed with respect to scalar multiplication)*

Example

$H = \{\mathbf{0}\}$ is a subspace.

Example

The vector space of polynomials (of any degree), $\mathbb{P} \in \mathbb{F}(\mathbb{R})$, is a vector subspace of the vector space of real valued functions defined over $\mathbb{R}$ ($\mathbb{F}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$).

Vector subspace

Example

$H = \mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2 \not\subset \mathbb{R}^3$, for instance, the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$, but $\mathbf{u} \notin \mathbb{R}^3$.

Example

$H = \mathbb{R}^2 \times \{0\}$ is a subspace of $\mathbb{R}^3$ because all vectors of H are of the form $\mathbf{u} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3$. It is obvious that H "looks like" $\mathbb{R}^2$. This resemblance is mathematically called **isomorphism**.

Example

Any plane in 3D passing through the origin is a subspace of $\mathbb{R}^3$.
Any plane in 3D <u>not</u> passing through the origin is <u>not</u> a subspace of $\mathbb{R}^3$, because $\mathbf{0}$ does not belong to the plane.

Vector subspace

Theorem 2.1

If H is a vector subspace, then H is a vector space.

Proof

a) $\Rightarrow$ *4*

$a \equiv \mathbf{0} \in H$

$4 \equiv \exists \mathbf{0} \in V | \mathbf{u} + \mathbf{0} = \mathbf{u}$

b) $\Rightarrow$ *1*

$b \equiv \forall \mathbf{u}, \mathbf{v} \in H \quad \mathbf{u} + \mathbf{v} \in H$

$1 \equiv \mathbf{u} + \mathbf{v} \in V$

Since $H \subset V$ *and thanks to b)* $\Rightarrow$ *2,3,7,8,9,10*

$2 \equiv \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$3 \equiv (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

$7 \equiv c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

$8 \equiv (c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

$9 \equiv c(d\mathbf{u}) = (cd)\mathbf{u}$

$10 \equiv 1\mathbf{u} = \mathbf{u}$

Vector subspace

Proof (continued)

c) $\Rightarrow$ 6

$c \equiv \forall \mathbf{u} \in H,\ \forall c \in \mathbb{K} \quad c\mathbf{u} \in H$

$6 \equiv c\mathbf{v} \in V$

Proof of 5

Since H is a subset of V, we know that for every $\mathbf{u} \in H$ there exists a unique $\mathbf{w} \in V | \mathbf{u} + \mathbf{w} = \mathbf{0}$. The problem is whether or not $\mathbf{w}$ is in H. We also know that $\mathbf{w} = (-1)\mathbf{v}$, and by c), $\mathbf{w} \in H$.

(q.e.d.)

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Subspace spanned by a set of vectors

Example

Let $\mathbf{v}_1, \mathbf{v}_2 \in V$ be two vectors of a vector space, V. The subset

$$H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

is a subspace of V.

Proof

Any vector of H is of the form $\mathbf{v} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$ for some $\lambda_1, \lambda_2 \in \mathbb{K}$.

Proof a) $\mathbf{0} \in H$

Simply by setting $\lambda_1 = \lambda_2 = 0$, we get $\mathbf{0} \in H$

Proof b) $\mathbf{u} + \mathbf{v} \in H$

Let $\mathbf{u}, \mathbf{v} \in H \Rightarrow \left.\begin{array}{l} \mathbf{u} = \lambda_{1u}\mathbf{v}_1 + \lambda_{2u}\mathbf{v}_2 \\ \mathbf{v} = \lambda_{1v}\mathbf{v}_1 + \lambda_{2v}\mathbf{v}_2 \end{array}\right\} \Rightarrow$

$$\begin{array}{rcl} \mathbf{u} + \mathbf{v} & = & (\lambda_{1u}\mathbf{v}_1 + \lambda_{2u}\mathbf{v}_2) + (\lambda_{1v}\mathbf{v}_1 + \lambda_{2v}\mathbf{v}_2) \\ & = & (\lambda_{1u} + \lambda_{1v})\mathbf{v}_1 + (\lambda_{2u} + \lambda_{2v})\mathbf{v}_2 \in H \end{array}$$

Subspace spanned by a set of vectors

Proof c) $c\mathbf{u} \in H$
Let $\mathbf{u} \in H \Rightarrow$

$$\mathbf{u} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \Rightarrow c\mathbf{u} = c(\lambda_u\mathbf{v}_1 + \lambda_2\mathbf{v}_2) = c\lambda_u\mathbf{v}_1 + c\lambda_2\mathbf{v}_2 \in H$$

Theorem 3.1

Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p \in V$ be p vectors of a vector space, V. The subset

$$H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$$

is a subspace of V.
Proof
Analogous to the previous example.

Subspace spanned by a set of vectors

Example

Consider the set of vectors $\mathbb{R}^4 \supset H = \{(a-3b, b-a, a, b)\ \forall a, b \in \mathbb{R}\}$. Is it a vector subspace?

<u>*Solution*</u>

All vectors of H can be written as

$$H \ni \mathbf{u} = \begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore, $H = \mathrm{Span}\{(1,-1,1,0), (-3,1,0,1)\}$ and by the previous theorem, it is a vector subspace.

Exercises

Exercises

From Lay (3rd ed.), Chapter 4, Section 1:

- 4.1.1
- 4.1.4
- 4.1.5
- 4.1.6
- 4.1.19
- 4.1.32
- 4.1.37 (computer)

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Null space of a matrix

Example

Consider the matrix

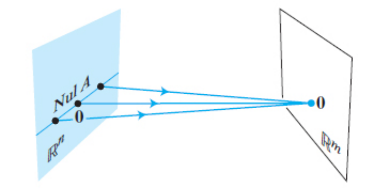
$$\begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}$$

The point $\mathbf{x} = (5, 3, -2)$ has the property that $A\mathbf{x} = \mathbf{0}$.

Definition 4.1 (Null space)

*The **null space of a matrix** $A \in \mathcal{M}_{m\times n}$ is the set of vectors*

$$\mathrm{Nul}\{A\} = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\}$$



Null space of a matrix

Example (continued)

$$\left(\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -5 & 9 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \end{array}\right)$$

Therefore

$$\mathrm{Nul}\{A\} = \{(-\tfrac{5}{2}x_3, -\tfrac{3}{2}x_3, x_3) \forall x_3 \in \mathbb{R}\}$$

The previous example ($\mathbf{x} = (5, 3, -2)$) is the point we obtain for $x_3 = -2$.

Null space of a matrix

Theorem 4.1

$\mathrm{Nul}\{A\}$ *is a vector subspace of* $\mathbb{R}^n$*.*

Proof

It is obvious that $\mathrm{Nul}\{A\} \subseteq \mathbb{R}^n$ *because* A *has* n *columns*

Proof a) $\mathbf{0} \in \mathrm{Nul}\{A\}$

$A\mathbf{0}_n = \mathbf{0}_m \Rightarrow \mathbf{0}_n \in \mathrm{Nul}\{A\}$

Proof b) $\mathbf{u} + \mathbf{v} \in \mathrm{Nul}\{A\}$

Let $\mathbf{u}, \mathbf{v} \in \mathrm{Nul}\{A\} \Rightarrow \left.\begin{array}{c} A\mathbf{u} = \mathbf{0} \\ A\mathbf{v} = \mathbf{0} \end{array}\right\} \Rightarrow$

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0} \Rightarrow \mathbf{u} + \mathbf{v} \in \mathrm{Nul}\{A\}$$

Proof c) $c\mathbf{u} \in \mathrm{Nul}\{A\}$

Let $\mathbf{u} \in H \Rightarrow$

$$A\mathbf{u} = \mathbf{0} \Rightarrow A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{0} = \mathbf{0} \Rightarrow c\mathbf{u} \in \mathrm{Nul}\{A\}$$

Null space of a matrix

Example

Let $H = \left\{ (a,b,c,d) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} a-2b+5c=d \\ c-a=b \end{array} \right\}$. Is H a vector subspace of $\mathbb{R}^4$?

Solution

We may rewrite the conditions of belonging to H as

$$\begin{array}{l} a-2b+5c=d \\ c-a=b \end{array} \quad \Rightarrow \begin{pmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \mathbf{0}$$

and, thanks to the previous theorem, H is a vector subspace of $\mathbb{R}^4$.

Null space of a matrix

Example (continued)

We can even provide a basis for H

$$\begin{pmatrix} 1 & -2 & 5 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The solution of $A\mathbf{x} = \mathbf{0}$ are all points of the form

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -c+d \\ 0 \\ c \\ d \end{pmatrix} = c \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Consequently $H = \mathrm{Span}\{(-1,0,1,0),(1,0,0,1)\}$.

Column space of a matrix

Definition 4.2 (Column space)

Let $A \in \mathcal{M}_{m\times n}$ a matrix and $\mathbf{a}_i \in \mathbb{R}^m$ $(i = 1, 2, ...n)$ its columns. The ***column space of the matrix*** *A is defined as*

$$\mathrm{Col}\{A\} = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, ...\mathbf{a}_n\} = \{\mathbf{b} \in \mathbb{R}^m | A\mathbf{x} = \mathbf{b} \textit{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

Theorem 4.2

The column space of a matrix is a subspace of $\mathbb{R}^m$ *Proof*
$\mathrm{Col}\{A\}$ is a set generated by a number of vectors and by Theorem 3.1 it is a subspace of $\mathbb{R}^m$.

Column space of a matrix

Example

Find a matrix A such that $\mathrm{Col}\{A\} = \{(6a-b, a+b, -7a) \forall a, b \in \mathbb{R}\}$

Solution We can express the points in $\mathrm{Col}\{A\}$ as

$$\mathrm{Col}\{A\} \ni \mathbf{x} = \begin{pmatrix} 6a-b \\ a+b \\ -7a \end{pmatrix} = a \begin{pmatrix} 6 \\ 1 \\ -7 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, $\mathrm{Col}\{A\} = \mathrm{Span}\{(6,1,-7),(-1,1,0)\}$. That is, these must be the two columns of A

$$A = \begin{pmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{pmatrix}$$

Comparison between the Null and the Column spaces

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Outline

Linear transformation

We have said that $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation, but it is not the only one.

Definition 5.1 (Linear transformation)

The transformation $T : V \to W$ between two vectors spaces V and W is a rule that for each vector $\mathbf{v} \in V$ assigns a unique vector $\mathbf{w} = T(\mathbf{v}) \in W$, such that

1. $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in V$
2. $T(c\mathbf{v}) = cT(\mathbf{v}) \quad \forall \mathbf{v} \in V, \forall c \in \mathbb{K}$

Example

For a matrix $A \in \mathcal{M}_{m\times n}$, we have that

$$\begin{array}{rcl} T : \mathbb{R}^n & \to & \mathbb{R}^m \\ \mathbf{x} & \to & A\mathbf{x} \end{array}$$

is a linear transformation (we can easily verify that T meets the two required conditions).

Linear transformation

Example

Consider the space of all continuous, real-valued functions defined over $\mathbb{R}$ whose all derivatives are also continuous. We will refer to this space as $C^\infty(\mathbf{R})$. For instance, all polynomials belong to this space, as well as any sin, cos function. It can be proved that $C^\infty(\mathbf{R})$ is a vector space.
Consider the transformation that assigns to each function in $C^\infty(\mathbf{R})$ its derivative

$$\begin{array}{rcl} D : C^\infty(\mathbf{R}) & \to & C^\infty(\mathbf{R}) \\ f & \to & D(f) \end{array}$$

is a linear transformation.
Proof

1. $D(f+g) = D(f) + D(g)$
2. $D(cf) = cD(f)$

Kernel and range of transformation

Definition 5.2 (Kernel (Núcleo))

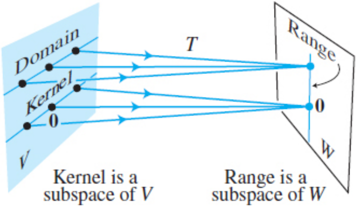
*The **kernel of a transformation** T is the set of all vectors such that*

$$\mathrm{Ker}\{T\} = \{\mathbf{v} \in V \,|\, T(\mathbf{v}) = \mathbf{0}\}$$

Definition 5.3 (Range (Imagen))

*The **range of a transformation** T is the set of all vectors such that*

$$\mathrm{Range}\{T\} = \{\mathbf{w} \in W \,|\, \exists \mathbf{v} \in V \;\; T(\mathbf{v}) = \mathbf{w}\}$$



Kernel and range of transformation

Example (continued)

$\mathrm{Ker}\{T\} = \mathrm{Nul}\{A\}$
$\mathrm{Ker}\{D\} = \{f(x) = c\}$ because $D(c) = 0$

Theorem 5.1

If $T(\mathbf{x}) = A\mathbf{x}$, then

$$\begin{aligned} \mathrm{Ker}\{T\} &= \mathrm{Nul}\{A\} \\ \mathrm{Range}\{T\} &= \mathrm{Col}\{A\} \end{aligned}$$

Exercises

From Lay (3rd ed.), Chapter 4, Section 2:

- 4.2.3
- 4.2.9
- 4.2.11
- 4.2.30
- 4.2.31

Outline

5 Vector spaces

Linear independence

Definition 6.1 (Linear independence)

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ *is* ***linearly independent*** *iff the only solution to the equation*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = \mathbf{0}$$

is the trivial solution ($c_1 = c_2 = ... = c_p = 0$)*. The set is linearly dependent if there exists another solution to the equation.*

Watch out that we cannot simply put all vectors as columns of a matrix A and solve $A\mathbf{c} = \mathbf{0}$ because this is only valid for vectors in $\mathbb{R}^n$, but it is not valid for any vector space.

Linear independence

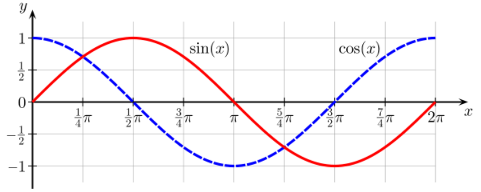
Example

- $\{\mathbf{v}_1\}$ is linearly dependent if $\mathbf{v}_1 = \mathbf{0}$.
- $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if $\mathbf{v}_2 = c\mathbf{v}_1$.
- $\{\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ is linearly dependent.

Example

In the vector space of continuous functions over $\mathbb{R}$, $C(\mathbb{R})$, the vectors $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are independent because

$$f_2(x) \neq cf_1(x)$$



Linear independence

Theorem 6.1

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$*, with* $\mathbf{v}_1 \neq \mathbf{0}$ *is* ***linearly dependent*** *if any of the vectors* $\mathbf{v}_j$ *(*$j > 1$*) is linearly dependent on the previous ones* $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{j-1}\}$*.*

Example

In the vector space of polynomials, consider the vectors $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = 4 - x$. The set $\{p_0(x), p_1(x), p_2(x)\}$ is linearly dependent because

$$p_2(x) = 4p_0(x) - p_1(x) \Rightarrow p_1(x) - 4p_0(x) + p_2(x) = 0$$

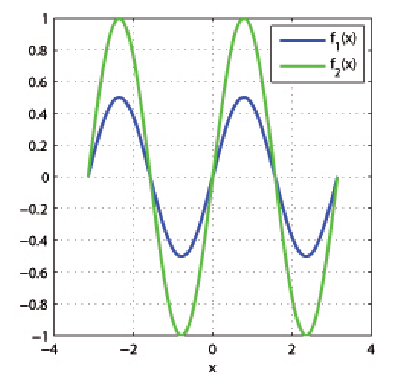
Linear independence

Example

In the vector space of continuous functions, consider the vectors $f_1(x) = \sin(x)\cos(x)$ and $f_2(x) = \sin(2x)$. The set $\{f_1(x), f_2(x)\}$ is linearly dependent because $f_2(x) = 2f_1(x)$

MATLAB:

```
x=[-pi:0.001:pi]
f1=sin(x).*cos(x);
f2=sin(2*x);
plot(x,f1,x,f2)
```



Basis of a subspace

Definition 6.2 (Basis of a subspace)

A set of vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ *is a basis of the vector subspace* H *iff*

1. B *is a linearly independent set of vectors*
2. $H = \mathrm{Span}\{B\}$

In other words, a basis is a non-redundant set of vectors that span H.

Example

Let A be an invertible matrix. By Theorem 5.1 and 11.5 of Chapter 3 (the invertible matrix theorem), we know that the columns of A span $\mathbb{R}^n$ and that they are linearly independent. Consequently, the columns of A are a basis of $\mathbb{R}^n$.

Basis of a subspace

Example

The standard basis of $\mathbb{R}^n$ are the columns of I_n

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix} \quad \dots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix}$$

Example

Let $\mathbf{v}_1 = (3, 0, -6)$, $\mathbf{v}_2 = (-4, 1, 7)$, $\mathbf{v}_3 = (-2, 1, 5)$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of $\mathbb{R}^3$?
Solution
This question is the same as whether A is invertible with

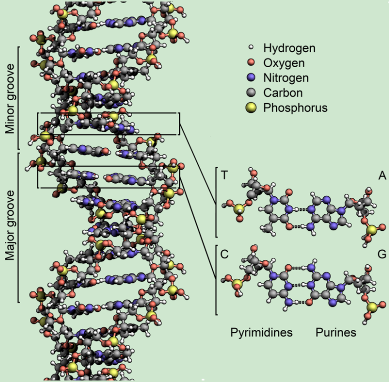
$$A = \begin{pmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{pmatrix} \Rightarrow |A| = 6 \Rightarrow \exists A^{-1}$$

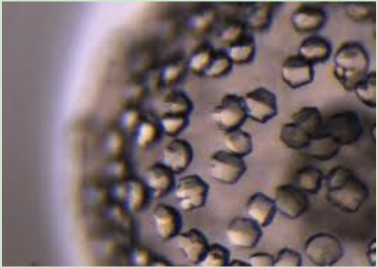
Because A is invertible, we have that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

Basis of a subspace

Example: DNA Structure

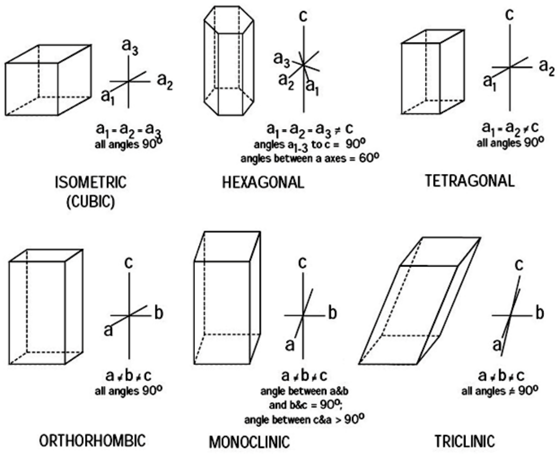
In 1953, Rosalind Franklin, James Watson and Francis Crick determined the 3D structure of DNA using data coming from X-ray diffraction of crystallized DNA. Watson and Crick received the Nobel prize in physiology and medicine in 1962 (Franklin died 1958).





Example: DNA Structure (continued)

Three-dimensional crystals repeat a certain motif all over the space following a crystal lattice. The vectors that define the crystal lattice are a basis of $\mathbb{R}^3$



Basis of a subspace

Example

$B = \{1, x, x^2, x^3, ...\}$ is the standard basis of the vector space of polynomials $\mathbb{P}$.
Proof

1. B is linearly independent:

   $$\forall x \in \mathbb{R} \quad c_0 1 + c_1 x + c_2 x^2 + c_3 x^3 + ... = 0 \Rightarrow c_0 = c_1 = c_2 = ... = 0$$

   The only way that a polynomial of degree whichever is 0 for all values of x is that the coefficients of the polynomial are all 0.

2. $\mathbb{P} = \mathrm{Span}\{B\}$:
   It is obvious that any polynomial can be written as a linear combination of elements of B (in fact, this is they way we normally do).

Basis of a subspace

Example

$H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (0, 2, -1)$, $\mathbf{v}_2 = (2, 2, 0)$, $\mathbf{v}_3 = (6, 16, -5)$. Find a basis of H

Solution

All vectors in H are of the form:

$$H \ni \mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

We realize that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, therefore, $\mathbf{v}_3$ is redundant:

$$\begin{aligned} H \ni \mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(5\mathbf{v}_1 + 3\mathbf{v}_2) \\ &= (c_1 + 5c_3)\mathbf{v}_1 + (c_2 + 3c_3)\mathbf{v}_2 \\ &= c_1'\mathbf{v}_1 + c_2'\mathbf{v}_2 \end{aligned}$$

It suffices to construct our basis with $\mathbf{v}_1$ and $\mathbf{v}_2$.

Basis of a subspace

Theorem 6.2 (Spanning set theorem (conjunto generador))

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ be a set of vectors and $H = \mathrm{Span}\{S\}$. Then,

1. *If $\mathbf{v}_k$ is a linear combination of the rest, then the set $S - \{\mathbf{v}_k\}$ still generates H.*
2. *If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis of H.*

Proof

1. *Assume that the linear combination that explains $\mathbf{v}_k$ is*
$$\mathbf{v}_k = a_1\mathbf{v}_1 + ... + a_{k-1}\mathbf{v}_{k-1} + a_{k+1}\mathbf{v}_{k+1} + ... + a_p\mathbf{v}_p$$
Consider any vector in H
$$\begin{array}{rcl} \mathbf{x} & = & c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p \\ & = & (c_1 + a_1)\mathbf{v}_1 + ... + (c_{k-1} + a_{k-1})\mathbf{v}_{k-1} + \\ & & (c_{k+1} + a_{k+1})\mathbf{v}_{k+1} + ... + (c_p + a_p)\mathbf{v}_p \end{array}$$
That is we can express $\mathbf{x}$ not using $\mathbf{v}_k$.
2. *Step 1: If S is a linearly independent set, then S is the basis of H.*
Step 2: If S is not, using the previous point we can remove a vector to produce S' that still generates H (go to Step 1).

Outline

5 Vector spaces
- Definition (a)
- Vector subspace (a)
- Subspace spanned by a set of vectors (a)
- Null space and column space of a matrix (b)
- Kernel and range of a linear transformation (b)
- Linearly independent sets and bases (b)
- Bases for $\mathrm{Nul}\{A\}$ and $\mathrm{Col}\{A\}$ (c)
- Coordinate system (c)
- Dimension of a vector space (d)
- Rank of a matrix (d)
- Change of basis (d)

Basis for $\mathrm{Nul}\{A\}$

Example

Let $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$
We solve the equation system $A\mathbf{x} = \mathbf{0}$ to find

$$(A|\mathbf{0}) \sim \left(\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

we have coloured the pivot columns from which learn

$$\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5 \\ x_3 &= -2x_4 + 2x_5 \end{aligned} \quad \Rightarrow \mathrm{Nul}\{A\} \ni \mathbf{x} = \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix}$$

Basis for $\mathrm{Nul}\{A\}$

Example (continued)

$$\mathrm{Nul}\{A\} \ni \mathbf{x} = \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Finally the basis for $\mathrm{Nul}\{A\}$ is

$$\mathrm{Nul}\{A\} = \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Basis for $\mathrm{Col}\{A\}$

Example

Consider A as in the previous example. We had

$$A \sim \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B$$

Let's call this latter matrix B. Non-pivot columns of B can be written as a linear combination of the pivot columns:

$$\begin{aligned} \mathbf{b}_2 &= -2\mathbf{b}_1 \\ \mathbf{b}_4 &= -\mathbf{b}_1 + 2\mathbf{b}_3 \\ \mathbf{b}_5 &= 3\mathbf{b}_1 - 2\mathbf{b}_3 \end{aligned}$$

Basis for $\mathrm{Col}\{A\}$

Example (continued)

Since row operations do not change the linear dependences among matrix columns, we can derive the same relationships for matrix A

$$\begin{aligned} \mathbf{a}_2 &= -2\mathbf{a}_1 \\ \mathbf{a}_4 &= -\mathbf{a}_1 + 2\mathbf{a}_3 \\ \mathbf{a}_5 &= 3\mathbf{a}_1 - 2\mathbf{a}_3 \end{aligned}$$

Finally, the basis of $\mathrm{Col}\{A\}$ is $\{\mathbf{a}_1, \mathbf{a}_3\}$.

$$\mathrm{Col}\{A\} = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_3\} = \mathrm{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\}$$

Basis for $\mathrm{Col}\{A\}$

Theorem 7.1

The pivot columns of A constitute a basis for $\mathrm{Col}\{A\}$.

Proof

Let B the reduced echelon form of A.

1. *The pivot columns of B form a linearly independent set because none of its elements can be expressed as a linear combination of the elements before each one of them.*
2. *The dependence relationships among columns are not affected by row operations. Therefore, the corresponding pivot columns of A are also linearly independent and, consequently, a basis of* $\mathrm{Col}\{A\}$.

Two views of a basis

As small as possible, as large as possible

1. The Spanning Set Theorem states that the basis is as small as possible as long as it spans the required subspace.
2. The basis has the maximum amount of vectors spanning the required subspace. If we add one more, the new set is not linearly independent.

Example

- $\{(1,0,0),(2,3,0)\}$ is a set of 2 linearly independent vectors. But it cannot span $\mathbb{R}^3$ because for this we need 3 vectors.
- $\{(1,0,0),(2,3,0),(4,5,6)\}$ is a set of 3 linearly independent vectors that spans $\mathbb{R}^3$, so it is a basis of $\mathbb{R}^3$.
- $\{(1,0,0),(2,3,0),(4,5,6),(7,8,9)\}$ is a set of 4 linearly dependent vectors that spans $\mathbb{R}^3$, so it cannot be a basis.

Exercises

Exercises

From Lay (3rd ed.), Chapter 4, Section 3:

- 4.3.1
- 4.3.2
- 4.3.8
- 4.3.12
- 4.3.24
- 4.3.31
- 4.3.32
- 4.3.33
- 4.3.37 (computer)

Outline

5 Vector spaces

- Definition (a)
- Vector subspace (a)
- Subspace spanned by a set of vectors (a)
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- Linearly independent sets and bases (b)
- Bases for $\mathrm{Nul}\{A\}$ and $\mathrm{Col}\{A\}$ (c)
- Coordinate system (c)
- Dimension of a vector space (d)
- Rank of a matrix (d)
- Change of basis (d)

Coordinate system

An important reason to assign a basis to a vector space V is that it makes V to "behave" as $\mathbb{R}^n$ through, what is called, a coordinate system.

Theorem 8.1 (The unique representation theorem)

Let $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ a basis of the vector space V, and consider any vector $\mathbf{v} \in V$. There exists a unique set of scalars such that

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_n\mathbf{b}_n$$

Proof

Let assume that there exists another set of scalars such that

$$\mathbf{v} = c_1'\mathbf{b}_1 + c_2'\mathbf{b}_2 + ... + c_n'\mathbf{b}_n$$

Subtracting both equations we have

$$\mathbf{0} = (c_1 - c_1')\mathbf{b}_1 + (c_2 - c_2')\mathbf{b}_2 + ... + (c_n - c_n')\mathbf{b}_n$$

But since the vectors $\mathbf{b}_i$ form a basis and are linearly independent, it must be

$$(c_1 - c_1') = (c_2 - c_2') = (c_n - c_n') = 0$$

Coordinate system

Proof (continued)

Finally, $c_1 = c'_1$, $c_2 = c'_2$, ..., $c_n = c'_n$ which is a contradiction with the hypothesis that there were two different sets of scalars representing the vector. Consequently, the set of scalars must be unique.

Definition 8.1 (Coordinates)

Let $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ a basis of the vector space V, and consider any vector $\mathbf{v} \in V$. The ***coordinates*** *of $\mathbf{v}$ in B are the c_i coefficients such that*

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + ... + c_n\mathbf{b}_n \Rightarrow [\mathbf{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{pmatrix}$$

The transformation $T : V \to \mathbb{R}^n$ such that $T(\mathbf{x}) = [\mathbf{x}]_B$ is called the ***coordinate mapping***.

Coordinate system

Example

Let $B = \{(1,0),(1,2)\}$ be a basis of $\mathbb{R}^2$ and $[\mathbf{x}]_B = (-2,3)$, then

$$\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 = -2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\6\end{pmatrix}$$

In fact $(1,6)$ are the coordinates of $\mathbf{x}$ in the standard basis $\{\mathbf{e}_1,\mathbf{e}_2\}$

$$\mathbf{x} = 1\mathbf{e}_1 + 6\mathbf{e}_2 = 1\begin{pmatrix}1\\0\end{pmatrix} + 6\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\6\end{pmatrix}$$

That is, the point $\mathbf{x}$ does not change, but depending on the coordinate system employed, we "see" it with different coordinates.

Coordinate system

Example (continued)

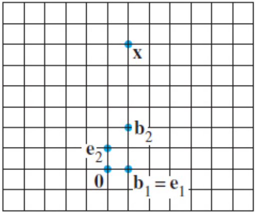


FIGURE 1 Standard graph paper.

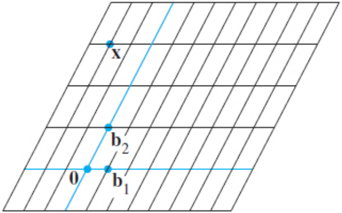
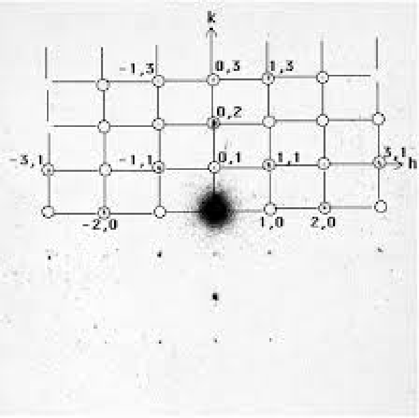


FIGURE 2 $\mathcal{B}$-graph paper.

Coordinate system

Example: X-ray diffraction

In ths figure we see how a X-ray diffraction pattern of a crystal is “indexed”.



Coordinates in $\mathbb{R}^n$

If we have a point $\mathbf{x}$ in $\mathbb{R}$ we can easily find its coordinates in any basis, as in the following example.

Example

Let $\mathbf{x} = (4, 5)$ and the basis $B = \{(2, 1), (-1, 1)\}$. We need to find c_1 and c_2 such that

$$\mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 \Rightarrow \begin{pmatrix} 4 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

From which we can easily derive that $c_1 = 3$ and $c_2 = 2$.

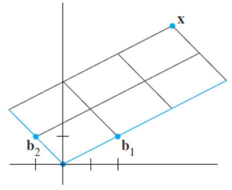


FIGURE 4
The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $(3, 2)$.

Change of basis

Change from the standard basis to an arbitrary basis

Note that the previous equation system is of the form

$$\mathbf{x} = P_B[\mathbf{x}]_B$$

where P_B is called the **change-of-coordinates matrix** and its columns are the vectors of the basis B (consequently, it is invertible). We find the coordinates of the vector $\mathbf{x}$ in the basis B as

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x}$$

Change between two arbitrary bases

Let's say we know the coordinates of a point in some basis, B_1, and we want to know its coordinates in some other basis, B_2. We may use

$$\mathbf{x} = P_{B_1}[\mathbf{x}]_{B_1} = P_{B_2}[\mathbf{x}]_{B_2} \Rightarrow [\mathbf{x}]_{B_2} = P_{B_2}^{-1}P_{B_1}[\mathbf{x}]_{B_1}$$

Coordinate mapping

Theorem 8.2 (The coordinate mapping is an isomorphism between V and $\mathbb{R}^n$)

The coordinate mapping is a bijective, linear transformation.

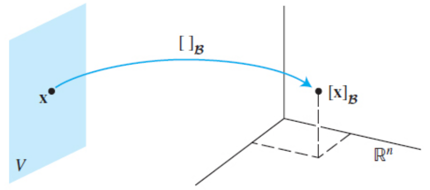


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

Corollary

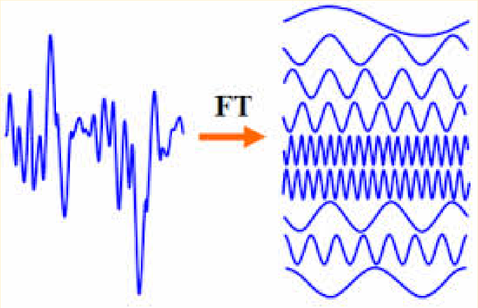
Since the coordinate mapping is a linear transformation it extends to linear combinations

$$[a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_p\mathbf{u}_p]_B = a_1[\mathbf{u}_1]_B + a_2[\mathbf{u}_2]_B + \dots + a_p[\mathbf{u}_p]_B$$

Coordinate mapping

Consequences

Any operation in V can be performed in $\mathbb{R}^n$ and then go back to V.
For spaces of functions, this opens a new door to analyze functions (signals, images, ...) in $\mathbb{R}^n$ using the appropriate basis: Fourier transform, wavelet transform, Discrete Cosine Transform, ...



Coordinate mapping

Example

Consider the space of polynomials of degree 2, $\mathbb{P}_2$. any polynomial in this space is of the form

$$p(t) = a_0 + a_1 t + a_2 t^2$$

If we choose the standard basis in $\mathbb{P}_2$ that is

$$B = \{1, t, t^2\}$$

Then, we have the coordinate mapping

$$T(p(t)) = [p]_B = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

that is an isomorphism from $\mathbb{P}_2$ onto $\mathbb{R}^3$.

Coordinate mapping

Example (continued)

Now we can perform any reasoning in $\mathbb{P}_2$ by studying an analogous problem in $\mathbb{R}^3$. For instance, let's study if the following polynomials are linearly independent

$$\begin{array}{rcl c l} p_1(t) & = & 1+2t^2 & \Rightarrow & [p_1(t)]_B = (1,0,2) \\ p_2(t) & = & 4+t+5t^2 & \Rightarrow & [p_2(t)]_B = (4,1,5) \\ p_3(t) & = & 3+2t & \Rightarrow & [p_3(t)]_B = (3,2,0) \end{array}$$

We simply need to see if the corresponding coordinates in $\mathbb{R}^3$ are linearly independent

$$\begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Looking at the non-pivot columns we learn that

$$p_3(t) = -5p_1(t) + 2p_2(t)$$

Finally, we conclude that the 3 polynomials are not linearly independent.

Coordinate mapping

Example

Consider $\mathbf{v}_1 = (3, 6, 2)$, $\mathbf{v}_2 = (-1, 0, 1)$, $B = \{\mathbf{v}_1, \mathbf{v}_2\}$, and $H = \mathrm{Span}\{B\}$. H is isomorphic to $\mathbb{R}^2$ (because its points have only 2 coordinates). For instance, the coordinates of $\mathbf{x} = (3, 12, 7) \in H$ are $[\mathbf{x}]_B = (2, 3)$.

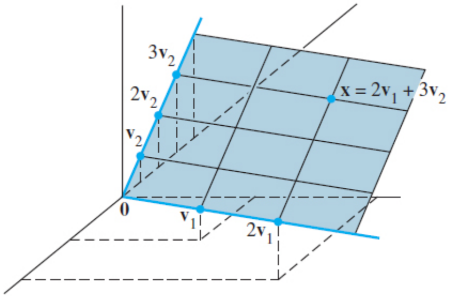


FIGURE 7 A coordinate system on a plane H in $\mathbb{R}^3$.

Coordinate mapping

Example

Consider $\mathbf{v}_1 = (3, 6, 2)$, $\mathbf{v}_2 = (-1, 0, 1)$, $B = \{\mathbf{v}_1, \mathbf{v}_2\}$, and $H = \mathrm{Span}\{B\}$. H is isomorphic to $\mathbb{R}^2$ (because its points have only 2 coordinates). For instance, the coordinates of $\mathbf{x} = (3, 12, 7) \in H$ are $[\mathbf{x}]_B = (2, 3)$.

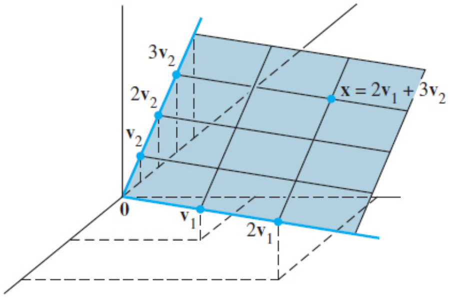


FIGURE 7 A coordinate system on a plane H in $\mathbb{R}^3$.

Exercises

Exercises

From Lay (3rd ed.), Chapter 4, Section 4:

- 4.4.3
- 4.4.8
- 4.4.9
- 4.4.13
- 4.4.17
- 4.4.19
- 4.4.22
- 4.4.24
- 4.4.25

Outline

Dimension of a vector space

We have just said that if the basis of a vector space V has n elements, then V is isomorphic to $\mathbb{R}^n$. n is a characteristic number of each space called the dimension.

Theorem 9.1

Let V be a vector space with a basis given by $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$. Then, any subset of V with more than n elements is linearly dependent.

Proof

Let S be a subset of V with $p > n$ vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$$

We now consider the set of coordinates of these vectors.

$$\{[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, ..., [\mathbf{v}_p]_B\}$$

They are $p > n$ vectors in $\mathbb{R}^n$ and, therefore, necessarily linearly dependent. That is, there exist $c_1, c_2, ..., c_p$, not all of them 0, such that

$$c_1[\mathbf{v}_1]_B + c_2[\mathbf{v}_2]_B + c_p[\mathbf{v}_p]_B = \mathbf{0} \in \mathbb{R}^n$$

Dimension of a vector space

Proof (continued)

If we now exploit the fact that the coordinate mapping is linear, then we have

$$[c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_p\mathbf{v}_p]_B = \mathbf{0} \in \mathbb{R}^n$$

Finally, we make use of the fact that the coordinate mapping is bijective

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_p\mathbf{v}_p = \mathbf{0} \in V$$

And, consequently, we have shown that the p vectors in S are linearly dependent.

Theorem 9.2

If a basis of a vector space has n vectors, then all other bases also have n vectors.

Proof

Let B_1 be a basis with n vectors of a vector space V. Let B_2 another basis of V. By the previous theorem, B_2 has at most n vectors. Let us assume now that B_2 has less than n vectors, then by the previous theorem B_1 would not be a basis. This is a contradiction with the fact that B_1 is a basis and, consequently, B_2 cannot have less than n vectors.

Dimension of a vector space

Definition 9.1

If the vector space V is spanned by a finite set of vectors, then V is ***finite-dimensional*** *and its* ***dimension*** *($\dim\{V\}$) is the number of elements of any of its bases. The dimension of $V = \{\mathbf{0}\}$ is 0. If V is not generated by a finite set of vectors, then it is* ***infinite-dimensional****.*

Example

$\dim\{\mathbb{R}^n\} = n$
$\dim\{\mathbb{P}_2\} = 3$ because one of its bases is $\{1, t, t^2\}$
$\dim\{\mathbb{P}\} = \infty$
$\dim\{\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}\} = 2$

Dimension of a vector space

Example: in $\mathbb{R}^3$

There is a single subspace of dimension 0 ($\{\mathbf{0}\}$)
There are infinite subspaces of dimension 1 (all lines going through the origin)
There are infinite subspaces of dimension 2 (all planes going through the origin)
There is a single subspace of dimension 3 ($\mathbb{R}^3$)

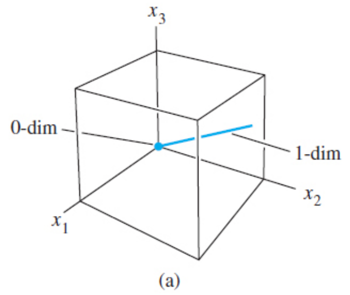


FIGURE 1 Sample subspaces of $\mathbb{R}^3$.

Dimension of a vector space

Theorem 9.3

Let $H \subseteq V$ be a vector subspace of a vector space V. Then,

$$\dim\{H\} \leq \dim\{V\}$$

Theorem 9.4

Let V a n-dimensional vector space ($n \geq 1$).

- *Any linearly independent subset of V with n elements is a basis.*
- *Any subset of V with n elements that span V is a basis.*

Dimension of a vector space

Theorem 9.5

Consider any matrix $A \in \mathcal{M}_{m\times n}$.

- $\dim\{\mathrm{Nul}\{A\}\}$ *is the number of free variables in the equation* $A\mathbf{x} = \mathbf{0}$.
- $\dim\{\mathrm{Col}\{A\}\}$ *is the number of pivot columns of* A.

Example

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \sim \begin{pmatrix} \color{blue}{1} & -2 & \color{blue}{0} & -1 & 3 \\ \color{blue}{0} & 0 & \color{blue}{1} & 2 & -2 \\ \color{blue}{0} & 0 & \color{blue}{0} & 0 & 0 \end{pmatrix}$$

The number of pivot columns of A is $2 = \dim\{\mathrm{Col}\{A\}\}$ (in blue), while the number of free variables is $3 = \dim\{\mathrm{Nul}\{A\}\}$ (the free variables are x_2, x_4 and x_5).

Exercises

Exercises

From Lay (3rd ed.), Chapter 4, Section 5:

- 4.5.1
- 4.5.13
- 4.5.21
- 4.5.25
- 4.5.26
- 4.5.27
- 4.5.28
- 4.5.31
- 4.5.32

Outline

5. Vector spaces
   - Definition (a)
   - Vector subspace (a)
   - Subspace spanned by a set of vectors (a)
   - Null space and column space of a matrix (b)
   - Kernel and range of a linear transformation (b)
   - Linearly independent sets and bases (b)
   - Bases for $\mathrm{Nul}\{A\}$ and $\mathrm{Col}\{A\}$ (c)
   - Coordinate system (c)
   - Dimension of a vector space (d)
   - **Rank of a matrix (d)**
   - Change of basis (d)

Rank of a matrix

The **rank of a matrix** is the number of linearly independent rows of that matrix. It can also be defined as the number of linearly independent columns of that matrix because both definitions yield the same number. We'll see a more formal definition below.

Definition 10.1 (Row space of a matrix)

Given a matrix $A \in \mathcal{M}_{m\times n}$, the ***row space*** *of A is the space spanned by all rows of A ($\mathrm{Row}\{A\} \subseteq \mathbb{R}^n$).*

Theorem 10.1

$$\mathrm{Row}\{A\} = \mathrm{Col}\{A^T\}$$

Rank of a matrix

Theorem 10.2

If a matrix A is row equivalent to another matrix B, then $\mathrm{Row}\{A\} = \mathrm{Row}\{B\}$. If B is in a reduced echelon form, then the non-null rows of B form a basis of $\mathrm{Row}\{A\}$

Proof

Proof $\mathrm{Row}\{A\} \supseteq \mathrm{Row}\{B\}$

Since the rows of B are obtained by row operations on the rows of A, then any linear combination of the rows of B can be obtained as linear combinations of the rows of A.

Proof $\mathrm{Row}\{A\} \subseteq \mathrm{Row}\{B\}$

Since the row operations are reversible, then any linear combination of the rows of A can be obtained as linear combinations of the rows of B.

Proof non-null rows of B form a basis

They are linearly independent because any non-null row of B cannot be obtained as a linear combination of the rows below (because it is in echelon form and there are numbers in early columns that have 0s below)

Rank of a matrix

Example

$$A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix} \sim B = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot columns have been highlighted in blue. At this point we can already construct a basis for the row and column spaces of A

$$\begin{aligned} \mathbb{R}^5 \supset \mathrm{Row}\{A\} &= \mathrm{Span}\{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\} \\ \mathbb{R}^4 \supset \mathrm{Col}\{A\} &= \mathrm{Span}\{(-2,1,3,1),(-5,3,11,7),(0,1,7,5)\} \end{aligned}$$

To calculate the null space of A we need the reduced echelon form

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank of a matrix

Example (continued)

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= 2x_3 - 3x_5 \\ x_4 &= 5x_5 \end{aligned} \Rightarrow \mathrm{Nul}\{A\} \ni \mathbf{x} = x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{pmatrix}$$

Finally,

$$\mathbb{R}^5 \supset \mathrm{Nul}\{A\} = \mathrm{Span}\{(-1, 2, 1, 0, 0), (-1, -3, 0, 5, 1)\}$$

Rank of a matrix

Definition 10.2 (Rank of a matrix)

$$\mathrm{Rank}\{A\} = \dim\{\mathrm{Col}\{A\}\}$$

That is, by definition, $\mathrm{Rank}\{A\}$ *is the number of pivot columns of* A*.*

Rank of a matrix

Theorem 10.3 (Rank theorem)

For any matrix $A \in \mathcal{M}_{m\times n}$

1. $\dim\{\mathrm{Row}\{A\}\} = \dim\{\mathrm{Col}\{A\}\}$
2. $\mathrm{Rank}\{A\} + \dim\{\mathrm{Nul}\{A\}\} = n$

Proof

1. *Let* B *be the reduced echelon form of* A*. By definition* $\mathrm{Rank}\{A\}$ *is the number of pivot columns in* A *(that is the same as the number of pivot columns in* B*). Since* B *is in reduced echelon form, each of its non-zero rows has a column pivot and, consequently, the number of non-zero rows coincides with the number of pivot columns. The basis of* $\mathrm{Row}\{B\} = \mathrm{Row}\{A\}$ *must have as many elements as pivot columns.*
2. *From Theorem 9.5 we know that* $\mathrm{Null}\{A\}$ *is the number of free variables in* $A\mathbf{x} = \mathbf{0}$*, that is, the number of non-pivot columns of* B*. Consequently, we have*

$$\dim\{\mathrm{Col}\{A\}\} + \dim\{\mathrm{Nul}\{A\}\} = n$$

But by definition, $\mathrm{Rank}\{A\} = \dim\{\mathrm{Col}\{A\}\}$*, which proves the theorem.*

Rank of a matrix

Example

Let $A \in \mathcal{M}_{7\times 9}$. We know $\dim\{\mathrm{Nul}\{A\}\} = 2$. What is $\mathrm{Rank}\{A\}$?
According to the previous theorem

$$\mathrm{Rank}\{A\} = n - \dim\{\mathrm{Nul}\{A\}\} = 9 - 2 = 7$$

Example

Let $A \in \mathcal{M}_{6\times 9}$. Can it be $\dim\{\mathrm{Nul}\{A\}\} = 2$?
Let us presume that it can be $\dim\{\mathrm{Nul}\{A\}\} = 2$, then

$$\mathrm{Rank}\{A\} = n - \dim\{\mathrm{Nul}\{A\}\} = 9 - 2 = 7$$

But since A has only 6 rows, the maximum rank can only be 6 (not 7), and therefore, it must be $\dim\{\mathrm{Nul}\{A\}\} \geq 3$.

Rank of a matrix

Example

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{pmatrix} \Rightarrow \begin{array}{l} \mathrm{Nul}\{A\} = \{(0, x_2, 0) \quad \forall x_2 \in \mathbb{R}\} \\ \mathrm{Row}\{A\} = \{(x_1, 0, x_3) \quad \forall x_1, x_3 \in \mathbb{R}\} \\ \mathrm{Col}\{A\} = \{(x_2, x_2, x_3) \quad \forall x_2, x_3 \in \mathbb{R}\} \\ \mathrm{Nul}\{A^T\} = \{(x_1, -x_1, 0) \quad \forall x_1 \in \mathbb{R}\} \end{array}$$

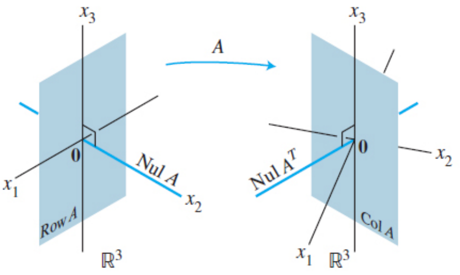


FIGURE 1 Subspaces determined by a matrix A.

Theorem 10.4 (The invertible matrix theorem (continued))

The following statements are equivalent to those in Theorems 5.1 and 11.5 of Chapter 3 (the invertible matrix theorem). Let $A \in \mathcal{M}_{n\times n}$

xix. *The columns of A form a basis of $\mathbb{R}^n$.*

xx. $\mathrm{Col}\{A\} = \mathbb{R}^n$.

xxi. $\dim\{\mathrm{Col}\{A\}\} = n$

xxii. $\mathrm{Rank}\{A\} = n$

xxiii. $\mathrm{Nul}\{A\} = \{\mathbf{0}\}$.

xxiv. $\dim\{\mathrm{Nul}\{A\}\} = 0$.

Proof vii $\Leftrightarrow$ xx
vii$\equiv$The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in \mathbb{R}^n$.
But $\mathrm{Col}\{A\}$ is the set of all $\mathbf{b}$'s for which $A\mathbf{x} = \mathbf{b}$ has a solution. Therefore, vii $\Rightarrow$ xx.
Proof xx $\Leftrightarrow$ xxi $\Leftrightarrow$ xxii
Because of the definition of rank.

Rank of a matrix

Proof v,viii $\Leftrightarrow$ xix

v$\equiv$The columns of A are linearly independent.

viii$\equiv$The columns of A span $\mathbb{R}^n$.

But both together are the definition of a basis for $\mathbb{R}^n$.

Proof xxi $\Leftrightarrow$ xxiv

Knowing xxi and thanks to the rank theorem 10.3, we can infer that $\dim\{\mathrm{Nul}\{A\}\} = n - n = 0$

Proof xxiv $\Leftrightarrow$ xxiii

The only subset with null dimension is $\{\mathbf{0}\}$.

Exercises

Exercises

From Lay (3rd ed.), Chapter 4, Section 6:

- 4.6.1
- 4.6.13
- 4.6.15
- 4.6.19
- 4.6.26
- 4.6.28
- 4.6.29
- 4.6.33
- 4.6.35

Outline

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   - Rank of a matrix (d)
   - Change of basis (d)

Change of basis

Example

Let us assume we have a vector $\mathbf{x}$ that has two different coordinates in two different coordinate systems B and C.

$$[\mathbf{x}]_B = (3, 1) \text{ and } [\mathbf{x}]_C = (6, 4)$$

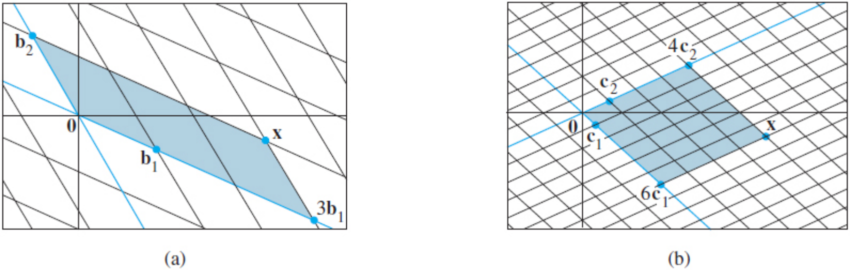


FIGURE 1 Two coordinate systems for the same vector space.

Change of basis

Example (continued)

Presume that for our example

$$\begin{aligned}\mathbf{b}_1 &= 4\mathbf{c}_1 + \mathbf{c}_2\\ \mathbf{b}_2 &= -6\mathbf{c}_1 + \mathbf{c}_2\end{aligned}$$

We can calculate the coordinates of the basis vectors B in the C coordinate system as

$$\begin{aligned}[\mathbf{b}_1]_C &= (4, 1)\\ [\mathbf{b}_2]_C &= (-6, 1)\end{aligned}$$

The coordinates of $\mathbf{x}$ in the basis B tell us

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$$

If we now apply the coordinate mapping transformation we have

$$[\mathbf{x}]_C = 3[\mathbf{b}_1]_C + [\mathbf{b}_2]_C = 3\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Change of basis

Example (continued)

Note that the columns of the matrix

$$\begin{pmatrix} 4 & -6 \\ 1 & 1 \end{pmatrix}$$

are the coordinates of each one of the elements of the basis B expressed in the coordinate system C, and that the overall change of coordinates has the form

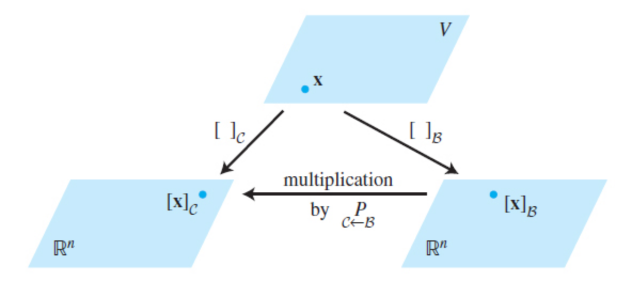
$$[\mathbf{x}]_C = \begin{pmatrix} 4 & -6 \\ 1 & 1 \end{pmatrix} [\mathbf{x}]_B$$

Change of basis

Theorem 11.1 (Change of basis)

Let $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n\}$ be two bases of the vector space V. We can transform coordinates from one coordinate system to the other by multiplying by a single, invertible $n \times n$ matrix, called $P_{C \leftarrow B}$ whose columns are the coordinates of the vectors of B in the basis C.

$$[\mathbf{x}]_C = P_{C \leftarrow B}[\mathbf{x}]_B$$



Change of basis

Corollary

To convert from C coordinates back to B coordinates we simply have to invert the transformation.

$$P_{B\leftarrow C} = P^{-1}_{C\leftarrow B}$$

Corollary

Consider the standard base in V given by $E = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$. The matrix to convert the coordinates from B to E is simply

$$P_{E\leftarrow B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{pmatrix}$$

Consequently, we have that for two different bases

$$\mathbf{x} = P_{E\leftarrow B}[\mathbf{x}]_B = P_{E\leftarrow C}[\mathbf{x}]_C$$

Finally,

$$[\mathbf{x}]_C = P^{-1}_{E\leftarrow C} P_{E\leftarrow B}[\mathbf{x}]_B$$

Change of basis

Numerical trick

Given the two basis B and C we can easily find the coordinates of B in the basis C in the following way. Let us define two matrices $\mathcal{B}$ and $\mathcal{C}$ whose columns are the elements of the basis. Then

$$(\mathcal{C}|\mathcal{B}) \sim (I_n|P_{C\leftarrow B})$$

Example

Let's say we are given $\mathbf{b}_1 = (-9, 1)$, $\mathbf{b}_2 = (-5, -1)$, $\mathbf{c}_1 = (1, -4)$, $\mathbf{c}_2 = (3, -5)$.

$$\left(\begin{array}{cc|cc} 1 & 3 & -9 & 5 \\ -4 & -5 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & 3 \end{array}\right)$$

Then, $P_{C\leftarrow B} = \begin{pmatrix} 6 & 4 \\ -5 & 3 \end{pmatrix}$.

Exercises

Exercises

From Lay (3rd ed.), Chapter 4, Section 7:

- 4.7.1
- 4.7.9

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5. Vector spaces
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