Chapter 6. Eigenvalues and eigenvectors

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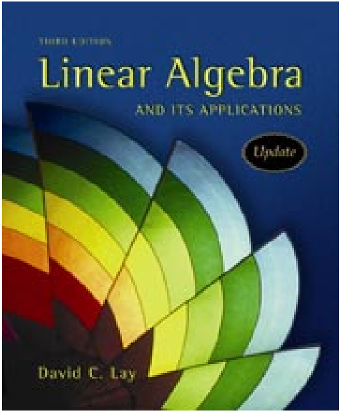
Outline

6. Eigenvalues and eigenvectors
   - Definition (a)
   - Characteristic equation (a)
   - Diagonalization (b)
   - Eigenvectors and linear transformations (b)
   - Complex eigenvalues (c)

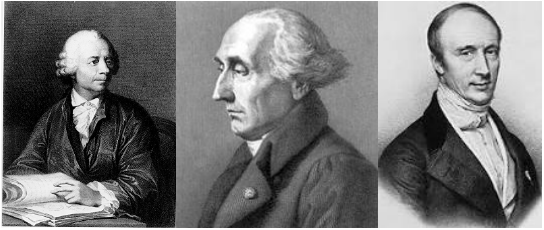
References



D. Lay. Linear algebra and its applications (3rd ed). Pearson (2006). Chapter 5.

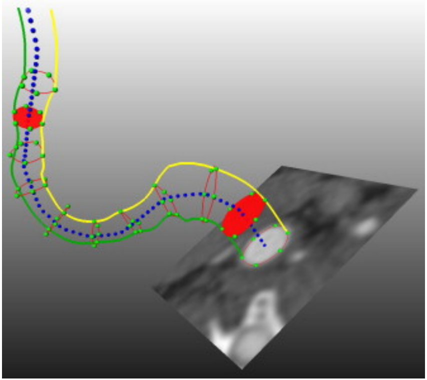
A little bit of history

Eigenvalues (or "proper values") were first used in the study of the motion of rigid bodies through the inertia matrix by Leonhard Euler and Joseph-Louis Lagrange in the mid of XVIIIth century. Then Augustin-Louis Cauchy used it to analyze quadratic surfaces and conic sections in the early XIXth. Since then, they have found applications in most scientific problems.



Applications

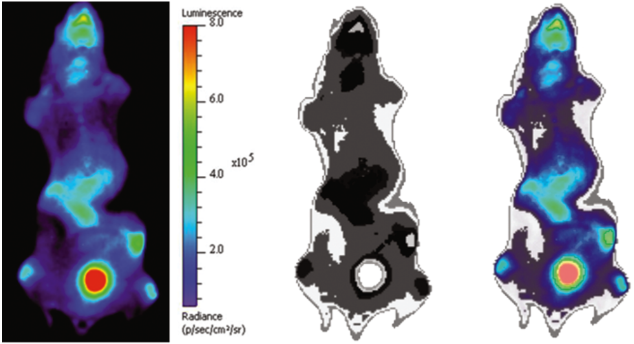
In this example eigenvalues are used to estimate the size of carotid in a volumetric image.



Hameeteman, K.; Zuluaga, M. A.; et al. *Evaluation framework for carotid bifurcation lumen segmentation and stenosis grading*. Med Image Anal, **2011**, 15, 477-488.

Applications

In this example eigenvalues were used as a part of another technique (Principal Component Analysis) to automatically analyze luminiscent images.



(a) (b) (c)

Spinelli, A.E., Boschi, F. *Unsupervised analysis of small animal dynamic Cerenkov luminescence imaging*. J Biomed Opt, **2011**, 16, 120506

Outline

6. Eigenvalues and eigenvectors
   - Definition (a)
   - Characteristic equation (a)
   - Diagonalization (b)
   - Eigenvectors and linear transformations (b)
   - Complex eigenvalues (c)

Eigenvalues and eigenvectors

Example

Consider the linear transformation $T(\mathbf{x}) = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \mathbf{x}$ on the vectors $\mathbf{u} = (-1, 1)$ and $\mathbf{v} = (2, 1)$

$$
\begin{aligned}
T(\mathbf{u}) &= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} \\
T(\mathbf{v}) &= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}
\end{aligned}
$$

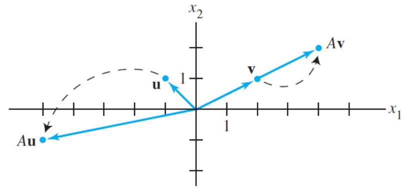


FIGURE 1 Effects of multiplication by A.

u is changing its direction and module, but **v** is only changing its module.

Eigenvalues and eigenvectors

Definition 1.1 (Eigenvalue and eigenvector)

*Given the matrix $A \in \mathcal{M}_{n\times n}$, λ is an **eigenvalue** of A if there exists a non-trivial solution $\mathbf{v} \in \mathbb{R}^n$ of the equation*

$$A\mathbf{v} = \lambda\mathbf{v}$$

*The solution $\mathbf{v}$ is the **eigenvector** associated to the eigenvalue λ.*

Example (continued)

In the previous example, $\mathbf{v}$ was an eigenvector with eigenvalue 2 (because $(2,1) \to (4,2)$, while $\mathbf{u}$ was not an eigenvector.

Eigenvalues and eigenvectors

Example

Show that $\lambda = 7$ is an eigenvalue of $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$.

Solution

We must find a solution of the equation $A\mathbf{v} = \lambda\mathbf{v}$, or what is the same

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \Rightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$$
$$\left(\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any vector of the form $\mathbf{v} = (v_1, v_1)$ satisfies the previous equation

Theorem 1.1

In general, eigenvectors are solution of the equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

*That is, all eigenvectors belong to $\mathrm{Nul}\{A - \lambda I\}$. This is called the **eigenspace**.*

Example (continued)

We see that we have a whole set of vectors associated to $\lambda = 7$, this is a subspace of the eigenspace:

$$\mathrm{Eigenspace}\{7\} = \{(v_1, v_1)\ \forall v_1 \in \mathbb{R}\}$$

It is a line passing through the origin with the direction $(1, 1)$.
The other eigenvalue of matrix A is $\lambda = -4$

$$\mathrm{Eigenspace}\{-4\} = \{(v_1, -\tfrac{5}{6} v_1)\ \forall v_1 \in \mathbb{R}\}$$

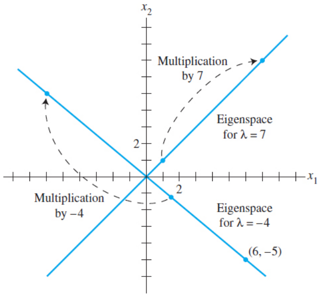


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

Eigenvalues and eigenvectors

Example

Knowing that $\lambda = 2$ is an eigenvalue of $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$, find a basis of its eigenspace.

Solution

$$A - 2I = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

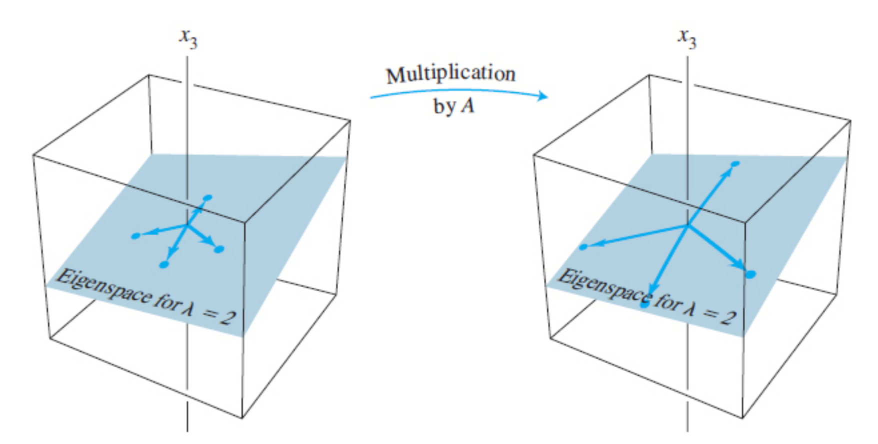
So any vector fulfilling this equation must satisfy

$$x_1 = \frac{1}{2}x_2 - 3x_3 \Rightarrow \mathrm{Eigenspace}\{2\} \ni \mathbf{x} = x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Finally the basis is formed by the vectors $(\frac{1}{2}, 1, 0)$ and $(-3, 0, 1)$.

Example (continued)

Within the eigenspace, A acts as a dilation.



Eigenvalues and eigenvectors

Theorem 1.2

The eigenvalues of a triangular matrix A are the elements of the main diagonal (a_{ii}, $i = 1, 2, ..., n$).

Proof

Consider the matrix $A - \lambda I$

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} & ... & a_{1n} \\ 0 & a_{22} - \lambda & a_{23} & ... & a_{2n} \\ 0 & 0 & a_{33} - \lambda & ... & a_{3n} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & a_{nn} - \lambda \end{pmatrix}$$

The equation system $A - \lambda I = \mathbf{0}$ has a non-trivial solution if at least 1 of the entries in the diagonal is 0. Therefore, it must be $\lambda = a_{ii}$ for some i. Varying i from 1 to n we obtain that all the elements in the main diagonal are the n eigenvalues of the matrix A.

Eigenvalues and eigenvectors

Example

The eigenvalues of $A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$ are $\lambda = 3, 0, 2$.

Theorem 1.3

Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_r$ be r eigenvectors associated to r different eigenvalues. Then, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ is linearly independent.

Proof

Let us assume that S is linearly dependent. Without loss of generality, we may assume that the first p $(p < r)$ are linearly independent, and that the $p+1$-th vector is dependent on the precedent vectors. Then, there must exist $c_1, c_2, ..., c_p$ not all of them zero such that

$$\mathbf{v}_{p+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p \tag{1}$$

Eigenvalues and eigenvectors

If we multiply both sides of the equation by A, then we have

$$\begin{aligned} A\mathbf{v}_{p+1} &= c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + ... + c_p A\mathbf{v}_p \\ \lambda_{p+1}\mathbf{v}_{p+1} &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + ... + c_p\lambda_p\mathbf{v}_p \end{aligned} \tag{2}$$

If we multiply Eq. (1) by λ_{p+1} and subtract from Eq. (2), we have

$$\mathbf{0} = c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{p+1})\mathbf{v}_2 + ... + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p$$

Since the first p vectors are linearly independent it must be for $i = 1, 2, ..., p$

$$c_i(\lambda_i - \lambda_{p+1}) = 0$$

Because all eigenvalues are different, then it must be $c_i = 0$ ($i = 1, 2, ..., p$). But this is a contradiction with the initial hypothesis that not all of them were 0. Consequently, the set S must be linearly independent. (q.e.d.)

Eigenvalues and eigenvectors

Difference equations

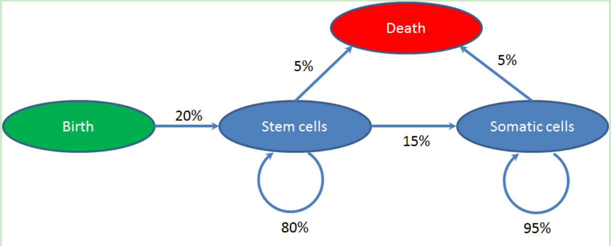
Let us assume we have two populations of cells: stem cells and mature cells. Everyday we measure the number of them and we observe that:

Stem cells:

- 80% of them have remained as stem cells
- 15% of them have differentiated into somatic cells
- 5% of them have died
- There are 20% new stem cells.

Somatic cells:

- 95% of them have remained as somatic cells
- 5% of them have died



Eigenvalues and eigenvectors

Difference equations (continued)

If we call $x_{stem}^{(k)}$ the number of stem cells on the day k, and $x_{somatic}^{(k)}$ the number of somatic cells the same day, then the following equation reflects the dynamics of the system:

$$\begin{pmatrix} x_{stem}^{(k+1)} \\ x_{somatic}^{(k+1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.15 & 0.95 \end{pmatrix} \begin{pmatrix} x_{stem}^{(k)} \\ x_{somatic}^{(k)} \end{pmatrix}$$

Let us assume that the day 0, there are $10,000$ stem cells, and 0 somatic cells. Then, the evolution over time is

$$\begin{pmatrix} x_{stem}^{(1)} \\ x_{somatic}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.15 & 0.95 \end{pmatrix} \begin{pmatrix} x_{stem}^{(0)} \\ x_{somatic}^{(0)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.15 & 0.95 \end{pmatrix} \begin{pmatrix} 10,000 \\ 0 \end{pmatrix} = \begin{pmatrix} 10,000 \\ 1,500 \end{pmatrix}$$

$$\begin{pmatrix} x_{stem}^{(2)} \\ x_{somatic}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.15 & 0.95 \end{pmatrix} \begin{pmatrix} x_{stem}^{(1)} \\ x_{somatic}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.15 & 0.95 \end{pmatrix} \begin{pmatrix} 10,000 \\ 1,500 \end{pmatrix} = \begin{pmatrix} 10,000 \\ 2,925 \end{pmatrix}$$

...

Eigenvalues and eigenvectors

Difference equations

The previous model is of the form

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}$$

The simplest way of constructing a solution of the previous equation is by taking an eigenvector $\mathbf{x}_1$ and its corresponding eigenvalue, λ:

$$\mathbf{x}^{(k)} = \lambda_1^k \mathbf{x}_1$$

This is actually a solution because:

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)} = A(\lambda_1^k \mathbf{x}_1) = \lambda_1^k (A\mathbf{x}_1) = \lambda_1^k (\lambda_1 \mathbf{x}_1) = \lambda_1^{k+1} \mathbf{x}_1$$

It turns out that any linear combination of eigenvectors is also a solution

$$\mathbf{x}^{(k)} = c_1 \lambda_1^k \mathbf{x}_1 + c_2 \lambda_2^k \mathbf{x}_2 + \ldots + c_n \lambda_n^k \mathbf{x}_n$$

Exercises

Exercises

From Lay (3rd ed.), Chapter 5, Section 1:

- 5.1.1
- 5.1.3
- 5.1.9
- 5.1.17
- 5.1.19
- 5.1.23
- 5.1.25
- 5.1.26
- 5.1.27

Outline

6 Eigenvalues and eigenvectors
- Definition (a)
- Characteristic equation (a)
- Diagonalization (b)
- Eigenvectors and linear transformations (b)
- Complex eigenvalues (c)

Characteristic equation

Example

Find the eigenvalues of $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$

Solution

We need to find scalar values λ such that the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has non-trivial solutions. By the Invertible Matrix theorem we know that this problem is equivalent to that of finding λ values such that

$$|A - \lambda I| = 0$$

In this case

$$\left| \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 9 = \lambda^2 + 4\lambda - 21 = 0$$

Characteristic equation

Example (continued)

$$\lambda^2 + 4\lambda - 21 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-21)}}{2 \cdot 1} = \begin{cases} -7 \\ 3 \end{cases}$$

Theorem 2.1 (The invertible matrix theorem (continued))

This theorem adds to the Theorems 5.1, 11.5 of Chapter 3 and 10.4 of Chapter 5.

xxv. $|A| \neq 0$.

xxvi. *0 is not an eigenvalue of A.*

Definition 2.1 (Characteristic equation)

A scalar λ is an eigenvalue of a matrix $A \in \mathcal{M}_{n\times n}$ iff it is solution of the ***characteristic equation***

$$|A - \lambda I| = 0$$

The determinant of $A - \lambda I$ is called the ***characteristic polynomial****.*

Characteristic equation

Example

Let us calculate the eigenvalues of $A = \begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (5-\lambda)^2, (3-\lambda)(1-\lambda) = 0$$

whose solutions are $\lambda = 5$ (with multiplicity 2), $\lambda = 3$, and $\lambda = 1$.

Example

Let us find the eigenvalues of a matrix whose characteristic polynomial is

$$|A - \lambda I| = \lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4(\lambda^2 - 4\lambda - 12) = \lambda^4(\lambda - 6)(\lambda + 2) = 0$$

whose solutions are $\lambda = 0$ (with multiplicity 4), $\lambda = 6$, and $\lambda = -2$.

Definition 2.2 (Similarity between matrices)

Given two matrices $A, B \in \mathcal{M}_{n\times n}$, A is ***similar*** *to B iff there exists an invertible matrix $P \in \mathcal{M}_{n\times n}$ such that*

$$B = P^{-1}AP$$

Watch out that *similarity* is not the same as *row equivalence* (A and B are row equivalent if there exists a E such that $B = EA$ being E invertible and the product of row operation matrices).

Characteristic equation

Theorem 2.2

If A is similar to B, then B is similar to A.

Proof

It suffices to take the definition of A similar to B and solve for B. If we multiply by P on the right

$$B = P^{-1}AP \Rightarrow PB = AP$$

Now, we multiply by P on the left (P^{-1} exists because P is invertible)

$$PB = AP \Rightarrow PBP^{-1} = A$$

and this is the definition of B being similar to A.

Characteristic equation

Theorem 2.3

If A and B are similar matrices, then they have the same characteristic polynomial.
Proof
If A is similar to B, then there exists an invertible matrix P such that

$$B = P^{-1}AP$$

If we subtract on both sides λI we have

$$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(A - \lambda I)P$$

Now taking the determinant of both sides

$$|B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P| = |P|^{-1}|A - \lambda I||P| = |A - \lambda I|$$

Characteristic equation

Theorem 2.4

If A and B are similar matrices, then they have the same characteristic polynomial.
Proof
If A is similar to B, then there exists an invertible matrix P such that

$$B = P^{-1}AP$$

If we subtract on both sides λI we have

$$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(A - \lambda I)P$$

Now taking the determinant of both sides

$$|B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P| = |P|^{-1}|A - \lambda I||P| = |A - \lambda I|$$

Exercises

Exercises

From Lay (3rd ed.), Chapter 5, Section 2:

- 5.2.1
- 5.2.9
- 5.2.18
- 5.2.19
- 5.2.20
- 5.2.23
- 5.2.24
- 5.2.28 (computer)

Outline

6. Eigenvalues and eigenvectors
   - Definition (a)
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Diagonalization

Definition 3.1 (Diagonalization)

*$A \in \mathcal{M}_{n\times n}$ is **diagonalizable** if there exists $P, D \in \mathcal{M}_{n\times n}$ (with P invertible and D diagonal) such that*

$$A = PDP^{-1}$$

Diagonalization simplifies the calculation of powers of A (A^k), is used to decouple dynamic systems, and in multivariate statistics to produce uncorrelated random variables.

Example

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \quad D^2 = \begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix} \; D^3 = \begin{pmatrix} 5^3 & 0 \\ 0 & 3^3 \end{pmatrix}$$

Diagonalization

Example

Let us assume that $A = PDP^{-1}$. Let us calculate calculate now the different powers of A

$A^2 = A \cdot A = (PDP^{-1})(PDP^{-1}) = (PD)(P^{-1}P)(DP^{-1}) = PDDP^{-1} = PD^2P^{-1}$
$A^3 = A^2 \cdot A = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$
...
$A^k = PD^kP^{-1}$

Let us particularize this result for $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$ that can be factorized with $P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ as $A = PDP^{-1}$.

$$A^k = PD^kP^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} =$$
$$\begin{pmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{pmatrix}$$

Diagonalization

Theorem 3.1 (Diagonalization theorem)

*$A \in \mathcal{M}_{n\times n}$ is **diagonalizable** iff A has n linearly independent eigenvectors. In this case, we may construct P by stacking the n eigenvectors, and D as a diagonal matrix with the corresponding eigenvalues.*

Proof

Consider the columns of $P = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \end{pmatrix}$ *and* $D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{pmatrix}$

Let us assume that $A = PDP^{-1}$ and we multiply by P on the right

$$\begin{aligned}
AP &= PD \\
A\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \end{pmatrix} &= \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \end{pmatrix}\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{pmatrix} \\
\begin{pmatrix} A\mathbf{p}_1 & A\mathbf{p}_2 & \dots & A\mathbf{p}_n \end{pmatrix} &= \begin{pmatrix} d_1\mathbf{p}_1 & d_2\mathbf{p}_2 & \dots & d_n\mathbf{p}_n \end{pmatrix}
\end{aligned}$$

Diagonalization

This implies that

$$\begin{aligned} A\mathbf{p}_1 &= d_1\mathbf{p}_1 \\ A\mathbf{p}_2 &= d_2\mathbf{p}_2 \\ &\ldots \\ A\mathbf{p}_n &= d_n\mathbf{p}_n \end{aligned}$$

But this is the definition of eigenvector, so the columns of P ($\mathbf{p}_i$) must be eigenvectors of A and d_i its corresponding eigenvalue. Since P is invertible, its columns must be linearly independent.

Diagonalization

Example

Diagonalize $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$.

<u>*Step 1:*</u> Find the eigenvalues of A

$$|A - \lambda I| = 0 \Rightarrow -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2 = 0$$

whose solutions are $\lambda = 1$ and $\lambda = -2$ (double).

<u>*Step 2:*</u> Find a linearly independent set of eigenvectors

$\underline{\lambda = 1}$

$$A - \lambda I = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Example (continued)

Step 2: Find a linearly independent set of eigenvectors

$\underline{\lambda = 1}$

$$A - \lambda I \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 = -x_2 \\ x_3 = -x_2 \end{array} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = -2}$

$$A - \lambda I = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} - \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 = -x_2 - x_3 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Diagonalization

Example (continued)

Step 3: Construct P and D

$$P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Step 4: Check everything is correct

P is invertible $|P| \neq 0$

$$|P| = 1$$

$A = PDP^{-1} \Rightarrow AP = PD$

$$AP = \begin{pmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \quad PD = \begin{pmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

Example (continued)

Step 4: Check everything is correct

P is invertible $|P| \neq 0$

MATLAB:

```
P=[1 -1 -1; -1 1 0; 1 0 1];
det(P)
```

$A = PDP^{-1} \Rightarrow AP = PD$

MATLAB:

```
A=[1 3 3; -3 -5 3; 3 3 1];
P=[1 -1 -1; -1 1 0; 1 0 1];
D=[1 0 0; 0 -2 0; 0 0 -2];
A*P
P*D
```

Diagonalization

Example

Diagonalize $A = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix}$.

Step 1: Find the eigenvalues of A

$$|A - \lambda I| = 0 \Rightarrow -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2 = 0$$

whose solutions are $\lambda = 1$ and $\lambda = -2$ (double). (Same eigenvalues as in the previous example)

Step 2: Find a linearly independent set of eigenvectors

$\underline{\lambda = 1}$

$$A - \lambda I = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Diagonalization

Example (continued)

Step 2: Find a linearly independent set of eigenvectors

$\underline{\lambda = 1}$

$$A - \lambda I \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{c} x_1 = x_3 \\ x_2 = -x_3 \end{array} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(The same eigenspace as in the previous example).

$\underline{\lambda = -2}$

$$A - \lambda I = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} - \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \Rightarrow \begin{array}{c} x_1 = -x_2 - \frac{3}{4}x_3 \\ \frac{1}{4}x_3 = 0 \end{array} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

(A cannot be diagonalized because there are not 3 linearly independent vectors)

Diagonalization

Theorem 3.2

If a $n \times n$ matrix has n different eigenvalues, then it is diagonalizable.

Proof

Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ be the n eigenvectors corresponding to the n different eigenvalues. The set

$$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$$

is linearly independent by Theorem 1.3 and A is diagonalizable by Theorem 3.1.

Example

Is $A = \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}$ diagonalizable?

Solution

A is a triangular matrix and its eigenvalues are 5, 0 and -2, all of them distinct, and by the previous theorem A is diagonalizable.

Diagonalization

Theorem 3.3

Let $A \in \mathcal{M}_{n\times n}$ with $p \leq n$ different eigenvalues. Let d_k be the dimension associated to the eigenvalue λ_k. Then,

1. *d_k is smaller or equal the multiplicity of λ_k.*
2. *A is diagonalizable iff d_k is equal to the multiplicity of λ_k. In this case,*
$$\sum_{k=1}^{p} d_k = n$$
3. *If A is diagonalizable and B_k are the bases of each one of the eigenspaces, then $\{B_1, B_2, ..., B_p\}$ is a basis of $\mathbb{R}^n$.*

Diagonalization

Example

Let $A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & 3 \end{pmatrix}$. Let's factorize it as $A = PDP^{-1}$. The eigenvalues and associated eigenvectors are

$$\begin{array}{lll} \lambda_1 = 5 \quad \leftrightarrow \quad \mathbf{v}_1 = \begin{pmatrix} -8 \\ 4 \\ 1 \\ 0 \end{pmatrix} & \mathbf{v}_2 = \begin{pmatrix} -16 \\ 4 \\ 0 \\ 1 \end{pmatrix} & \\ & & \Rightarrow \\ \lambda_2 = -3 \quad \leftrightarrow \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \end{array} \quad \begin{array}{l} P = \begin{pmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \\ D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \end{array}$$

Exercises

From Lay (3rd ed.), Chapter 5, Section 3:

- 5.3.1
- 5.3.23
- 5.3.27
- 5.3.28
- 5.3.29
- 5.3.31
- 5.3.32
- 5.3.33 (computer)

Outline

6 Eigenvalues and eigenvectors

- Definition (a)
- Characteristic equation (a)
- Diagonalization (b)
- Eigenvectors and linear transformations (b)
- Complex eigenvalues (c)

The matrix of a linear transformation

The objective of this section is to show that if A is diagonalizable ($A = PDP^{-1}$), then the transformation $T_A(\mathbf{x}) = A\mathbf{x}$ is essentially the same as $T_D(\mathbf{u}) = D\mathbf{u}$.

Definition 4.1 (The matrix of a linear transformation)

Consider a linear transformation between two vectors spaces $T : U \to V$. Let B be a basis of V, and C be a basis of W. Let $\mathbf{x} \in V$ and consider its coordinates $[\mathbf{x}]_B = (r_1, r_2, ..., r_n)$.

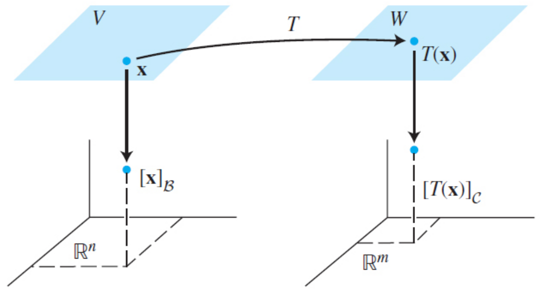


FIGURE 1 A linear transformation from V to W.

The matrix of a linear transformations

Let's analyze $\mathbf{x}$ and $T(\mathbf{x})$

$$\begin{array}{rcl} \mathbf{x} & = & r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + ... + r_n\mathbf{b}_n \Rightarrow \\ T(\mathbf{x}) & = & T(r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + ... + r_n\mathbf{b}_n) \quad \text{[T is linear]} \\ & = & r_1 T(\mathbf{b}_1) + r_2 T(\mathbf{b}_2) + ... + r_n T(\mathbf{b}_n) \end{array}$$

Now, let us consider the coordinates in C of the transformed vector

$$[T(\mathbf{x})]_C = r_1[T(\mathbf{b}_1)]_C + r_2[T(\mathbf{b}_2)]_C + ... + r_n[T(\mathbf{b}_n)]_C$$

We can write this equation in matrix form as

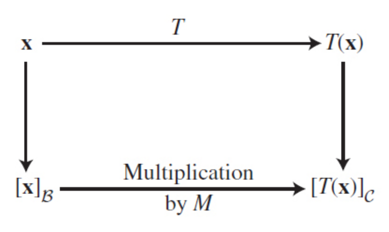
$$[T(\mathbf{x})]_C = M[\mathbf{x}]_B$$

where $M \in \mathcal{M}_{m\times n}$ is a matrix formed by the transformations of each one of the basis vectors in B

$$M = \begin{pmatrix} [T(\mathbf{b}_1)]_C & [T(\mathbf{b}_2)]_C & ... & [T(\mathbf{b}_n)]_C \end{pmatrix}$$

Matrix M is called the **matrix of T relative to the bases B and C**.

The matrix of a linear transformations



Example

Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ and

$$\begin{array}{rcl} T(\mathbf{b}_1) & = & 3\mathbf{c}_1 - 2\mathbf{c}_2 + 5\mathbf{c}_3 \\ T(\mathbf{b}_2) & = & 4\mathbf{c}_1 + 7\mathbf{c}_2 - \mathbf{c}_3 \end{array} \Rightarrow M = \begin{pmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{pmatrix}$$

Transformations from V into V

Definition 4.2 (B-matrix for T)

*If T is a transformation from V into V and B is a basis of V, then the matrix M is called the B-**matrix of** T.*

Example

Consider in the vector space of polynomials of degree 2 ($\mathbb{P}_2$), the derivative transformation

$$\begin{aligned} T: \quad & \mathbb{P}_2 \to \mathbb{P}_2 \\ & T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t \end{aligned}$$

Consider the standard basis of $\mathbb{P}_2$, $B = \{1, t, t^2\}$.

Example (continued)

Which is the B-transformation matrix?

Solution

$$T(1) = 0 \quad \rightarrow \quad [T(1)]_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(t) = 1 \quad \rightarrow \quad [T(t)]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T(t^2) = 2t \quad \rightarrow \quad [T(t^2)]_B = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Transformations from V into V

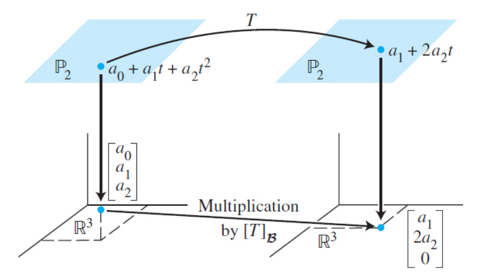
Example (continued)

Verify that $[T(\mathbf{x})]_B = M[\mathbf{x}]_B$

<u>*Solution*</u>

Given any polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ its coordinates are $[p(t)]_B = (a_0, a_1, a_2)$. The derivative of $p(t)$ is $T(p(t)) = a_1 + 2a_2 t$, then

$$[T(p(t))]_B = \begin{pmatrix} a_1 \\ 2a_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$



Theorem 4.1 (Diagonal matrix representation)

Suppose matrix A is diagonalizable ($A = PDP^{-1}$). If B is the basis of $\mathbb{R}^n$ formed by the columns of P, then D is the B-matrix of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

Proof

Let $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n$ be the columns of P so that $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ is a basis. We know that for any basis in $\mathbb{R}^n$

$$\mathbf{x} = P[\mathbf{x}]_B \Rightarrow [\mathbf{x}]_B = P^{-1}\mathbf{x}$$

Let $[T]_B$ be the transformation matrix in the basis B. We know that by definition

$$\begin{array}{rcll}
[T]_B & = & \begin{pmatrix}[T(\mathbf{b}_1)]_B & [T(\mathbf{b}_2)]_B & \ldots & [T(\mathbf{b}_n)]_B\end{pmatrix} & (T(\mathbf{x}) = A\mathbf{x}) \\
 & = & \begin{pmatrix}[A\mathbf{b}_1]_B & [A\mathbf{b}_2]_B & \ldots & [A\mathbf{b}_n]_B\end{pmatrix} & \textit{(change of coordinates)} \\
 & = & \begin{pmatrix}P^{-1}A\mathbf{b}_1 & P^{-1}A\mathbf{b}_2 & \ldots & P^{-1}A\mathbf{b}_n\end{pmatrix} & \textit{(matrix multiplication)} \\
 & = & P^{-1}A\begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n\end{pmatrix} & \textit{(definition of } P\textit{)} \\
 & = & P^{-1}AP = D &
\end{array}$$

Transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$

Example

Let $T(\mathbf{x}) = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \mathbf{x}$. Find a basis B in which the B-matrix of T is diagonal.

Solution

We diagonalize A as $A = PDP^{-1}$, with $P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$. We may change vectors $\mathbf{x}$ to the basis $B = \{(1,-1),(1,-2)\}$ by applying

$$\mathbf{u} = P^{-1}\mathbf{x}$$

Then, in this new basis, T can be applied as

$$T(\mathbf{u}) = D\mathbf{u} = DP^{-1}\mathbf{x}$$

If we now, come back to the original basis

$$T(\mathbf{x}) = PT(\mathbf{u}) = PDP^{-1}\mathbf{x} = A\mathbf{x}$$

Understanding D as the transformation matrix in some basis gives us insight on its effect (in this example, an anisotropic dilation).

Similar matrices

Definition 4.3 (Similar matrices)

A and C are ***similar matrices*** *iff there exists another matrix P such that $A = PCP^{-1}$. Given the transformation $T(\mathbf{x}) = A\mathbf{x}$, C is the B-matrix of the transformation T, when B is the basis defined by the columns of the matrix P.*

Conversely, if B is any basis and P is the matrix formed by the vectors in the basis B, then the B-matrix of the transformation T is $P^{-1}AP$.

$$\begin{array}{ccc}
\mathbf{x} & \xrightarrow[\text{by } A]{\text{Multiplication}} & A\mathbf{x} \\
{\scriptstyle \text{Multiplication by } P^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle \text{Multiplication by } P} \\
[\mathbf{x}]_{\mathcal{B}} & \xrightarrow[\text{by } C]{\text{Multiplication}} & [A\mathbf{x}]_{\mathcal{B}}
\end{array}$$

Similar matrices

Example

Let $A = \begin{pmatrix} 4 & -9 \\ 4 & 8 \end{pmatrix}$, $T(\mathbf{x}) = A\mathbf{x}$ and $\mathbf{b}_1 = (3,2)$, $\mathbf{b}_2 = (2,1)$. A is not diagonalizable but the basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ has the property that $[T]_B$ is triangular (it is said to be in Jordan form). According to the previous definition, the B-matrix of the transformation T is

$$[T]_B = P^{-1}AP = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 & -9 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

Numerical note

An easy way to compute $P^{-1}AP$ once we have AP is to find a row equivalent matrix

$$\left(\begin{array}{c|c} P & AP \end{array} \right) \sim \left(\begin{array}{c|c} I & P^{-1}AP \end{array} \right)$$

Exercises

From Lay (3rd ed.), Chapter 5, Section 4:

- 5.4.1
- 5.4.3
- 5.4.5
- 5.4.13
- 5.4.18
- 5.4.22
- 5.4.23
- 5.4.25
- 5.4.27 (computer)

Outline

6 Eigenvalues and eigenvectors

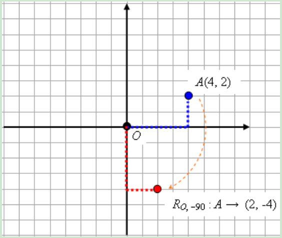
- Definition (a)
- Characteristic equation (a)
- Diagonalization (b)
- Eigenvectors and linear transformations (b)
- Complex eigenvalues (c)

Complex eigenvalues

Complex eigenvalues are always related to a rotation around a certain axis.

Example

Consider the linear transformation $T(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$ is a rotation of 90°.



Obviously, there cannot be any real eigenvector since all the vectors are rotating. All eigenvalues are complex:

$$|A - \lambda I| = 0 = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

Complex eigenvalues

Example (continued)

Let's see what happens if we allow applying the transformation on complex vectors:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix}$$
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Complex eigenvalues

Example

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} \end{pmatrix}$.

Solution

To find the eigenvalues we solve the characteristic equation:

$$0 = |A - \lambda I| = \begin{vmatrix} \frac{1}{2} - \lambda & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} - \lambda \end{vmatrix} = \lambda^2 - \frac{8}{5}\lambda + 1 \Rightarrow \lambda = \frac{4}{5} \pm \frac{3}{5} i$$

MATLAB: `A=[1/2 -3/5; 3/4 11/10]; l=eigs(A)`

Complex eigenvalues

Example (continued)

$\underline{\lambda_1 = \frac{4}{5} - \frac{3}{5}i}$

$$
\begin{aligned}
A - \lambda_1 I &= \begin{pmatrix} \frac{1}{2} - (\frac{4}{5} - \frac{3}{5}i) & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} - (\frac{4}{5} - \frac{3}{5}i) \end{pmatrix} = \begin{pmatrix} -\frac{3}{10} + \frac{3}{5}i & -\frac{3}{5} \\ \frac{3}{4} & \frac{3}{10} + \frac{3}{5}i \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & \frac{2}{5} + \frac{4}{5}i \\ 0 & 0 \end{pmatrix} \Rightarrow x_1 = -(\frac{2}{5} + \frac{4}{5}i)x_2 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} -2 - 4i \\ 5 \end{pmatrix}
\end{aligned}
$$

MATLAB:

```
A_lI=A-l(1)*eye(2);
A_lI(1,:)=A_lI(1,:)/A_lI(1,1)
A_lI(2,:)=A_lI(2,:)-A_lI(1,:)*A_lI(2,1)
```

$\underline{\lambda_2 = \frac{4}{5} + \frac{3}{5}i = \lambda_1^*}$

$$
A - \lambda_2 I \sim \begin{pmatrix} 1 & \frac{2}{5} - \frac{4}{5}i \\ 0 & 0 \end{pmatrix} \Rightarrow x_1 = -(\frac{2}{5} - \frac{4}{5}i)x_2 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} -2 + 4i \\ 5 \end{pmatrix} = \mathbf{v}_1^*
$$

Example (continued)

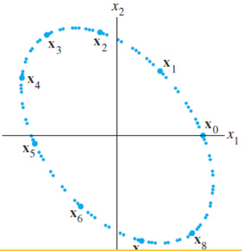
The application of A on $\mathbb{R}^2$ is a rotation. To see this, we may start with $\mathbf{x}_0 = (2, 0)$ and calculate

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix}$$

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} .5 & -.6 \\ .75 & 1.1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} -.4 \\ 2.4 \end{bmatrix}$$

$$\mathbf{x}_3 = A\mathbf{x}_2, \ldots$$

Figure 1 shows $\mathbf{x}_0, \ldots, \mathbf{x}_8$ as larger dots. The smaller dots are the locations of $\mathbf{x}_9, \ldots, \mathbf{x}_{100}$. The sequence lies along an elliptical orbit. ■



Complex eigenvalues

Definition 5.1 (Conjugate of a vector and matrix)

*The **conjugate of a vector** is defined as*

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix} \Rightarrow \mathbf{v}^* = \begin{pmatrix} v_1^* \\ v_2^* \\ \dots \\ v_n^* \end{pmatrix}$$

*In the same way, the **conjugate of a matrix** is defined as*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} a_{11}^* & a_{12}^* & \dots & a_{1n}^* \\ a_{21}^* & a_{22}^* & \dots & a_{2n}^* \\ \dots & \dots & \dots & \dots \\ a_{m1}^* & a_{m2}^* & \dots & a_{mn}^* \end{pmatrix}$$

Theorem 5.1 (Properties)

$$(r\mathbf{v})^* = r^*\mathbf{v}^* \qquad (AB)^* = A^*B^*$$
$$(A\mathbf{v})^* = A^*\mathbf{v}^* \qquad (rA)^* = r^*A^*$$

Eigenanalysis of a real matrix that acts on $\mathbb{C}^n$

Theorem 5.2

Let $A \in \mathcal{M}_{n\times n}$ be a matrix with real coefficients. If λ is an eigenvalue of A, then λ^* is also an eigenvalue. If $\mathbf{v}$ is an eigenvector associated to λ, then $\mathbf{v}^*$ is an eigenvector associated to λ^*.

Proof

If λ is an eigenvalue and $\mathbf{v}$ one of its eigenvectors, then we know that

$$A\mathbf{v} = \lambda\mathbf{v}$$

If we now conjugate both sides

$$(A\mathbf{v})^* = (\lambda\mathbf{v})^* \Rightarrow A\mathbf{v}^* = \lambda^*\mathbf{v}^*$$

(Remind that A has real coefficients and that's why $A^* = A$).
The previous equation means that $\mathbf{v}^*$ is also an eigenvector of A and that λ^* is its eigenvalue.

Eigenanalysis of a real matrix that acts on $\mathbb{C}^n$

Example

Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Its eigenvalues are $\lambda = a \pm bi$ and the corresponding eigenvectors $\mathbf{v} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} a+bi \\ b-ai \end{pmatrix} = (a+bi) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} a-bi \\ b+ai \end{pmatrix} = (a-bi) \begin{pmatrix} 1 \\ i \end{pmatrix}$$

In particular if $a = \cos(\phi)$ and $b = \sin(\phi)$, then we have a rotation matrix whose eigenvalues are

$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \Rightarrow \lambda = \cos(\phi) \pm \sin(\phi) i = e^{\pm i\phi}$$

Eigenanalysis of a real matrix that acts on $\mathbb{C}^n$

Example on Slide 60 (continued)

Let $A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} \end{pmatrix}$. Consider $\lambda_1 = \frac{4}{5} - \frac{3}{5}i$ and its corresponding eigenvector $\mathbf{v}_1 = (-2-4i, 5)$. Now, we construct the matrix

$$P = \begin{pmatrix} \mathrm{Re}\{\mathbf{v}_1\} & \mathrm{Im}\{\mathbf{v}_1\} \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ 5 & 0 \end{pmatrix}$$

and make a change of basis to the basis whose vectors are the columns of P:

$$C = P^{-1}AP = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \cos(36.87^\circ) & -\sin(36.87^\circ) \\ \sin(36.87^\circ) & \cos(36.87^\circ) \end{pmatrix}$$

That is, C is a pure rotation and thanks to the change of basis we obtain an elliptical rotation as shown in Slide 62.

Eigenanalysis of a real matrix that acts on $\mathbb{C}^n$

Theorem 5.3

Let A be a real, 2×2 matrix with complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector in $\mathbb{C}^2$. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} \mathrm{Re}\{\mathbf{v}\} & \mathrm{Im}\{\mathbf{v}\} \end{pmatrix}$$

and

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

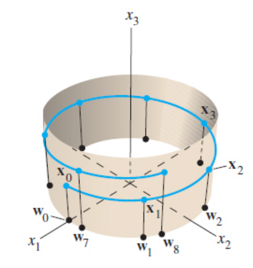
Proof
It makes use of

$$\mathrm{Re}\{A\mathbf{v}\} = A\mathrm{Re}\{\mathbf{v}\}$$
$$\mathrm{Im}\{A\mathbf{v}\} = A\mathrm{Im}\{\mathbf{v}\}$$

Eigenanalysis of a real matrix that acts on $\mathbb{C}^n$

Example: Rotations extend to higher dimensions

Consider $A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1.07 \end{pmatrix}$. This is the rotation previously described in the XY plane plus a scaling in the Z direction. Any point in the XY (for instance, $\mathbf{w}_0 = (2, 0, 0)$) plane rotates within the plane. Any point outside the plane (for instance, $\mathbf{x}_0 = (2, 0, 1)$ rotates in XY and shifts along Z). The following figure shows the successive application of A on $\mathbf{w}_0$ and $\mathbf{x}_0$.



Exercises

From Lay (3rd ed.), Chapter 5, Section 5:

- 5.5.1
- 5.5.7
- 5.5.13
- 5.5.23
- 5.5.24
- 5.5.25
- 5.5.26
- 5.5.27

Outline

6. Eigenvalues and eigenvectors
   - Definition (a)
   - Characteristic equation (a)
   - Diagonalization (b)
   - Eigenvectors and linear transformations (b)
   - Complex eigenvalues (c)