Chapter 9. Linear algebra applications in geometry

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Biomedical Engineering

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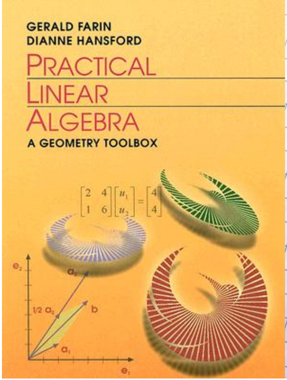
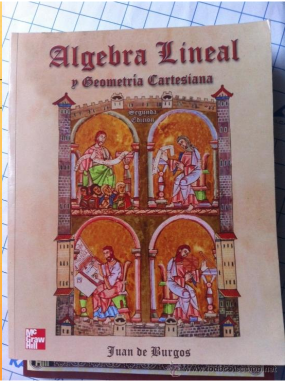
Outline

9. Linear algebra applications in geometry
   - Local and global coordinates
   - Points and vectors
   - Lines in 2D
   - Affine maps in 2D
   - Conic sections in 2D
   - 3D Geometry
   - Quadrics in 3D

References





G. Farin, D. Hansford. Practical Linear Algebra: a geometry toolbox. A.K. Peters (2005).
J. de Burgos. Álgebra lineal y geometría cartesiana. McGraw Hill 2ª Ed. (2000)

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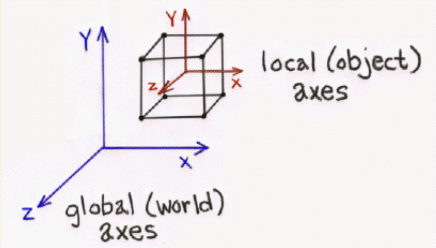
Local and global coordinates

Reference

Farin and Hansford, Chapter 1

Local and global coordinates

In real applications we may need to distinguish between local and global coordinates.



And we need some way of transforming one into the other. This is nothing more than a change of basis.

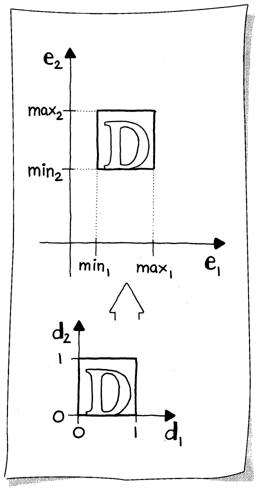
Local and global coordinates

Shift and scale

In Vector Graphics it is common to design objects in a local coordinate system ($\mathbf{d}$) and, then, place, rotate and scale the object in the global coordinate system ($\mathbf{e}$). We need some transformation to go from one space to the other.

For the first component, d_1, we note that we go from a local interval $[0, 1]$ to a global interval $[min_1, max_1]$. We may easily perform the transformation as

$$\frac{d_1 - 0}{1 - 0} = \frac{e_1 - min_1}{max_1 - min_1} \Rightarrow$$
$$e_1 = min_1 + (max_1 - min_1) d_1$$



Local and global coordinates

Shift and scale

The more general transformation maps the local interval $[min_{d1}, max_{d1}]$ to the global interval $[min_{e1}, max_{e1}]$. This is achieved with transformation

$$e_1 = min_{e1} + \frac{max_{e1} - min_{e1}}{max_{d1} - min_{d1}} d_1$$

The same kind of transformation is applied to the second component $(d_2 \to e_2)$. Putting everything in matrix notation we have

$$\mathbf{e} = \begin{pmatrix} min_{e1} \\ min_{e2} \end{pmatrix} + \begin{pmatrix} \frac{max_{e1} - min_{e1}}{max_{d1} - min_{d1}} & 0 \\ 0 & \frac{max_{e2} - min_{e2}}{max_{d2} - min_{d2}} \end{pmatrix} \mathbf{d}$$

This transformation is of the form

$$\mathbf{e} = T(\mathbf{d}) = \mathbf{e}_{min} + A\mathbf{d}$$

that is not a linear transformation because of the shift (e.g., show that $T(\mathbf{d}_1 + \mathbf{d}_2) \neq T(\mathbf{d}_1) + T(\mathbf{d}_2)$).

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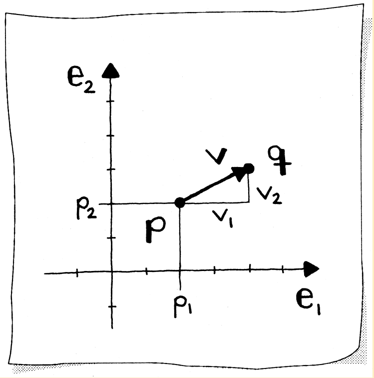
Points and vectors

Reference

Farin and Hansford, Chapter 2

Points and vectors

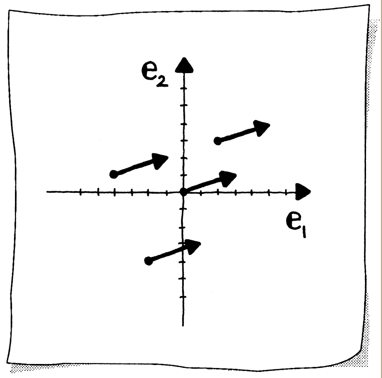
We also need to distinguish between points and vectors. Both are represented as a list of coordinates. Informally, a point indicates a location in space, while a vector indicates a direction (orientation+sense) in space. In this example, we have two points, **p** and **q**, and a vector **v** that goes from **p** to **q**. We may talk about the length of a vector, but not of a point.



Points and vectors

Points and vectors

In this example we have multiple copies of the same vector (since they all have the same direction and magnitude). In Physics, forces are vectors that are applied to objects that are located at points. In this figure we would see the same force applied to different objects.



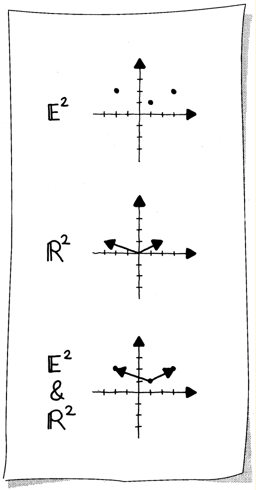
Points and vectors

Points and vectors

More formally, points belong to an Euclidean space while vectors belong to a vector space.

$$\mathbf{p}, \mathbf{q} \in \mathbb{E}^2$$
$$\mathbf{v} \in \mathbb{R}^2$$

Although we may represent both spaces in the same figure and we may define operations using both kinds of spaces. The goal of distinguishing between points and vectors is to distinguish between operations that depend on the coordinate system and operations that do not.



Operations on points and vectors

Coordinate independent operations

$$
\begin{array}{rll}
- : & \mathbb{E}^2 \times \mathbb{E}^2 \to \mathbb{R}^2 & \mathbf{v} = \mathbf{q} - \mathbf{p} \\
+ : & \mathbb{E}^2 \times \mathbb{R}^2 \to \mathbb{E}^2 & \mathbf{p} = \mathbf{q} + \mathbf{v} \\
+ : & \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 & \mathbf{v} = \mathbf{u} + \mathbf{w} \\
\cdot : & \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 & \mathbf{v} = r\mathbf{u}
\end{array}
$$

Coordinate dependent operations

$$
\begin{array}{rll}
+ : & \mathbb{E}^2 \times \mathbb{E}^2 \to \mathbb{E}^2 & \mathbf{t} = \mathbf{p} + \mathbf{q} \\
\cdot : & \mathbb{R} \times \mathbb{E}^2 \to \mathbb{E}^2 & \mathbf{q} = r\mathbf{p}
\end{array}
$$

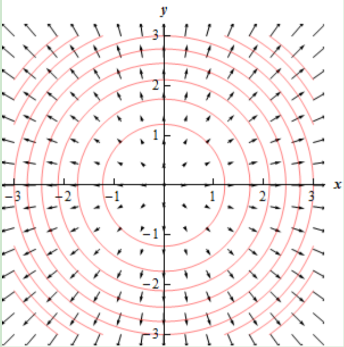
Vector fields

Vector fields

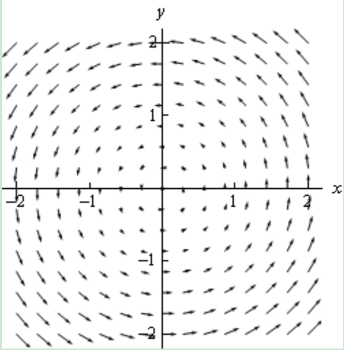
Any function that assigns a vector to a point $f: \mathbb{E}^2 \to \mathbb{R}^2 \quad \mathbf{v} = f(\mathbf{p})$

Example

$f(x, y) = (x, y)$



$f(x, y) = (-y, x)$



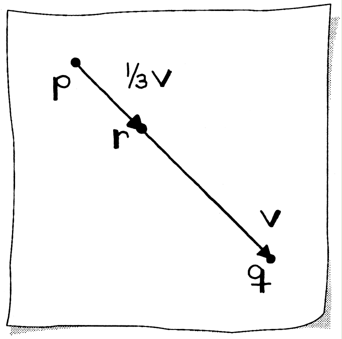
Combinations of points

Barycentric combinations

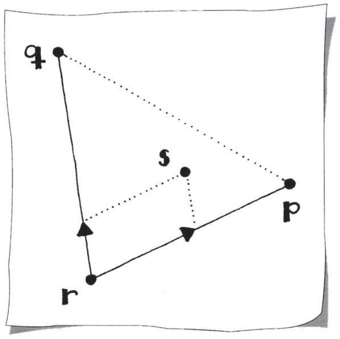
A weighted sum of points where the weights add up to 1 is called a barycentric combination

Example

$$\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{q} = \mathbf{p} + t(\mathbf{q}-\mathbf{p})$$



$$\mathbf{s} = t_1\mathbf{r} + t_2\mathbf{p} + t_3\mathbf{q}$$



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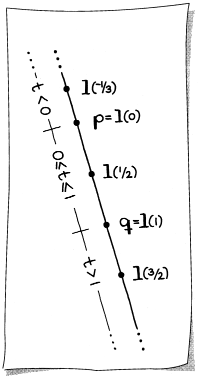
Lines in 2D

Reference

Farin and Hansford, Chapter 3

Parametric equation of a line

- Given two points:
  $\mathbf{l}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) \quad t \in \mathbb{R}$
- Given point and vector:
  $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v} \quad t \in \mathbb{R}$



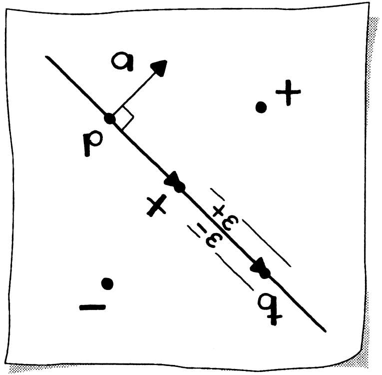
Lines in 2D

Implicit equation of a line

- Given a point and the normal direction:
  $\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0$

  In 2D:

$$(a_1, a_2) \cdot (x_1 - p_1, x_2 - p_2) = 0 \Rightarrow$$
$$ax_1 + bx_2 + c = 0$$



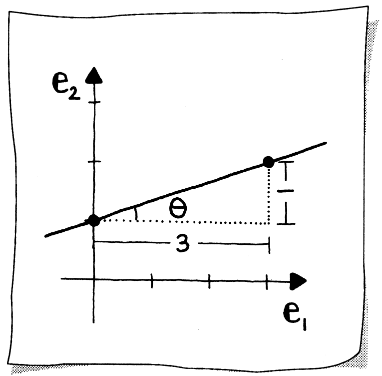
Lines in 2D

Explicit equation of a line

- Given a point and slope:
  In 2D:

$$x_2 = p_2 + m(x_1 - p_1)$$
$$x_2 = mx_1 + b$$
$$x_2 = (\tan \Theta)x_1 + b$$

But it is not a good representation for vertical lines.



Lines in 2D

Distance of a point to a line

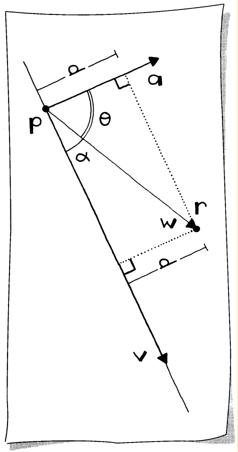
- Implicit line:
  Line: $\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0$
  Point: $\mathbf{r}$

  Let $\mathbf{w} = \mathbf{r} - \mathbf{p}$ and calculate:

  $$\mathbf{a} \cdot \mathbf{w} = \|\mathbf{a}\|\|\mathbf{w}\| \cos(\theta)$$

  Analyzing the figure we note that $\cos(\theta) = \frac{d}{\|w\|}$. Then

  $$\mathbf{a} \cdot \mathbf{w} = \|\mathbf{a}\| d \Rightarrow d = \frac{\mathbf{a} \cdot \mathbf{w}}{\|\mathbf{a}\|}$$

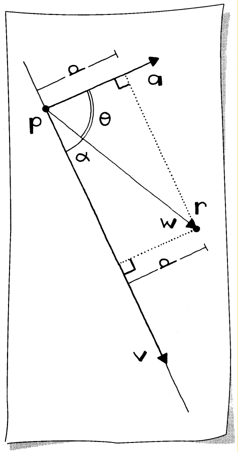


Lines in 2D

Distance of a point to a line

- Parametric line:
  Line: $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$
  Point: $\mathbf{r}$

  Let $\mathbf{w} = \mathbf{r} - \mathbf{p}$ and calculate:
  $$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos(\alpha)$$
  Analyzing the figure we note that $\sin(\alpha) = \frac{d}{\|w\|} = \sqrt{1 - \cos^2(\alpha)}$. Then
  $$d = \|\mathbf{w}\|\sqrt{1 - \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)^2}$$



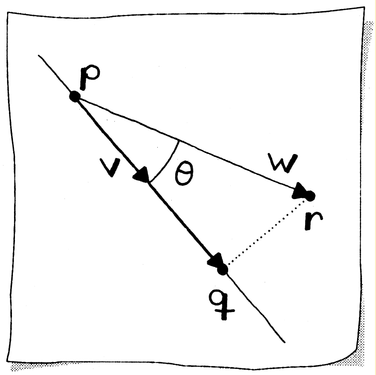
Lines in 2D

The foot of a point

- <u>Parametric line</u>:
  Line: $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$
  Point: $\mathbf{r}$

  Let $\mathbf{w} = \mathbf{r} - \mathbf{p}$. The closest point within the line to $\mathbf{r}$ is

$$\mathbf{q} = \mathbf{p} + \mathrm{Proj}_{\mathbf{v}}\{\mathbf{w}\} = \mathbf{p} + \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|^2}\mathbf{v}$$



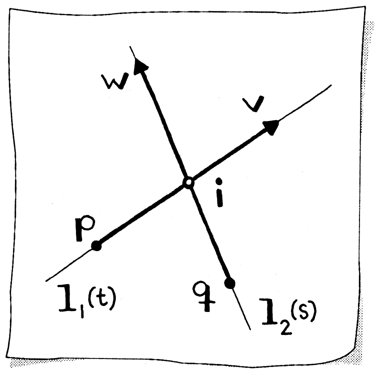
Lines in 2D

The intersection of two lines

- Parametric lines:
  Line 1: $\mathbf{l}_1(t) = \mathbf{p} + t\mathbf{v}$
  Line 2: $\mathbf{l}_2(s) = \mathbf{q} + s\mathbf{w}$

  We need to solve the equation system

$$\mathbf{l}_1(t) = \mathbf{l}_2(s)$$
$$\mathbf{p} + t\mathbf{v} = \mathbf{q} + s\mathbf{w}$$
$$\begin{pmatrix} \mathbf{v} & -\mathbf{w} \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} = \mathbf{q} - \mathbf{p}$$



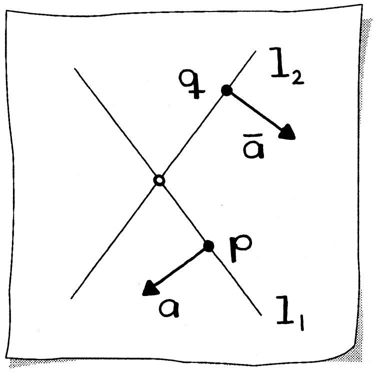
Lines in 2D

The intersection of two lines

- Implicit lines:
  Line 1: $\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0$
  Line 1: $\bar{\mathbf{a}} \cdot (\mathbf{x} - \mathbf{q}) = 0$

  We need to find $\mathbf{x}$ satisfying both equations at the same time

$$\mathbf{a}^T\mathbf{x} - \mathbf{a}^T\mathbf{p} = 0$$
$$\bar{\mathbf{a}}^T\mathbf{x} - \bar{\mathbf{a}}^T\mathbf{q} = 0$$
$$\begin{pmatrix} \mathbf{a}^T \\ \bar{\mathbf{a}}^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{a}^T\mathbf{p} \\ \bar{\mathbf{a}}^T\mathbf{q} \end{pmatrix}$$



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Affine maps in 2D

Reference

Farin and Hansford, Chapter 6

Affine change of coordinates

We transform the point $\mathbf{x}$ into point $\mathbf{x}'$. Note that the matrix multiplication is performed on vectors, not on points

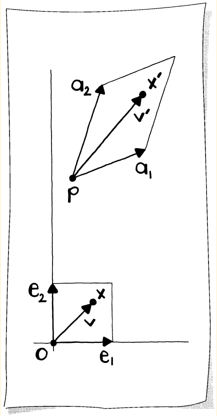
$$\mathbf{v} = \mathbf{x} - \mathbf{o}$$
$$\mathbf{v}' = A\mathbf{v}$$
$$\mathbf{x}' = \mathbf{p} + \mathbf{v}'$$

In total

$$\mathbf{x}' = \mathbf{p} + A(\mathbf{x} - \mathbf{o})$$

We may go back by

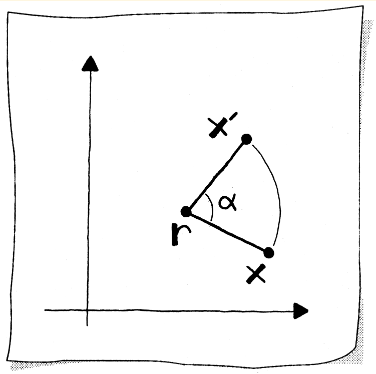
$$\mathbf{x} = \mathbf{o} + A^{-1}(\mathbf{x}' - \mathbf{p})$$



Affine maps in 2D

Translations and rotations

- Translation: $\mathbf{x}' = \mathbf{p} + (\mathbf{x} - \mathbf{o})$
- Rotation: $\mathbf{x}' - \mathbf{r} = R_\alpha(\mathbf{x} - \mathbf{r})$



Affine maps in 2D

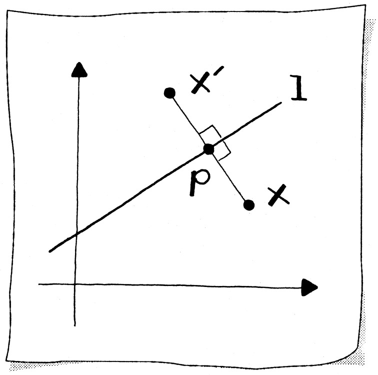
Mirrors and compositions

- Mirror:

$$\mathbf{p} = \frac{1}{2}(\mathbf{x} + \mathbf{x}')$$
$$\mathbf{x}' = 2\mathbf{p} - \mathbf{x}$$

- Compositions:

$$\mathbf{x}' = \mathbf{o}' + A(\mathbf{x} - \mathbf{o})$$
$$\mathbf{x}'' = \mathbf{o}'' + A'(\mathbf{x}' - \mathbf{o}')$$
$$\overline{\mathbf{x}'' = \mathbf{o}'' + A'A(\mathbf{x} - \mathbf{o})}$$



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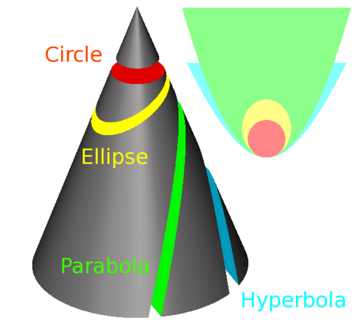
Conic sections

Reference

Juan de Buegos (2000), Capítulo 11

Conic sections

The circle, the ellipse, the parabola, and the hyperbola are all curves stemming from a section of a cone.



Conic sections

Conic sections

They are all second order curves

$$\underbrace{ax_1^2 + bx_2^2 + cx_1x_2}_{\text{2nd order}} + \underbrace{dx_1 + ex_2}_{\text{1st order}} + \underbrace{f}_{\text{0th order}} = 0$$

By renaming the coefficients, we may rewrite it as

$$a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 + 2b_1x_1 + 2b_2x_2 + c = 0$$
$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 2 \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c = 0$$
$$\mathbf{x}^T A\mathbf{x} + 2B\mathbf{x} + c = 0$$

Compare this to the more widely known equation of the parabola $y = ax^2 + bx + c$. Finally, we can write it in a very compact form

$$\tilde{\mathbf{x}}^T M\tilde{\mathbf{x}} = \begin{pmatrix} x_1 & x_2 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = 0$$

Conic sections

Definition 5.1 (Conic sections)

*A **conic section** or **conics** is the locus (lugar geométrico) of all points satisfying*

$$\tilde{\mathbf{x}}^T M \tilde{\mathbf{x}} = 0$$

Definition 5.2 (Conic equality)

*Two conics $\tilde{\mathbf{x}}^T M_1 \tilde{\mathbf{x}} = 0$ and $\tilde{\mathbf{x}}^T M_2 \tilde{\mathbf{x}} = 0$ are the **same** if*

$$M_1 = kM_2$$

for some real number k.

Definition 5.3 (Degenerate and ordinary conics)

*A conic section is **degenerate** if*

$$\det\{M\} = 0$$

*A conic section is **ordinary**, if it is not degenerate.*

Conic sections

Examples of ordinary conics

Circumphere	$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Parabola	$y^2 = 2px$

Examples of degenerate conics

Two lines	$x^2 - y^2 = (x - y)(x + y) = 0$
Two lines	$x^2 - 4 = (x - 2)(x + 2) = 0$
Two lines (superposed)	$x^2 = 0$
Two complex lines	$x^2 + y^2 = (x - iy)(x + iy) = 0$

Intersection of a conics and a line

Intersection of a conics and a line

Consider the parametric equation of a line in homogeneous coordinates

$$\tilde{\mathbf{l}}(t) = \begin{pmatrix} l_1(t) \\ l_2(t) \\ 1 \end{pmatrix} = \begin{pmatrix} p_1 + tv_1 \\ p_2 + tv_2 \\ 1 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \tilde{\mathbf{p}} + t\tilde{\mathbf{v}}$$

We need to find a point in the line (i.e., t) such that

$$\begin{gathered} \tilde{\mathbf{l}}(t)^T M \tilde{\mathbf{l}}(t) = 0 \\ (\tilde{\mathbf{p}} + t\tilde{\mathbf{v}})^T M (\tilde{\mathbf{p}} + t\tilde{\mathbf{v}}) = 0 \\ \tilde{\mathbf{v}}^T M \tilde{\mathbf{v}} t^2 + 2\tilde{\mathbf{v}}^T M \tilde{\mathbf{p}} t + \tilde{\mathbf{p}}^T M \tilde{\mathbf{p}} = 0 \end{gathered}$$

This is a second order equation in t. If there is no solution, then the line does not intersect the conics. If there is only 1 solution, then the line is **tangent** to the conics. If there are 2 solutions, then the line intersects the conics (the line is **secant** to the conics, *secante*).

Reduced equation of a conics

Let λ_1 and λ_2 be the eigenvalues of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$. Then, there exists a basis in which the conics can be expressed as

$\lambda_1 \neq 0, \lambda_2 \neq 0$	$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\det\{M\}}{\det\{A\}} = 0$	Ellipses, hyperbolas, pairs of intersecting lines.
$\lambda_1 = 0, \lambda_2 \neq 0$ $\det\{M\} \neq 0$	$y^2 = 2\sqrt{-\frac{\det\{M\}}{\lambda_2^3}}x$	Parabolas
$\lambda_1 = 0, \lambda_2 \neq 0$ $\det\{M\} = 0$	$y^2 = k$	Pairs of parallel lines

General classification of conics

Definition 5.4 (Signature of a quadratic form)

Consider a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ *and its diagonalization such that*

$$Q(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$$

The signature of $Q(\mathbf{x})$ *is* (n_0, n_+, n_-) *where* n_0 *is the number of null* λ *coefficients,* n_+ *the number of positve* λ *coefficients, and* n_- *the number of negative* λ *coefficients.*

Theorem 5.1

The signature of a quadratic form is invariant to changes of basis, i.e., it only depends on Q.

Definition 5.5 (Signature of a matrix)

The signature of a symmetric matrix is the signature of its associated quadratic form.

General classification of conics

General classification of conics

A	M	**Conics**
$\det\{A\} > 0$	$\mathrm{Sig}\{M\} = (0,1,2)$ or $(0,2,1)$	(Real) Ellipse
	$\mathrm{Sig}\{M\} = (0,3)$ or $(0,3,0)$	Empty set (or imaginary ellipse)
	$\mathrm{Det}\{M\} = 0$	A point (or the intersection of two imaginary lines)
$\det\{A\} < 0$	$\det\{M\} \neq 0$	Hyperbola
	$\det\{M\} = 0$	Two secant (real) lines
$\det\{A\} = 0$	$\det\{M\} \neq 0$	Parabola
	$\det\{M\} = 0$	Two parallel (real) lines

Geometric transformations

Geometric transformations

- Shift: Shift the center to $\hat{\mathbf{c}} = (c_1, c_2, 0)$

$$(\tilde{\mathbf{x}} - \hat{\mathbf{c}})^T M_1 (\tilde{\mathbf{x}} - \hat{\mathbf{c}}) = 0$$

- Rotate: Rotate the conics with a rotation matrix R:

$$(R\tilde{\mathbf{x}})^T M_1 (R\tilde{\mathbf{x}}) = 0$$
$$\tilde{\mathbf{x}}^T (R^T M_1 R)\tilde{\mathbf{x}} = 0$$

with $R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

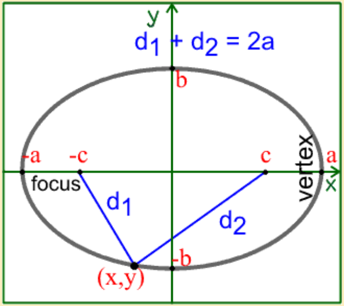
Ellipse

Ellipse

Reduced equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Parametric equation:
$x = a \cos t$
$y = b \sin t$
$t \in [0, 2\pi)$

Interfocal distance: $d(F, F') = 2c$
where
$a^2 + b^2 = c^2$



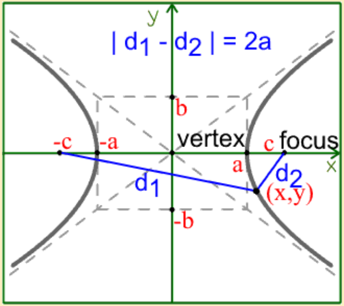
Hyperbola

Hyperbola

Reduced equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parametric equation: $x = \pm a \cosh t$
$y = b \sinh t$
$t \in \mathbb{R}$

Interfocal distance: $d(F, F') = 2c$
where
$a^2 + b^2 = c^2$



(Calculus note)

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2} \qquad \sinh x = \frac{e^{x}-e^{-x}}{2}$$

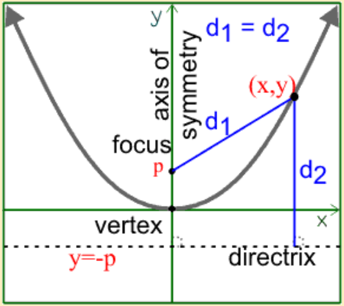
$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$

Parabola

Parabola

Reduced equation: $y^2 = 2px$

Parametric equation: $\begin{cases} x = \frac{t^2}{2p} \\ y = t \\ t \in \mathbb{R} \end{cases}$



Outline

9 Linear algebra applications in geometry

- Local and global coordinates
- Points and vectors
- Lines in 2D
- Affine maps in 2D
- Conic sections in 2D
- 3D Geometry
- Quadrics in 3D

Cross product

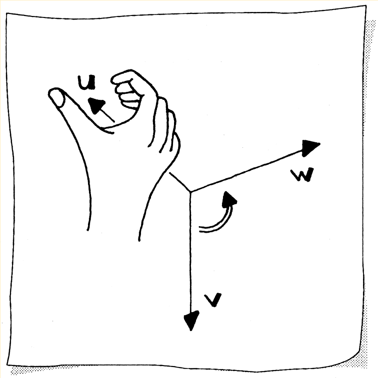
Reference

Farin and Hansford, Chapter 10

Cross product

The cross product is defined for 3D vectors as

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



Properties:

$$\mathbf{u} \perp \mathbf{v} \text{ and } \mathbf{u} \perp \mathbf{w}$$
$$\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|\|\mathbf{w}\| - (\mathbf{v} \cdot \mathbf{w})^2$$
$$\mathbf{v} \times (c\mathbf{v}) = \mathbf{0}$$
$$\mathbf{v} \times (c\mathbf{w}) = (c\mathbf{v}) \times \mathbf{w} = c(\mathbf{v} \times \mathbf{w})$$
$$\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$$
$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

Cross product

Example

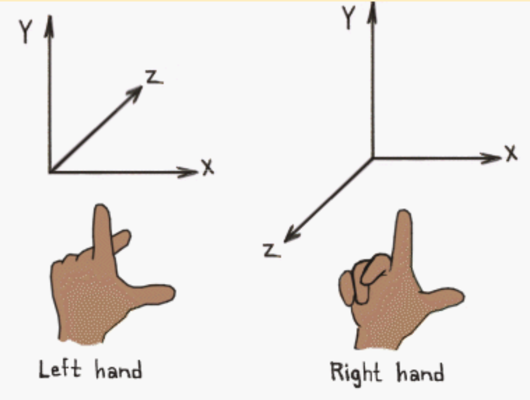
$$\mathbf{u} = \mathbf{e}_1 \times \mathbf{e}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{e}_3$$

$$\mathbf{u} = \mathbf{e}_2 \times \mathbf{e}_1 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{e}_3$$

Cross product

Coordinate systems

- Right-handed:
  $\mathbf{x} \times \mathbf{y} = \mathbf{z}$
  $\mathbf{y} \times \mathbf{z} = \mathbf{x}$
  $\mathbf{z} \times \mathbf{x} = \mathbf{y}$
- Left-handed:
  $\mathbf{x} \times \mathbf{y} = -\mathbf{z}$
  $\mathbf{y} \times \mathbf{z} = \mathbf{x}$
  $\mathbf{z} \times \mathbf{x} = -\mathbf{y}$

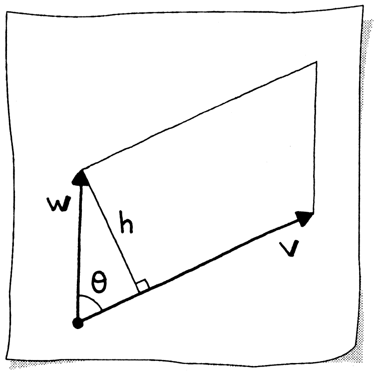


Cross product

Area of parallelogram

The norm of $\mathbf{v} \times \mathbf{w}$ is the area of the parallelogram formed by **u** and **v** and is equal to:

$$A = \|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\theta)$$



Lines

Parametric equation of a line

A line is defined in 3D (and nD) by two points or a point and a vector

- Given two points:
  $\mathbf{l}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) \quad t \in \mathbb{R}$
- Given point and vector:
  $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v} \quad t \in \mathbb{R}$

Giving a point and a perpendicular vector does no longer work.

Planes

Implicit equation of a planes

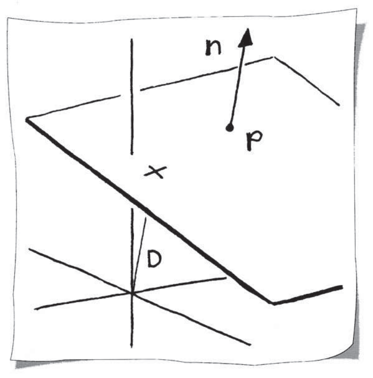
A plane is defined in 3D by a point and a perpendicular vector

- Given a point and the normal direction:
  $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

  In 3D:

  $$(n_1, n_2, n_3) \cdot (x_1 - p_1, x_2 - p_2, x_3 - p_3) = 0 \Rightarrow$$
  $$Ax_1 + Bx_2 + Cx_3 + D = 0$$

  The absolute value of D in the implicit equation is the distance of the plane to the coordinate system origin.



Hyperplanes

Hyperplanes

A hyperplane of $\mathbb{R}^n$ is an affine space of a dimension $n-1$. For instance

$\mathbb{R}^n$	Dimension	Dimension of hyperplane	Hyperplane name
$\mathbb{R}^2$	2D	1	Line
$\mathbb{R}^3$	3D	2	Plane
$\mathbb{R}^n$	nD	n-1	Hyperplane

All hyperplanes are defined by a point ($\mathbf{p}$) and a normal vector ($\mathbf{n}$)

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

Distance of a point to a plane (hyperplane)

The distance between a point $\mathbf{r}$ and a plane (or hyperplane) is given by

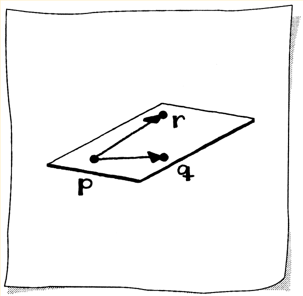
$$d = \frac{\mathbf{n} \cdot (\mathbf{r} - \mathbf{p})}{\|\mathbf{n}\|}$$

Planes

Parametric equation of a plane

A plane can also be defined in 3D (and nD) by a point and two in-plane vectors

- Given a point and two in-plane vectors:
  $P(s,t) = \mathbf{p} + s\mathbf{v} + t\mathbf{w} \quad \forall s,t \in \mathbb{R}$
- Given three points:
  $P(s,t) = \mathbf{p} + s(\mathbf{q} - \mathbf{p}) + t(\mathbf{r} - \mathbf{p}) \quad \forall s,t \in \mathbb{R}$



Scalar triple product

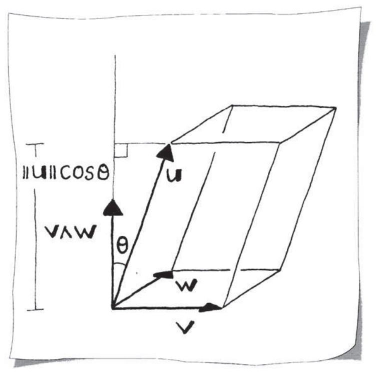
Scalar triple product

The volume of a parallelepiped can be measured with the scalar triple product

$$V = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

Properties:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$$



Distance between two lines

Given two lines in parametric form

$$\mathbf{l}_1(s_c) = \mathbf{p}_0 + s_c\mathbf{u} \ \mathbf{l}_2(t_c) = \mathbf{q}_0 + t_c\mathbf{v}$$

The distance between the two lines is the length of the vector $\mathbf{w}_c$ that is perpendicular to both lines. $\mathbf{w}_c$ is defined by two points: one in line 1 ($\mathbf{x}_1$) and another one in line 2 ($\mathbf{x}_2$):

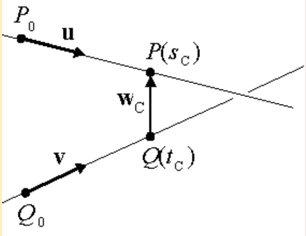
$$\mathbf{w}_c = \mathbf{x}_2 - \mathbf{x}_1 = \mathbf{q}_0 + t_c\mathbf{v} - (\mathbf{p}_0 + s_c\mathbf{u})$$

The conditions on $\mathbf{w}_c$ are:

$$\mathbf{w}_c \cdot \mathbf{u} = 0 \text{ and } \mathbf{w}_c \cdot \mathbf{v} = 0$$

After reorganizing the terms

$$\begin{pmatrix} \|\mathbf{u}\|^2 & -\mathbf{u}\cdot\mathbf{v} \\ \mathbf{u}\cdot\mathbf{v} & \|\mathbf{v}\|^2 \end{pmatrix} \begin{pmatrix} s_c \\ t_c \end{pmatrix} = \begin{pmatrix} (\mathbf{p}_0 - \mathbf{q}_0)\cdot\mathbf{u} \\ (\mathbf{p}_0 - \mathbf{q}_0)\cdot\mathbf{v} \end{pmatrix}$$



Intersection of two lines

Intersection of two lines

The two lines in the previous slide intersect if $\mathbf{x}_1 = \mathbf{x}_2$. We also note that the two lines intersect if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{p}_0 - \mathbf{q}_0$ are in the same plane, or what is the same they are linearly dependent

$$\left|\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{p}_0 - \mathbf{q}_0\end{pmatrix}\right| = 0$$

Intersection of a line and a plane

Intersection of a line and a plane

- Parametric line, implicit plane:

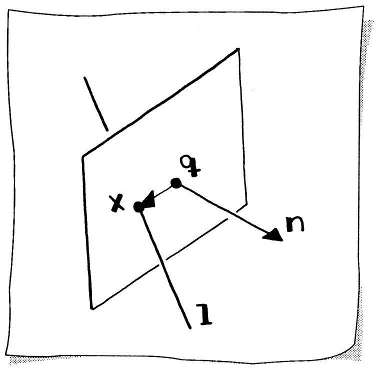
$$\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$$
$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{q}) = 0$$

For the intersection we need to find t such that

$$\mathbf{n} \cdot (\mathbf{p} + t\mathbf{v} - \mathbf{q}) = 0$$

whose solution is

$$t = \frac{\mathbf{n} \cdot (\mathbf{q} - \mathbf{p})}{\mathbf{n} \cdot \mathbf{v}}$$
$$\mathbf{x} = \mathbf{p} + \frac{\mathbf{n} \cdot (\mathbf{q} - \mathbf{p})}{\mathbf{n} \cdot \mathbf{v}} \mathbf{v}$$



Intersection of a line and a plane

Intersection of a line and a plane

- Parametric line, parametric plane:

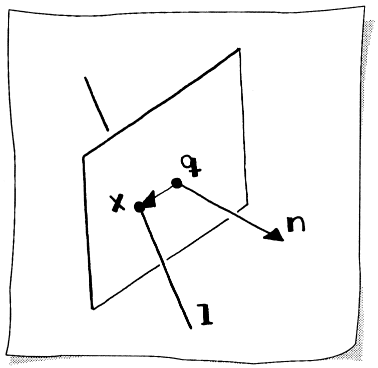
$$\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$$
$$\mathbf{P}(t_1, t_2) = \mathbf{q} + t_1\mathbf{u} + t_2\mathbf{w}$$

We need to find t, t_1 and t_2 such that

$$\mathbf{p} + t\mathbf{v} = \mathbf{q} + t_1\mathbf{u} + t_2\mathbf{w}$$

Reorganizing the terms:

$$\begin{pmatrix}\mathbf{u} & \mathbf{w} & -\mathbf{v}\end{pmatrix}\begin{pmatrix}t_1 \\ t_2 \\ t\end{pmatrix} = \mathbf{p} - \mathbf{q}$$



Intersection of a line and a triangle

- Parametric line, 3 points of a triangle:

$$\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$$

$$\mathbf{P}(t_1, t_2) = \mathbf{p}_1 + t_1(\mathbf{p}_2 - \mathbf{p}_1) + t_2(\mathbf{p}_3 - \mathbf{p}_1)$$
$$t_1, t_2 \in [0, 1], t_1 + t_2 \leq 1$$

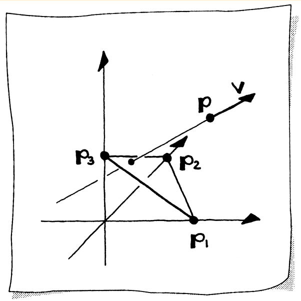
We need to find t, t_1 and t_2 such that

$$\mathbf{p} + t\mathbf{v} = \mathbf{p}_1 + t_1(\mathbf{p}_2 - \mathbf{p}_1) + t_2(\mathbf{p}_3 - \mathbf{p}_1)$$

Reorganizing the terms:

$$\begin{pmatrix} \mathbf{p}_2 - \mathbf{p}_1 & \mathbf{p}_3 - \mathbf{p}_1 & -\mathbf{v} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t \end{pmatrix} = \mathbf{p} - \mathbf{p}_1$$

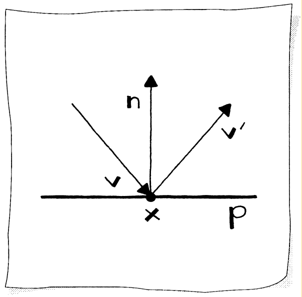
The intersection point is within the triangle if $t_1, t_2 \in [0, 1], t_1 + t_2 \leq 1$.



Reflection

Reflection

- Reflection:
  This situation is encountered, for instance, in reflected light rays. By inspecting the figure we note that
  $$\mathbf{n} \cdot \mathbf{v} = -\mathbf{n} \cdot \mathbf{v}'$$
  On the other side, it must also be
  $$c\mathbf{n} = \mathbf{v}' - \mathbf{v}$$
  We have two unknowns c and $\mathbf{v}$ and two equations. After some manipulation we reach
  $$\mathbf{v}' = \mathbf{v} - 2(\mathbf{n} \cdot \mathbf{n}^T)\mathbf{v}$$



Intersection of three planes

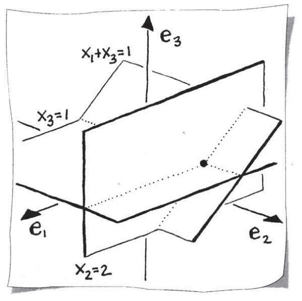
- Implicit equations:
  For each of the planes, we have

$$\mathbf{n}_1 \cdot (\mathbf{x} - \mathbf{p}_1) = 0 \Rightarrow \mathbf{n}_1^T \mathbf{x} = \mathbf{n}_1^T \mathbf{p}_1$$
$$\mathbf{n}_2 \cdot (\mathbf{x} - \mathbf{p}_2) = 0 \Rightarrow \mathbf{n}_2^T \mathbf{x} = \mathbf{n}_2^T \mathbf{p}_2$$
$$\mathbf{n}_3 \cdot (\mathbf{x} - \mathbf{p}_3) = 0 \Rightarrow \mathbf{n}_3^T \mathbf{x} = \mathbf{n}_3^T \mathbf{p}_3$$

  Gathering all together

$$\begin{pmatrix} \mathbf{n}_1^T \\ \mathbf{n}_2^T \\ \mathbf{n}_3^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{n}_1^T \mathbf{p}_1 \\ \mathbf{n}_2^T \mathbf{p}_2 \\ \mathbf{n}_3^T \mathbf{p}_3 \end{pmatrix}$$

  In non-degenerate situations, this equation system has a unique solution that is the intersection point. Otherwise, the planes may intersect in one line, two lines, three lines, or even in a plane (if the three planes are the same plane).



Intersection of two planes

Intersection of two planes

- Implicit equations:
  For each of the planes, we have

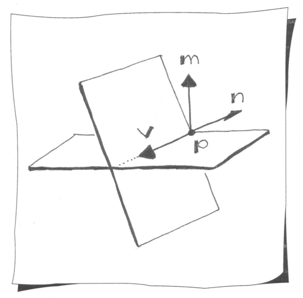
$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{p}_1)=0 \Rightarrow \mathbf{n}^T\mathbf{x}=\mathbf{n}^T\mathbf{p}_1$$
$$\mathbf{m}\cdot(\mathbf{x}-\mathbf{p}_2)=0 \Rightarrow \mathbf{m}^T\mathbf{x}=\mathbf{m}^T\mathbf{p}_2$$

The two planes intersect in a line of the form

$$\mathbf{l}(t)=\mathbf{p}+t(\mathbf{n}\times\mathbf{m})$$

To find $\mathbf{p}$ we solve the equation system

$$\begin{pmatrix}\mathbf{n}^T\\ \mathbf{m}^T\\ (\mathbf{n}\times\mathbf{m})^T\end{pmatrix}\mathbf{x}=\begin{pmatrix}\mathbf{n}^T\mathbf{p}_1\\ \mathbf{m}^T\mathbf{p}_2\\ 0\end{pmatrix}$$



Outline

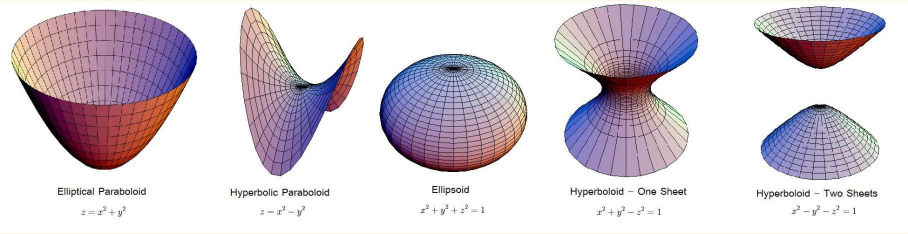
Quadrics

Reference

Juan de Buegos (2000), Capítulo 12

Quadrics

Quadrics are 3D surfaces that meet a second order equation.



Quadrics in the Wikipedia

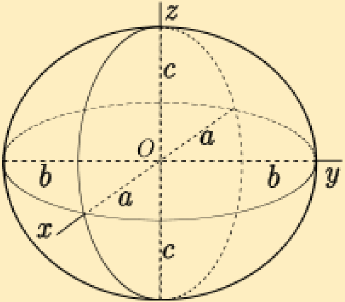
Ellipsoid

Ellipsoid

Reduced equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Parametric equation:
$$x = a \cos u \sin v$$
$$y = b \sin u \sin v$$
$$z = c \cos v$$
$$u, v \in [0, 2\pi)$$

Cuts along X, Y and Z are ellipses.



Hyperboloid of one sheet

Hyperboloid of one sheet

Reduced equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

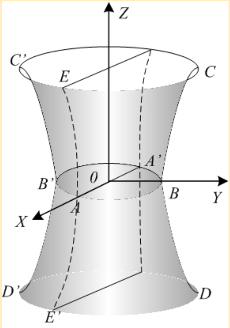
Parametric equation:

$$x = a\sqrt{1+u^2}\cos v$$
$$y = b\sqrt{1+u^2}\sin v$$
$$z = cu$$

Parametric equation:

$$x = a\cosh u \cos v$$
$$y = b\cosh u \sin v$$
$$z = c\sinh u$$

$$v \in [0, 2\pi),\ u \in \mathbb{R}$$



Cuts along X and Y are hyperbolas.
Cuts along Z are ellipses.

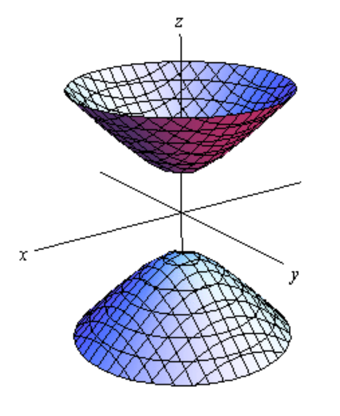
Hyperboloid of two sheets

Hyperboloid of two sheets

Reduced equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

Parametric equation:
$$x = a \sinh u \cos v$$
$$y = b \sinh u \sin v$$
$$z = c \cosh u$$
$$v \in [0, 2\pi),\ u \in \mathbb{R}$$

Cuts along X and Y are hyperbolas.
Cuts along Z are ellipses.

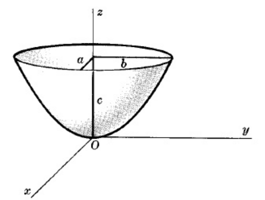


Elliptic paraboloid

Elliptic paraboloid

Reduced equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0$

Parametric equation:
$$x = a\sqrt{u}\cos v$$
$$y = b\sqrt{u}\sin v$$
$$z = cu$$
$$v \in [0, 2\pi),\ u \in [0, \infty)$$



Cuts along X and Y are parabolas.
Cuts along Z are ellipses.

Hyperbolic paraboloid

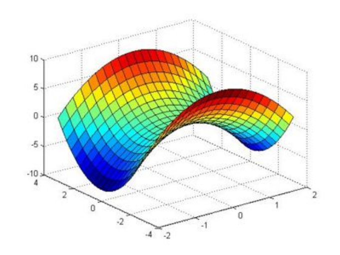
Hyperbolic paraboloid

Reduced equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0$

Parametric equation:

$$x = a\sqrt{u}\cosh v$$
$$y = b\sqrt{u}\sinh v$$
$$z = cu$$
$$u, v \in \mathbb{R}$$

Cuts along Y are parabolas.
Cuts along Z are hyperbolas.

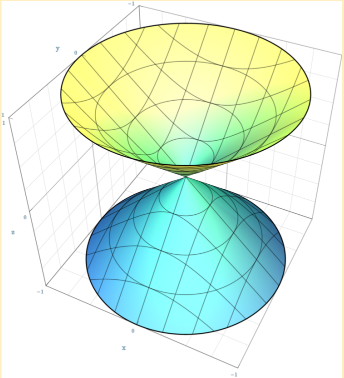


Cone

Cone

Reduced equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

Parametric equation:
$$x = au\cos v$$
$$y = bu\sin v$$
$$z = cu$$
$$v \in [0, 2\pi), u \in \mathbb{R}$$



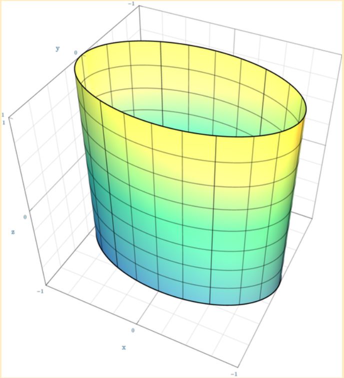
Cuts along Y are parabolas.
Cuts along Z are ellipses.

Elliptic cylinder

Elliptic cylinder

Reduced equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Parametric equation:
$$x = a \cos v$$
$$y = b \sin v$$
$$z = u$$
$$v \in [0, 2\pi), u \in \mathbb{R}$$



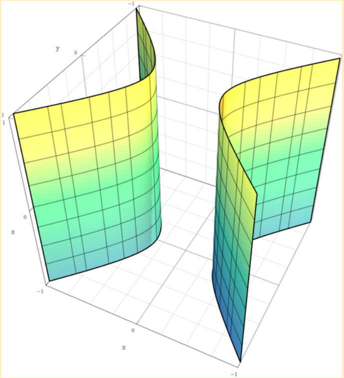
Cuts along X and Y are pairs of lines.
Cuts along Z are ellipses.

Hyperbolic cylinder

Hyperbolic cylinder

Reduced equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parametric equation:
$$x = a\cosh v$$
$$y = b\sinh v$$
$$z = u$$
$$u, v \in \mathbb{R}$$



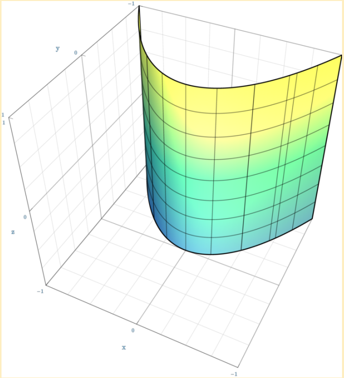
Cuts along X and Y are pairs of lines.
Cuts along Z are hyperbolas.

Parabolic cylinder

Parabolic cylinder

Reduced equation: $\frac{x^2}{a^2} - \frac{y}{b} = 0$

Parametric equation:
$$x = au$$
$$y = bu^2$$
$$z = v$$
$$u, v \in \mathbb{R}$$



Cuts along X and Y are pairs of lines or single lines.
Cuts along Z are parabolas.

Quadrics

Definition 7.1

Quadrics All quadrics can be written as

$$\sum_{i,j=1}^{3} a_{ij} x_i x_j + 2 \sum_{i=1}^{3} b_i x_i + c = 0$$
$$\tilde{\mathbf{x}}^T M \tilde{\mathbf{x}} = 0$$

with $a_{ij} = a_{ji}$ *and*

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{12} & a_{22} & a_{23} & b_2 \\ a_{13} & a_{23} & a_{33} & b_3 \\ b_1 & b_2 & b_3 & c \end{pmatrix} \text{ and } \tilde{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}$$

Quadrics

Definition 7.2 (Quadrics equality)

Two quadrics $\tilde{\mathbf{x}}^T M_1 \tilde{\mathbf{x}} = 0$ *and* $\tilde{\mathbf{x}}^T M_2 \tilde{\mathbf{x}} = 0$ *are the* ***same*** *if*

$$M_1 = k M_2$$

for some real number k.

Definition 7.3 (Degenerate or ordinary quadrics)

A quadrics is ***degenerate*** *if* $\det\{M\} = 0$ *(e.g., cones, cylinders and pairs of planes). It is ordinary if it is not degenerate (e.g., ellipsoids, paraboloids, hyperboloids)*

Examples of degenerate quadrics

$x^2 - y^2 = 0 = (x-y)(x+y)$	A pair of planes
$x^2 + y^2 = 0 = (x-iy)(x+iy)$	A pair of imaginary planes
$x^2 - 1 = 0 = (x-1)(x+1)$	A pair of planes
$x^2 + y^2 - 25 = 0$	Cylinder of radius 5

General classification of quadrics

General classification of quadrics

Let λ_1, λ_2 and λ_3 be the eigenvalues of A. Then, there exists a basis such that the reduced equation of the quadrics is

Condition	**Quadrics**
$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$	$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \frac{\det\{M\}}{\det\{A\}} = 0$ Ellipsoids, hyperboloids and cones
$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$ $\det\{M\} \neq 0$	$\lambda_1 x^2 + \lambda_2 y^2 = 2\sqrt{-\frac{\det\{M\}}{\lambda_1 \lambda_2}} z$ Paraboloid
$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$ $\det\{M\} = 0$	$\lambda_1 x^2 + \lambda_2 y^2 = k$ Elliptical cylinder
$\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0$ $\mathrm{Rank}\{M\} = 3$	$y^2 = 2qx$ Parabolic cylinder
$\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0$ $\mathrm{Rank}\{M\} < 3$	$y^2 = k$ Pair of planes

Outline

9. Linear algebra applications in geometry
   - Local and global coordinates
   - Points and vectors
   - Lines in 2D
   - Affine maps in 2D
   - Conic sections in 2D
   - 3D Geometry
   - Quadrics in 3D