BAYES' THEOREM

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1. INTRODUCTION TO CONDITIONAL PROBABILITY

In order to analyze in more detail the Bayes' Theorem, we must first, give a brief introduction to the types of probability. We are going to consider two types of probability:

Conditional: Let Ω be the sample space of a random experiment, A and B two events with $P(B) \neq 0$, the conditional probability from event A to event B (to event B having occurred) is defined as:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

The conditional probability $P(A/B)$ is a probability defined on the set of events Ω, whose intersection with B is non-empty, therefore, it verifies all the properties of probability. Together with the concept of conditional probability appears the concept of independence.

Independence: Let two events A and B in the sample space Ω, they are said to be independent if the occurrence of one does not change the probability that the other will occur:

$$P(B/A) = P(B)$$

The independence of two events is not an intrinsic property of them, that is, it is not a property that depends on the nature of the events, but rather a property linked to their probabilities.

Once the concept of conditional probability has been introduced, we can explain Bayes' Theorem, as an example of conditional probability.

2. BAYES' THEOREM (Orduz Camacho et al., 2015)

The importance of Bayes in the history of probability theory with which we are concerned is decisive. This author is the initiator of one of the most important parts of this theory when he begins to apply the method of obtaining the probabilities of the causes for which an event that has been observed may have been produced.

Also known as "Mathematical Theory of Probability", this theorem explains **how to obtain the probability of an event *A* given that *B* has occurred**. That is, establishing the prior probability of event A and the probability that event B would have occurred given event A. This was stated as follows:

Bayes' Theorem

This theorem refers to the "calculation of the conditional probability of event A given that B has occurred.

If $A_1, A_2, \ldots, A_n$ are exhaustive and exclusive events such that $P(Ai) > 0$,

$\forall\ i = 1,2, \ldots, n$ then for any event A such that $P(A) > 0$, it is true:

Two events A and B are exclusive if $P(A \cap B) = 0$. A family of events: $A_1, A_2, \ldots, A_n$ is exhaustive if $\cap A_i = \Omega$

General case:

$$P(A_i/A) = \frac{P(A/A_i)P(A_i)}{\sum_{i=1}^{n} P(A/A_i)P(A_i)}$$

The proof of this result is very simple and is based on the definition of conditional probability and the theorem of total probabilities

Conditional probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Total probabilities theorem

Let $A_1, A_2, \ldots, A_n$ be mutually exclusive events with non-zero probability, such that $A_1 \cup A_2 \cup \ldots \cup A_n = \Omega$. If B is an event in Ω, then:

$$P(B) = \sum_{i=1}^{n} P(B/A_i)P(A_i)_i$$

Example for two exclusive events:

$$P(A/B) = \frac{P(A)P(B/A)}{P(A)P(B/A) + P(A')P(B/A)'}$$

3. APPLICATIONS OF BAYES' THEOREM

Application 1: Diagnosis of having tuberculosis (Girón & Bernardo, 2000)

A tuberculin test was applied to a group of people with respiratory disorders to determine if they had the disease. From previous studies, the test's sensitivity and specificity are known, given by:

- Sensitivity: $P(+/Sick) = 0.98$
- Specificity: $P(-/Healthy) = 0.95$

Therefore, the probability of false positives is *P(+/Healthy)* = 1 – 0.95 = 0.05; the probability of false negatives is *P(-/Sick)* = 1 – 0.98 = 0.02. The conditional probabilities of the experiment are summarized in the following table:

	Positive (+)	**Negative (-)**
Sick	0.98	0.02
Healthy	0.05	0.95

"If the result of the test, applied to a person from that risk group, was Positive (+): What is the evidence in favour of that person being sick, *P(Sick/+)*?" **Note:** *P(Sick)* = 0.01.

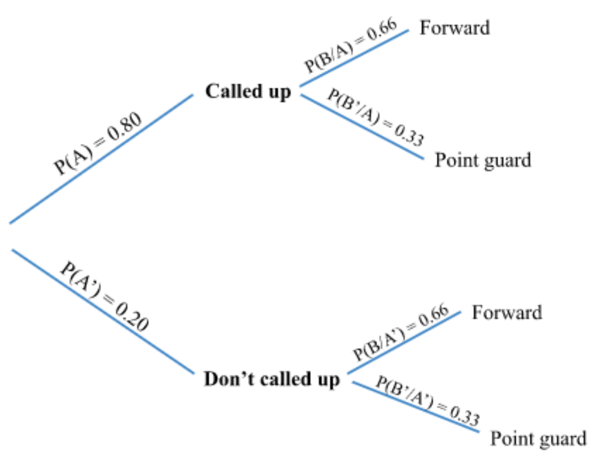
$$\boldsymbol{P(Sick/+)} = \frac{P(Sick)P(+/Sick)}{P(Sick)P(+/Sick)\ + P(Healthy)P(+/Healthy)} = \frac{0.01 * 0.98}{(0.01 * 0.98) + (0.99 * 0.05)}$$
$$= \mathbf{0.1653} = \mathbf{16.53}\ \%$$

The result, which is apparently shocking, is that, only **16.53%** of the people who tested positive would be really sick. The majority of those who tested positive (83.47%) would be false positives.

<u>Application 2:</u> Personal experience

A basketball team is made up of 15 players. To play the match, only 12 of them will be called up. Among the 12 who play, 8 play as forwards, and 4 as point guards. From the 3 left, 2 are forwards and 1 point guard. What probability do I have of being called up if I play in the forward position?

We have to consider the following probabilities:



Now, we calculate the probability of "being called up playing as forward".

$$P(A/B) = \frac{P(A)P(B/A)}{P(A)P(B/A) + P(A')P(B/A)'} = \frac{0.80 * 0.66}{0.80 * 0.66 + 0.20 * 0.33} = 0.888$$
$$= \mathbf{88.8}\%$$

<u>Application 3</u> (Triola, 2010):

An aircraft emergency locator transmitter (ELT) is a device designed to transmit a signal in the case of a crash. The **Altigauge** Manufacturing Company makes 80% of the ELTs, the **Bryant** Company makes 15% of them, and the **Chartair** Company makes the other 5%. The ELTs made by Altigauge have a 4% rate of defects, the Bryant ELTs have a 6% rate of defects, and the Chartair ELTs have a 9% rate of defects (which helps to explain why Chartair has the lowest market share).

a) If an ELT is randomly selected from the general population of all ELTs, find the probability that it was made by the Altigauge Manufacturing Company.
b) If a randomly selected ELT is then tested and is found to be defective, find the probability that it was made by the Altigauge Manufacturing Company.

We use the following notation:

A = ELT manufactured by Altigauge
B = ELT manufactured by Bryant
C = ELT manufactured by Chartair
D = ELT is defective
D* = ELT is not defective (or it is good)

a) If an ELT is randomly selected from the general population of all ELTs, the probability that it was made by Altigauge is **0.8** (because Altigauge manufactures 80% of them).
b) If we now have the additional information that the ELT was tested and was found to be defective, we want to revise the probability from part (a) so that the new information can be used. We want to find the value of *P(A/D),* which is the probability that the ELT was made by the Altigauge company given that it is defective. Based on the given information, we know these probabilities:

P(A) = 0.80 because Altigauge makes 80% of the ELTs
P(B) = 0.15 because Bryant makes 15% of the ELTs
P(C) = 0.05 because Chartair makes 5% of the ELTs
P(D/A) = 0.04 because 4% of the Altigauge ELTs are defective
P(D/B) = 0.06 because 6% of the Bryant ELTs are defective
P(D/C) = 0.09 because 9% of the Chartair ELTs are defective

Here is Bayes' theorem extended to include three events corresponding to the selection of ELTs from the three manufacturers (A, B, C):

$$P(A/D) = \frac{P(A)P(D/A)}{[P(A)P(D/A)] + [P(B)P(D/B)] + [P(C)P(D/C)]} =$$

$$= \frac{0.80 * 0.04}{[0.80 * 0.04] + [0.15 * 0.06] + [0.05 * 0.09]} = 0.703 = \mathbf{70.3\%}$$

4. USEFUL INFORMATION

Additional information to better understand the Bayes' Theorem are:

- In the next review, they analysed **Bayes' Theorem as a research tool in biomedical sciences**: Okeh, U. M., & Ugwu, A. C. (2008). Bayes' theorem: A paradigm research tool in biomedical sciences. African Journal of Biotechnology, 7(25).

- For a more **teaching and visual explanation**, you can see the following video: https://www.youtube.com/watch?v=9wCnvr7Xw4E.